Radiation Processes in High Energy Astrophysics

A Physical Approach with Applications

- Fundamentals
- Thomson Scattering
- Bremsstrahlung
- Synchrotron Radiation
- Inverse Compton Scattering
- Synchro-Compton Radiation
- Relativistic Beaming
- Particle Acceleration

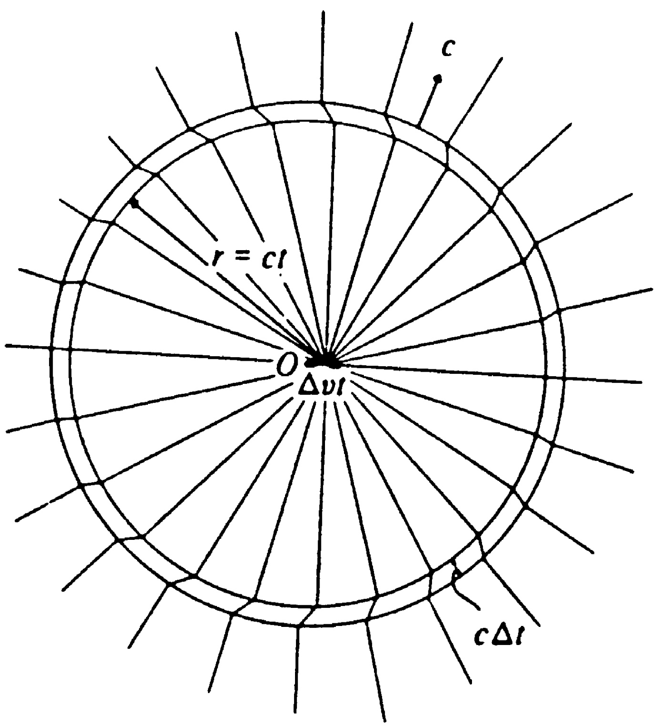
Basic Radiation Concepts

Much of what we need to understand radiation processes in X-ray and γ-ray astronomy can be derived using classical electrodynamics and central to that development is the physics of the radiation of accelerated charged particles. The central relation is the *radiation loss rate of an accelerated charged particle* in the non-relativistic limit

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{rad}} = \frac{|\ddot{\boldsymbol{p}}|^2}{6\pi\varepsilon_0 c^3} = \frac{q^2|\ddot{\boldsymbol{r}}|^2}{6\pi\varepsilon_0 c^3}. \quad (1)$$

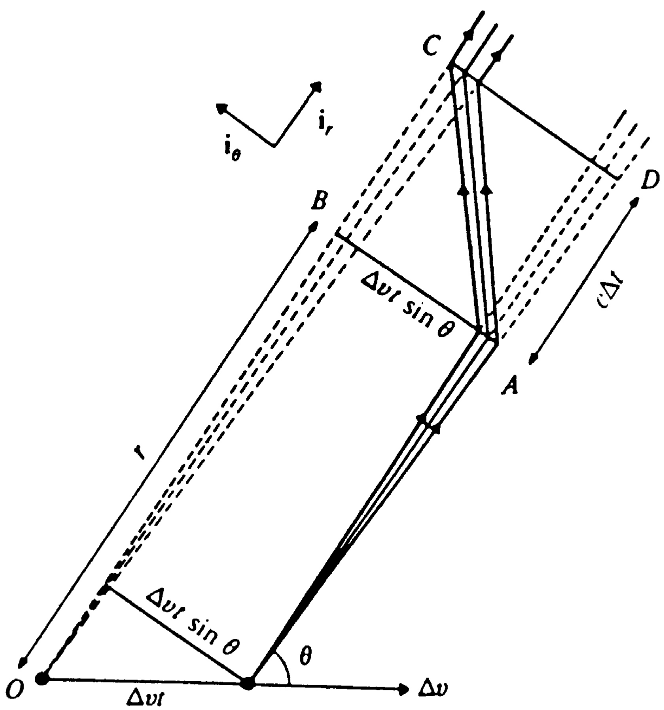
$\boldsymbol{p} = q\boldsymbol{r}$ is the *dipole moment* of the accelerated electron with respect to some origin. This formula is very closely related to the radiation rate of a dipole radio antenna and so is often referred to as the radiation loss rate for *dipole radiation*. Note that I will use *SI units* in all the derivations, although it will be necessary to convert the results into the conventional units used in X-ray and γ-ray astronomy when they are confronted with observations. Thus, I will normally use metres, kilograms, teslas and so on.

The radiation of an accelerated charged particle
J.J. Thomson's treatment (1906, 1907)



Consider a charge q stationary at the origin O of some inertial frame of reference S at time $t = 0$. The charge then suffers a small acceleration to velocity Δv in the short time interval Δt. After a time t, we can distinguish between the field configuration inside and outside a sphere of radius $r = ct$ centred on the origin of S. Outside this sphere, the field lines do not yet know that the charge has moved away from the origin and so the field lines are radial, centred on O. Inside this sphere, the field lines are radial about the origin of the frame of reference centred on the moving charge. Between these two regions, there is a thin shell of thickness $c\,\Delta t$ in which we join up corresponding electric field lines.

Radiation of an accelerated charged particle (2)



There must be a component of the electric field in the $\boldsymbol{i}_\theta$ direction. This 'pulse' of electromagnetic field is propagated away from the charge at the speed of light and is the energy loss of the accelerated charged particle.

The increment in velocity Δv is very small, $\Delta v \ll c$, and therefore it can be assumed that the field lines are radial at $t = 0$ and also at time t in the frame of reference S.

Consider a small cone of electric field lines at angle θ with respect to the acceleration vector of the charge at $t = 0$ and at some later time t when the charge is moving at a constant velocity Δv. We join up electric field lines through the thin shell of thickness $c\,\mathrm{d}t$ as shown in the diagram.

The radiation of an accelerated charged particle (3)

The strength of the E_θ component of the field is given by number of field lines per unit area in the $\boldsymbol{i}_\theta$ direction. From the geometry of the diagram, the E_θ field component is given by the relative sizes of the sides of the rectangle $ABCD$, that is

$$E_\theta / E_r = \Delta v \, t \sin\theta / c\Delta t. \tag{2}$$

E_r is given by Coulomb's law,

$$E_r = q/4\pi\varepsilon_0 r^2 \quad \text{where} \quad r = ct, \tag{3}$$

and so

$$E_\theta = \frac{q(\Delta v/\Delta t)\sin\theta}{4\pi\varepsilon_0 c^2 r}. \tag{4}$$

$\Delta v/\Delta t$ is the acceleration $\ddot{r}$ of the charge and hence

$$E_\theta = \frac{q\ddot{r}\sin\theta}{4\pi\varepsilon_0 c^2 r}. \tag{5}$$

The radiation of an accelerated charged particle (4)

Notice that the radial component of the field decreases as r^{-2}, according to Coulomb's law, but the field in the pulse decreases only as r^{-1} because the field lines become more and more stretched in the E_θ-direction, as can be seen from (2). Alternatively we can write $p = qr$, where p is the dipole moment of the charge with respect to some origin, and hence

$$E_\theta = \frac{\ddot{p} \sin\theta}{4\pi\varepsilon_0 c^2 r}. \tag{6}$$

This is a pulse of electromagnetic radiation and hence the energy flow per unit area per second at distance r is given by the Poynting vector $\boldsymbol{E} \times \boldsymbol{H} = E^2/Z_0$, where $Z_0 = (\mu_0/\varepsilon_0)^{1/2}$ is the impedance of free space. The rate loss of energy through the solid angle dΩ at distance r from the charge is therefore

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{rad}} \mathrm{d}\Omega = \frac{|\ddot{\boldsymbol{p}}|^2 \sin^2\theta}{16\pi^2 Z_0 \varepsilon_0^2 c^4 r^2} r^2 \mathrm{d}\Omega = \frac{|\ddot{\boldsymbol{p}}|^2 \sin^2\theta}{16\pi^2 \varepsilon_0 c^3} \mathrm{d}\Omega. \tag{7}$$

The radiation of an accelerated charged particle (5)

To find the total radiation rate, we integrate over all solid angles, that is, we integrate over θ with respect to the direction of the acceleration. Integrating over solid angle means integrating over $\mathrm{d}\Omega = 2\pi \sin\theta \, \mathrm{d}\theta$ and so

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{rad}} = \int_0^\pi \frac{|\ddot{\boldsymbol{p}}|^2 \sin^2\theta}{16\pi^2 \varepsilon_0 c^3} \, 2\pi \sin\theta \, \mathrm{d}\theta. \tag{8}$$

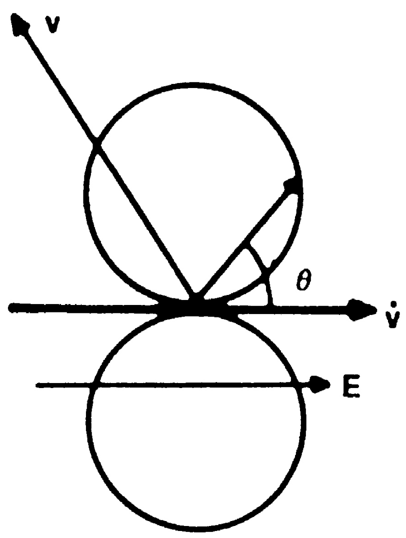
We find the key result

$$\boxed{-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{rad}} = \frac{|\ddot{\boldsymbol{p}}|^2}{6\pi\varepsilon_0 c^3} = \frac{q^2 |\ddot{\boldsymbol{r}}|^2}{6\pi\varepsilon_0 c^3}.} \tag{9}$$

This result is sometimes called *Larmor's formula* – precisely the same result comes out of the full theory. These formulae embody the three essential properties of the radiation of an accelerated charged particle.

The Properties of Dipole Radiation

$$-\left(\frac{dE}{dt}\right)_{rad} = \frac{q^2|\ddot{\boldsymbol{r}}|^2}{6\pi\varepsilon_0 c^3}\,.$$



1. The total radiation rate is given by Larmor's formula. The acceleration is the *proper acceleration* of the particle and the loss rate is measured in its instantaneous rest frame.
2. The *polar diagram* of the radiation is of *dipolar* form, that is, the electric field strength varies as $\sin\theta$ and the power radiated per unit solid angle varies as $\sin^2\theta$, where θ is the angle with respect to the acceleration vector of the particle. Notice that there is no radiation along the acceleration vector and the field strength is greatest at right angles to the acceleration vector.
3. The radiation is *polarised* with the electric field vector lying in the direction of the acceleration vector of the particle, as projected onto a sphere at distance r from the charged particle.

Radiation of Accelerated Electron
Improved Version

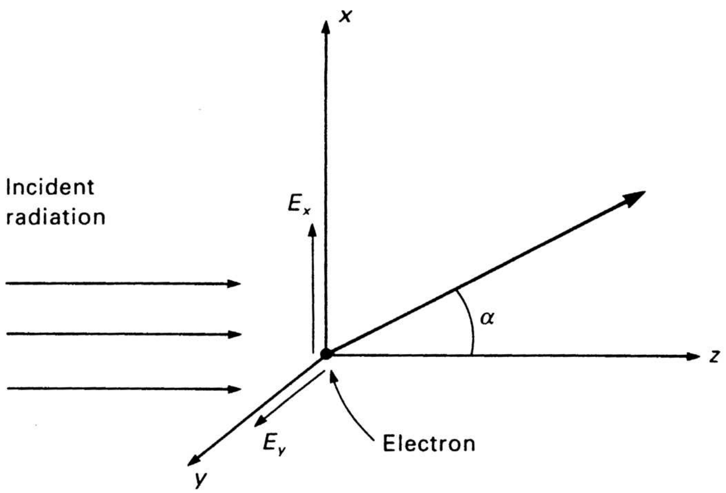
See *High Energy Astrophysics,3rd edition, Section 6.2.3*, for details.

- Write down Maxwell's equations in free space.
- Introduce the scalar and vector potentials and ϕ and $\boldsymbol{A}$ respectively.
- Select the *Lorentz gauge* so that the wave equations for ϕ and $\boldsymbol{A}$ have the same structure with source terms associated with charges and currents respectively.
- Solve for $\boldsymbol{A}$ and then select the solutions for $\boldsymbol{E}$ and $\boldsymbol{B}$ which fall off as $1/r$.
- This gets us back to

$$E_\theta = \frac{\ddot{p}\sin\theta}{4\pi\varepsilon_0 c^2 r}.$$

- Proceed as before.

Example - Thomson Scattering



Thomson scattering is the scattering of electromagnetic waves by free electrons in the classical limit. Thomson first published the formula for the *Thomson cross-section* in 1906 in connection with the scattering of X-rays.

We seek the formula describing the scattering of a beam of radiation incident upon a stationary electron. We assume that the beam of incident radiation propagates in the positive z-direction. Without loss of generality, we arrange the geometry of the scattering so that the scattering angle α lies in the $x - z$ plane. In the case of unpolarised radiation, we resolve the electric field strength into components of equal intensity in the $\boldsymbol{i}_x$ and $\boldsymbol{i}_y$ directions.

Thomson Scattering

The electric fields experienced by the electron in the x and y directions, $E_x = E_{x0} \exp(\mathrm{i}\omega t)$ and $E_y = E_{y0} \exp(\mathrm{i}\omega t)$ respectively, cause the electron to oscillate and the accelerations in these directions are:

$$\ddot{r}_x = eE_x/m_\mathrm{e} \quad \ddot{r}_y = eE_y/m_\mathrm{e}. \tag{10}$$

We can therefore enter these accelerations into the radiation formula (9) which shows the angular dependance of the emitted radiation upon the polar angle θ. Let us treat the x-acceleration first. In this case, we can use the formula (9) directly with the substitution $\alpha = \pi/2 - \theta$. Therefore, the intensity of radiation scattered through angle θ into the solid angle $\mathrm{d}\Omega$ is

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_x \mathrm{d}\Omega = \frac{e^2|\ddot{r}_x|^2 \sin^2\theta}{16\pi^2\varepsilon_0 c^3}\,\mathrm{d}\Omega = \frac{e^4|E_x|^2}{16\pi^2 m_\mathrm{e}^2 \varepsilon_0 c^3}\cos^2\alpha\,\mathrm{d}\Omega. \tag{11}$$

Thomson Scattering

We have to take time averages of E_x^2 and we find that $\overline{E_x^2} = E_{x0}^2/2$, where E_{x0} is the maximum field strength of the wave. We sum over all waves contributing to the E_x-component of radiation and express the result in terms of the incident energy per unit area upon the electron. The latter is given by Poynting's theorem, $\boldsymbol{S}_x = (\boldsymbol{E} \times \boldsymbol{H}) = c\varepsilon_0 E_x^2 \boldsymbol{i}_z$. Again, we take time averages and find that the contribution to the intensity in the direction α from the x-component of the acceleration is $S_x = \sum_i c\varepsilon_0 E_{x0}^2/2$. Therefore

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_x \mathrm{d}\Omega = \frac{e^4 \cos^2\alpha}{16\pi^2 m_\mathrm{e}^2 \varepsilon_0 c^3} \sum_i \overline{E_x^2}\, \mathrm{d}\Omega = \frac{e^4 \cos^2\alpha}{16\pi^2 m_\mathrm{e}^2 \varepsilon_0^2 c^4} S_x\, \mathrm{d}\Omega. \qquad (12)$$

Thomson Scattering

Now let us look at the scattering of the E_y-component of the incident field. From the geometry of the previous diagram, it can be seen that the radiation in the $x - z$ plane from the acceleration of the electron in the y-direction corresponds to scattering at $\theta = 90^\circ$ and so the scattered intensity in the α-direction is

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_y \mathrm{d}\Omega = \frac{e^4}{16\pi^2 m_\mathrm{e}^2 \varepsilon_0^2 c^4} S_y \, \mathrm{d}\Omega. \tag{13}$$

The total scattered radiation into $\mathrm{d}\Omega$ is the sum of these components (notice that we add the intensities of the two independent field components).

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right) \mathrm{d}\Omega = \frac{e^4}{16\pi^2 m_\mathrm{e}^2 \varepsilon_0^2 c^4} (1 + \cos^2\alpha) \frac{S}{2} \, \mathrm{d}\Omega \tag{14}$$

where $S = S_x + S_y$ and $S_x = S_y$ for unpolarised radiation. We now express the scattered intensity in terms of a differential scattering cross-section $\mathrm{d}\sigma_\mathrm{T}$ in the following way. We define the scattered intensity in direction α by the following relation

$$\frac{\mathrm{d}\sigma_\mathrm{T}(\alpha)}{\mathrm{d}\Omega} = \frac{\text{energy radiated per unit time per unit solid angle}}{\text{incident energy per unit time per unit area}}\,. \tag{15}$$

Thomson Scattering

Since the total incident energy is S, the differential cross-section for Thomson scattering is

$$d\sigma_\mathrm{T}(\alpha) = \frac{e^4}{16\pi^2\varepsilon_0^2 m_\mathrm{e}^2 c^4}\frac{(1+\cos^2\alpha)}{2}\,d\Omega. \tag{16}$$

In terms of the *classical electron radius* $r_\mathrm{e} = e^2/4\pi\varepsilon_0 m_\mathrm{e} c^2$, this can be expressed

$$d\sigma_\mathrm{T} = \frac{r_\mathrm{e}^2}{2}(1+\cos^2\alpha)\,d\Omega. \tag{17}$$

To find the total cross-section, we integrate over all angles α,

$$\sigma_\mathrm{T} = \int_0^\pi \frac{r_\mathrm{e}^2}{2}(1+\cos^2\alpha)\,2\pi\sin\alpha\,d\alpha = \frac{8\pi}{3}r_\mathrm{e}^2 = \frac{e^4}{6\pi\varepsilon_0^2 m_\mathrm{e}^2 c^4}. \tag{18}$$

$$\boxed{\sigma_\mathrm{T} = 6.653\times\ 10^{-29}\,\mathrm{m}^2.} \tag{19}$$

This is Thomson's famous result for the total cross-section for scattering by stationary free electrons and is justly referred to as the *Thomson cross-section*.

Thomson Scattering

- The scattering is symmetric with respect to the scattering of angle α. Thus as much radiation is scattered backwards as forwards.

- Another useful calculation is the scattering cross-section for 100% polarised emission. We can work this out by integrating the scattered intensity (11) over all angles.

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_x = \frac{e^2|\ddot{\boldsymbol{r}}_x|^2}{16\pi^2\varepsilon_0 c^3}\int \sin^2\theta\, 2\pi \sin\theta\, \mathrm{d}\theta = \left(\frac{e^4}{6\pi\varepsilon_0^2 m_\mathrm{e}^2 c^4}\right) S_x. \tag{20}$$

We find the same total cross-section for scattering as before because it does not matter how the electron is forced to oscillate. The only important quantity is the total intensity incident upon it and it does not matter how anisotropic the radiation is. This result can be written in terms to the energy density of radiation u_rad in which the electron is located,

$$u_\mathrm{rad} = \sum_i u_i = \sum_i S_i/c, \tag{21}$$

and hence

$$-(\mathrm{d}E/\mathrm{d}t) = \sigma_\mathrm{T} c u_\mathrm{rad}. \tag{22}$$

- Thomson scattering is one of the most important processes which impedes the escape of photons from any region. We write down the expression for the energy scattered by the electron in terms of the number density N of photons of frequency ν so that

$$-\frac{\mathrm{d}(Nh\nu)}{\mathrm{d}t} = \sigma_{\mathrm{T}} c N h\nu. \tag{23}$$

There is no change of energy of the photons in the scattering process and so, if there are N_{e} electrons per unit volume, the number density of photons decreases exponentially with distance

$$\begin{aligned} -\frac{\mathrm{d}N}{\mathrm{d}t} &= \sigma_{\mathrm{T}} c N_{\mathrm{e}} N \qquad -\frac{\mathrm{d}N}{\mathrm{d}x} = \sigma_{\mathrm{T}} N_{\mathrm{e}} N \\ N &= N_0 \exp\left(-\int \sigma_{\mathrm{T}} N_{\mathrm{e}}\, \mathrm{d}x\right). \end{aligned} \tag{24}$$

Thus, the *optical depth* of the medium to Thomson scattering is

$$\tau = \int \sigma_{\mathrm{T}} N_{\mathrm{e}}\, \mathrm{d}x. \tag{25}$$

- In this process, the photons are scattered in random directions and so they perform a random walk, each step corresponding to the *mean free path* λ_{T} of the photon through the electron gas where $\lambda_{\mathrm{T}} = (\sigma_{\mathrm{T}} N_{\mathrm{e}})^{-1}$.

- A distinctive feature of the process is that the scattered radiation is polarised, even if the incident beam of the radiation is unpolarised. This can be understood intuitively because all the E-vectors of the unpolarised beam lie in the $x - y$ plane. Therefore, in the case of observing the electron precisely in the $x - y$ plane, the scattered radiation is 100% polarised. On the other hand, if we look along the z-direction, we observe unpolarised radiation. If we define the degree of polarisation as,

$$\Pi = \frac{I_{\mathrm{max}} - I_{\mathrm{min}}}{I_{\mathrm{max}} + I_{\mathrm{min}}}, \tag{26}$$

and so by a simple calculation the fractional polarisation of the radiation is

$$\Pi = \frac{1 - \cos^2\alpha}{1 + \cos^2\alpha}. \tag{27}$$

This is therefore a means of producing polarised radiation from an initially unpolarised beam.

A useful relativistic invariant

The energy loss rate by radiation $\mathrm{d}E/\mathrm{d}t$ is a Lorentz invariant between inertial frames.

Expert version. The total energy emitted in the form of radiation is the 'time' component of the momentum four-vector $[E/c, \boldsymbol{p}]$ and $\mathrm{d}t$ is the time component of the displacement four-vector $[\mathrm{d}t, \mathrm{d}\boldsymbol{r}]$. Therefore, both the energy $\mathrm{d}E$ and the time interval $\mathrm{d}t$ transform in the same way between inertial frames of reference and so their ratio $\mathrm{d}E/\mathrm{d}t$ is also an invariant.

Gentler version. In the *instantaneous rest frame* of the accelerated charged particle, dipole radiation is emitted with zero net momentum, as may be seen from the polar diagram of dipole radiation. Therefore its four-momentum can be written $[\mathrm{d}E'/c, 0]$. This radiation is emitted in the interval of proper time $\mathrm{d}t'$ which has four vector $[\mathrm{d}t', 0]$. We may now use the inverse Lorentz transformation to relate $\mathrm{d}E'$ and $\mathrm{d}t'$ to $\mathrm{d}E$ and $\mathrm{d}t$.

$$\mathrm{d}E = \gamma\,\mathrm{d}E' \qquad \mathrm{d}t = \gamma\,\mathrm{d}t'\ , \tag{28}$$

and hence

$$\boxed{\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}E'}{\mathrm{d}t'}\ .} \tag{29}$$

Radiation of Accelerated Relativistic Electron

We can derive from this result the radiation rate as observed by the external observer who measures the velocity and acceleration of the electron to be $\boldsymbol{a}$ and $\boldsymbol{v}$ respectively, the proper acceleration measured in the instantaneous rest frame of the electron being $\boldsymbol{a}_0$. Then, from the above results,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}E'}{\mathrm{d}t'} = \frac{e^2|\boldsymbol{a}_0|^2}{6\pi\varepsilon_0 c^3}. \tag{30}$$

To relate $\boldsymbol{a}_0$, $\boldsymbol{a}$ and $\boldsymbol{v}$, it is simplest to equate the norms of the four-accelerations of the accelerated electron in the frames S and S′. I leave it as an exercise to the reader to show that

$$\boldsymbol{a}_0^2 = \gamma^4\left[\boldsymbol{a}^2 + \gamma^2\left(\frac{\boldsymbol{v}\cdot\boldsymbol{a}}{c}\right)^2\right] \tag{31}$$

and so

$$\boxed{\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\text{in S}} = \frac{e^2\gamma^4}{6\pi\varepsilon_0 c^3}\left[\boldsymbol{a}^2 + \gamma^2\left(\frac{\boldsymbol{v}\cdot\boldsymbol{a}}{c}\right)^2\right]\ .} \tag{32}$$

Radiation of Accelerated Relativistic Electron (2)

Another useful exercise is to resolve $\boldsymbol{a}$ parallel and perpendicular to $\boldsymbol{v}$ so that

$$\boldsymbol{a} = a_{\parallel}\boldsymbol{i}_{\parallel} + a_{\perp}\boldsymbol{i}_{\perp} \tag{33}$$

and then to show that the radiation rate is

$$\boxed{\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\text{in S}} = \frac{e^2\gamma^4}{6\pi\varepsilon_0 c^3}\left(|a_{\perp}|^2 + \gamma^2|a_{\parallel}|^2\right) \ .} \tag{34}$$

I have shown how these relations are obtained in *HEA3*. This is a useful expression for obtaining the loss rate due to synchrotron radiation very quickly (see later).

Parseval's theorem and the spectral distribution of the radiation of an accelerated electron

We need to be able to decompose the radiation field of the electron into its spectral components. *Parseval's theorem* provides an elegant method of relating the dynamical history of the particle to its radiation spectrum. We introduce the Fourier transform of the acceleration of the particle through the Fourier transform pair, which I write in symmetrical form:

$$\dot{\boldsymbol{v}}(t) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \dot{\boldsymbol{v}}(\omega) \exp(-\mathrm{i}\omega t)\, \mathrm{d}\omega \qquad (35)$$

$$\dot{\boldsymbol{v}}(\omega) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \dot{\boldsymbol{v}}(t) \exp(\mathrm{i}\omega t)\, \mathrm{d}t. \qquad (36)$$

According to Parseval's theorem, $\dot{\boldsymbol{v}}(\omega)$ and $\dot{\boldsymbol{v}}(t)$ are related by the following integrals:

$$\int_{-\infty}^{\infty} |\dot{\boldsymbol{v}}(\omega)|^2\, \mathrm{d}\omega = \int_{-\infty}^{\infty} |\dot{\boldsymbol{v}}(t)|^2\, \mathrm{d}t. \qquad (37)$$

This is proved in all textbooks on Fourier anlaysis.

Parseval's Theorem

We can therefore apply this relation to the energy radiated by a particle which has an *acceleration history* $\dot{\boldsymbol{v}}(t)$,

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}E}{\mathrm{d}t} = \int_{-\infty}^{\infty} \frac{e^2}{6\pi\varepsilon_0 c^3} |\dot{\boldsymbol{v}}(t)|^2 \, \mathrm{d}t = \int_{-\infty}^{\infty} \frac{e^2}{6\pi\varepsilon_0 c^3} |\dot{\boldsymbol{v}}(\omega)|^2 \, \mathrm{d}\omega. \tag{38}$$

Now, what we really want is $\int_0^\infty \ldots \mathrm{d}\omega$ rather than $\int_{-\infty}^\infty \ldots \mathrm{d}\omega$. Since the acceleration is a real function, another theorem in Fourier analysis tells us that

$$\int_0^{\infty} |\dot{\boldsymbol{v}}(\omega)|^2 \, \mathrm{d}\omega = \int_{-\infty}^{0} |\dot{\boldsymbol{v}}(\omega)|^2 \, \mathrm{d}\omega \tag{39}$$

and hence we find

$$\text{Total emitted radiation} = \int_0^{\infty} I(\omega) \, \mathrm{d}\omega = \int_0^{\infty} \frac{e^2}{3\pi\varepsilon_0 c^3} |\dot{\boldsymbol{v}}(\omega)|^2 \, \mathrm{d}\omega. \tag{40}$$

Parseval's Theorem

Therefore

$$I(\omega) = \frac{e^2}{3\pi\varepsilon_0 c^3} |\dot{\boldsymbol{v}}(\omega)|^2. \tag{41}$$

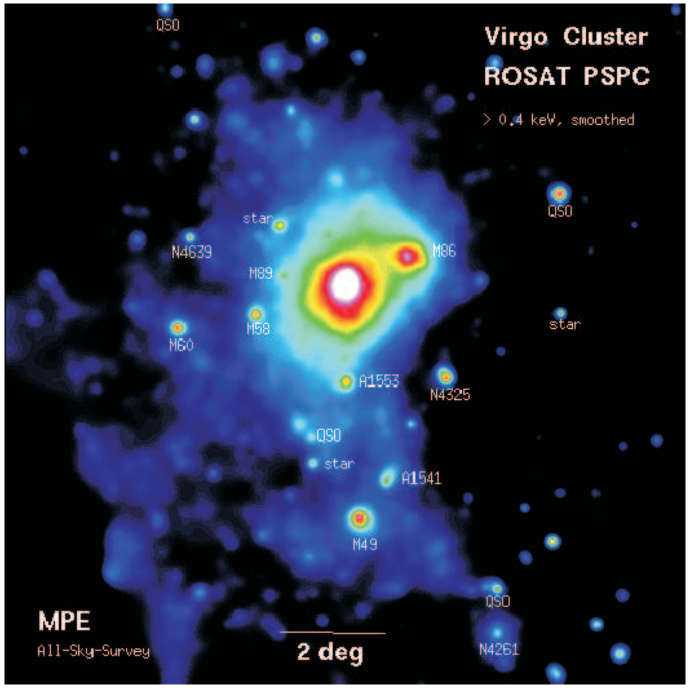
Note that this is the total energy per unit bandwidth emitted throughout the period during which the particle is accelerated. For a distribution of particles, this result must be integrated over all the particles contributing to the radiation at frequency ω.

This is also a very good result which often gives physical insight into the shape of the spectrum of emitted radiation.

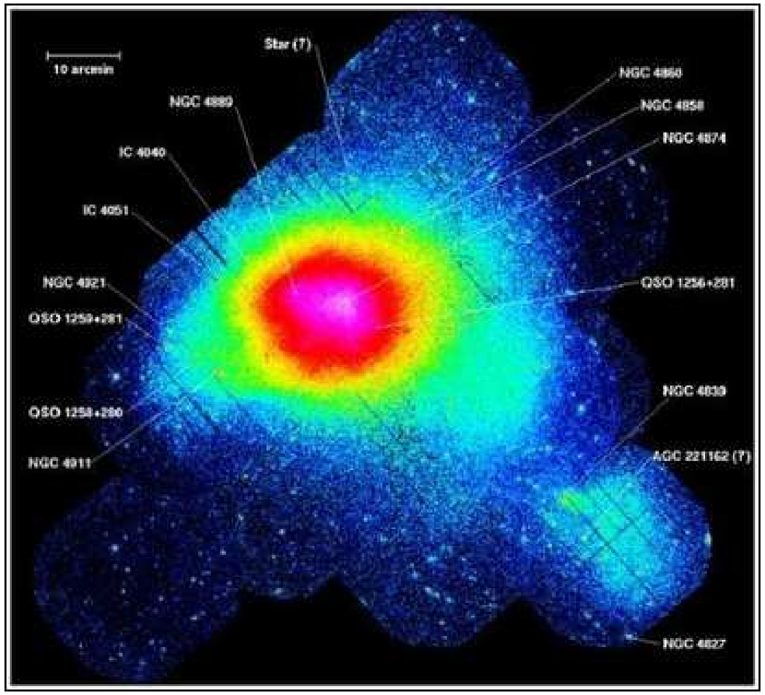
Bremsstrahlung

Bremsstrahlung is the radiation associated with the acceleration of electrons in the electrostatic fields of ions and the nuclei of atoms. In X-ray and γ-ray astronomy, the most important cases are those in which bremmstrahlung is emitted by very hot plasmas at $T \geq 10^6$ K, at which temperatures the hydrogen and helium atoms are fully ionised. We use the tools already introduced to derive classically the expressions for the bremsstrahlung emissivity of a hot plasma.

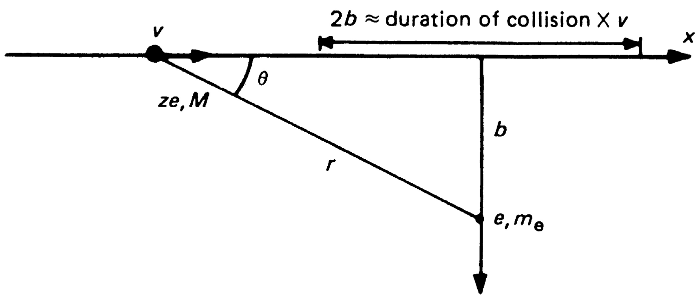
Virgo Cluster in X-rays (ROSAT)



Coma Cluster in X-rays (ROSAT)



Encounters between Charged Particles



The charge of the high energy particle is ze and its mass M.

Let us first study the 'collision' of a high energy proton or nucleus with the electrons of a fully ionised plasma. It is assumed that the nucleus is undeviated in the encounter with the electron; b, the distance of closest approach of the particle to the electron, is called the *collision parameter* of the interaction.

The total *momentum impulse* given to the electron in the encounter is $\int F\,\mathrm{d}t$. By symmetry, the forces parallel to the line of flight of the high energy particle cancel out and so we need only work out the component of force perpendicular to the line of flight.

$$F_{\perp} = \frac{ze^2}{4\pi\varepsilon_0 r^2}\sin\theta \quad ; \quad \mathrm{d}t = \frac{\mathrm{d}x}{v}. \qquad (42)$$

Encounters between Charged Particles

Changing variables to θ, $b/x = \tan\theta$, $r = b/\sin\theta$ and therefore

$$\mathrm{d}x = (-b/\sin^2\theta)\,\mathrm{d}\theta\ . \tag{43}$$

v is effectively constant and therefore

$$\int_{-\infty}^{\infty} F_\perp\,\mathrm{d}t = -\int_0^\pi \frac{ze^2}{4\pi\varepsilon_0 b^2}\sin^2\theta\frac{b\sin\theta}{v\sin^2\theta}\,\mathrm{d}\theta = -\frac{ze^2}{4\pi\varepsilon_0 bv}\int_0^\pi \sin\theta\,\mathrm{d}\theta \tag{44}$$

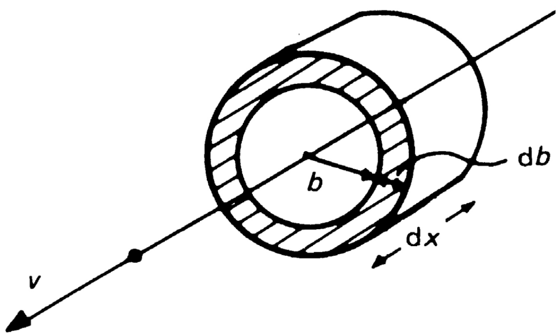
Therefore

$$\text{momentum impulse } p = \frac{ze^2}{2\pi\varepsilon_0 bv} \tag{45}$$

and the kinetic energy transferred to the electron is

$$\boxed{\frac{p^2}{2m_\mathrm{e}} = \frac{z^2e^4}{8\pi^2\varepsilon_0^2 b^2 v^2 m_\mathrm{e}} = \text{energy lost by high energy particle.}} \tag{46}$$

Encounters between Charged Particles



We want the *average energy loss per unit length* and so we need the number of collisions with collision parameters in the range b to $b + \mathrm{d}b$ and integrate over collision parameters. The total energy loss of the high energy particle, $-\mathrm{d}E$, is:

$$\begin{aligned} & (\text{number of electrons in volume } 2\pi b\,\mathrm{d}b\,\mathrm{d}x) \\ \times\ & (\text{energy loss per interaction}) \\ =\ & \int_{b_{\mathrm{min}}}^{b_{\mathrm{max}}} N_{\mathrm{e}}\, 2\pi b\, \mathrm{d}b\, \frac{z^2 e^4}{8\pi^2 \varepsilon_0^2 b^2 v^2 m_{\mathrm{e}}}\, \mathrm{d}x \end{aligned}$$

where N_{e} is the number density of electrons. I have included limits b_{max} and b_{min} to the range of collision parameters in this integral. Then,

$$-\frac{\mathrm{d}E}{\mathrm{d}x} = \frac{z^2 e^4 N_{\mathrm{e}}}{4\pi \varepsilon_0^2 v^2 m_{\mathrm{e}}} \ln\left(\frac{b_{\mathrm{max}}}{b_{\mathrm{min}}}\right). \qquad (47)$$

This process is closely related to the *ionisation losses* which we will meet again.

Gaunt Factors

Notice how the logarithmic dependence upon $b_{\mathrm{max}}/b_{\mathrm{min}}$ comes about. The closer the encounter, the greater the momentum impulse, $p \propto b^{-2}$. There are, however, more electrons at large distances $(\propto b\, \mathrm{d}b)$ and hence, when we integrate, we obtain only a logarithmic dependence of the energy loss rate upon the range of collision parameters.

Why introduce the limits b_{max} and b_{min}?

The reason is that the proper sum is very much more complicated and would take account of the acceleration of the electron by the high energy particle and include a proper quantum mechanical treatment of the interaction. Our approximate methods give rather good answers, however, because the limits b_{max} and b_{min} only appear inside the logarithm and hence need not be known very precisely.

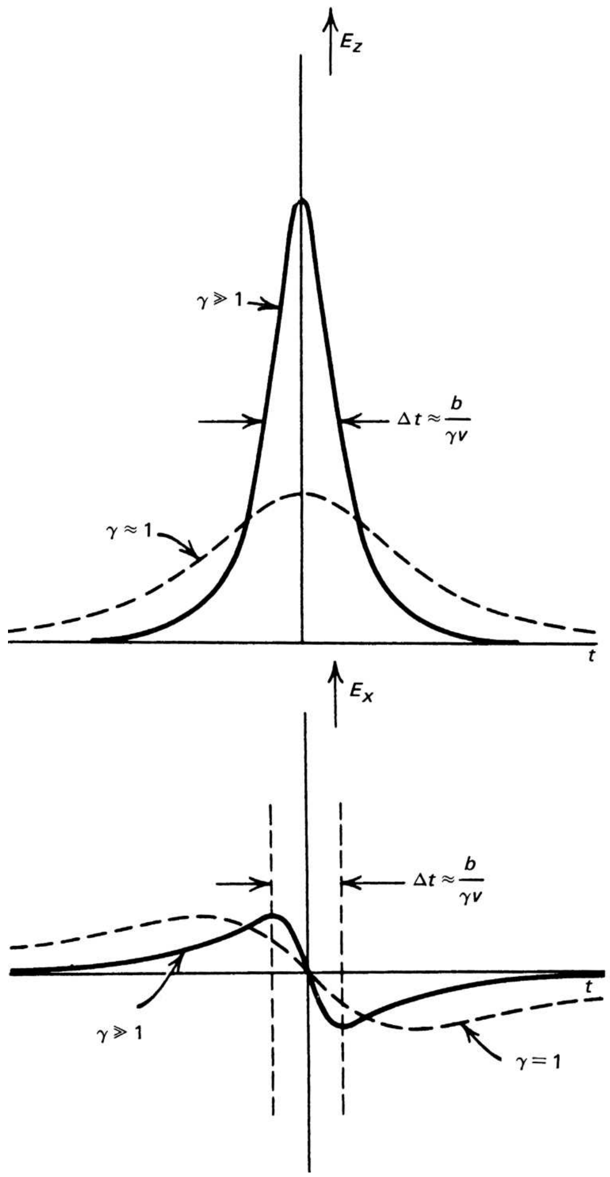
This is the simplest example of the type of calculation which needs to be carried out in working out energy transfers and accelerations of electrons and protons in fully ionised plasmas. The logarithmic term $\ln(b_{\mathrm{max}}/b_{\mathrm{min}})$ appears in the guise of what are often referred to as *Gaunt factors* and care has to be taken to use the correct values of b_{max} and b_{min} in different physical conditions. Similar forms of Gaunt factor appear in working out the spectrum of bremsstrahlung and the electrical conductivity of a plasma.

Spectrum and Energy Loss Rate of Bremsstrahlung

In the classical limit, bremsstrahlung is the emission of an electron accelerated in an electrostatic encounter with a nucleus. Electrons lose more energy in electron-electron collisions, but these do not result in the emission of dipole radiation since there is no net electric dipole moment associated with these encounters. Hence, the calculation

- Work out an expression for the acceleration of an electron in the electrostatic field of the nucleus. The roles of the particles in the calculation above are reversed – the electron is moving at a high speed past the stationary nucleus.
- Fourier transform of the acceleration of the electron and use Parseval's theorem to work out the spectrum of the emitted radiation.
- Integrate this result over all collision parameters and worry about suitable limits for b_{max} and b_{min}.
- In the case in which the electron is relativistic, transform back into the laboratory frame of reference.
- For a Maxwellian gas, integrate over the Maxwell distribtion.
- For a non-thermal distribution, integrate over the velocity or energy distribution.

Outline Calculation



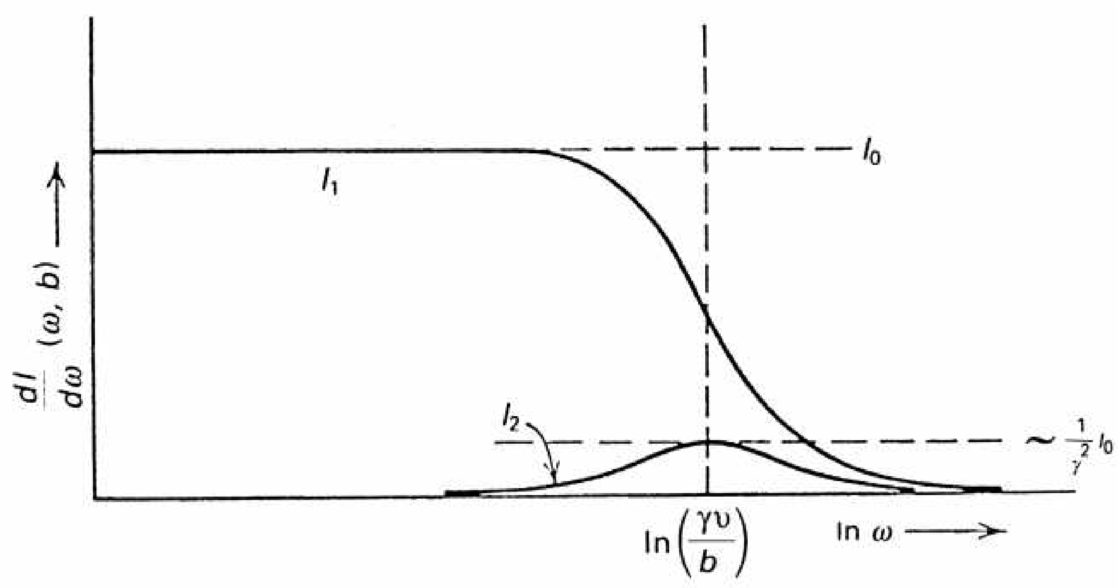
Both the relativistic and non-relativistic cases begin in the same way and so work out both cases simultaneously. The accelerations along the trajectory of the electron, $a_{\|}$, and perpendicular to it, $a_{\perp}$, *in its rest-frame* are given by

$$a_{\|} = \dot{v}_x = -\frac{eE_x}{m_\mathrm{e}} = \frac{\gamma Z e^2 v t}{4\pi\varepsilon_0 m_\mathrm{e}[b^2 + (\gamma v t)^2]^{3/2}}$$

$$a_{\perp} = \dot{v}_z = -\frac{eE_z}{m_\mathrm{e}} = \frac{\gamma Z e^2 b}{4\pi\varepsilon_0 m_\mathrm{e}[b^2 + (\gamma v t)^2]^{3/2}}$$

where Ze is the charge of the nucleus (see *HEA3*).

Outline Calculation



The radiation spectrum of the electron in an encounter with a charged nucleus is then

$$
\begin{aligned}
I(\omega) &= \frac{e^2}{3\pi\varepsilon_0 c^3}\left[|a_{\parallel}(\omega)|^2 + |a_{\perp}(\omega)|^2\right] \\
&= \frac{Z^2 e^6}{24\pi^4\varepsilon_0^3 c^3 m_e^2}\frac{\omega^2}{\gamma^2 v^2}\left[\frac{1}{\gamma^2}K_0^2\left(\frac{\omega b}{\gamma v}\right) + K_1^2\left(\frac{\omega b}{\gamma v}\right)\right] \qquad (48)
\end{aligned}
$$

where K_0 and K_1 are modified Bessel functions of order zero and one. This is the intensity spectrum which results from a single encounter between an electron and a nucleus with collision parameter b.

The Results

- The impulse perpendicular to the direction of travel contributes the greater intensity, even in the non-relativistic case $\gamma = 1$.

- The perpendicular component results in significant radiation at low frequencies.

- The spectrum is flat because the acceleration perpendicular to the line of flight is a brief impulse. The Fourier transform of a delta function is a flat spectrum.

- The turn-over corresponds roughly to the frequency which is the inverse of the duration of the collision.

Bremsstrahlung

Thus, *at high frequencies*, there is an exponential cut-off to the spectrum

$$I(\omega) = \frac{Z^2 e^6}{48\pi^3 \varepsilon_0^3 c^3 m_e^2 \gamma v^3} \left[\frac{1}{\gamma^2} + 1\right] \exp\left(-\frac{2\omega b}{\gamma v}\right) . \qquad (49)$$

The exponential cut-off tells us that there is little power emitted at frequencies greater than $\omega \approx \gamma v/b$.

The *low frequency spectrum* has the form

$$I(\omega) = \frac{Z^2 e^6}{24\pi^4 \varepsilon_0^3 c^3 m_e^2} \frac{1}{b^2 v^2} = K . \qquad (50)$$

To an excellent approximation, the low frequency spectrum is flat up to frequency $\omega = \gamma v/b$, above which the spectrum falls off exponentially.

Bremsstrahlung

Finally, we integrate over all collision parameters which contribute to the radiation at frequency ω. If the electron is moving relativistically, the number density of nuclei it observes is enhanced by a factor γ because of relativistic length contraction. Hence, in the moving frame of the electron, $N' = \gamma N$ where N is the number density of nuclei in the laboratory frame of reference. The number of encounters per second is $N'v$ and since, properly speaking, all parameters are measured in the rest frame of the electron, let us add superscript dashes to all the relevant parameters. The radiation spectrum in frame of the electron is therefore

$$I(\omega') = \int_{b'_{\mathrm{min}}}^{b'_{\mathrm{max}}} 2\pi b' \gamma N v K' \, \mathrm{d}b' \qquad (51)$$

$$= \frac{Z^2 e^6 \gamma N}{12\pi^3 \varepsilon_0^3 c^3 m_{\mathrm{e}}^2} \frac{1}{v} \ln\left(\frac{b'_{\mathrm{max}}}{b'_{\mathrm{min}}}\right). \qquad (52)$$

Non-relativistic and Thermal Bremsstrahlung

First, we evaluate the total energy loss rate by bremsstrahlung of a high energy but non-relativistic electron. We neglect the relativistic correction factors and hence obtain the low frequency radiation spectrum

$$I(\omega) = \frac{Z^2 e^6 N}{12\pi^3 \varepsilon_0^3 c^3 m_e^2} \frac{1}{v} \quad \ln \Lambda \tag{53}$$

where $\Lambda = b_{\text{max}}/b_{\text{min}}$. We have to make the correct choice of limiting collision parameters b_{max} and b_{min}.

For b_{max}, we note that we should only integrate out to those values of b for which $\omega b/v = 1$. For larger values of b, the radiation at frequency ω lies on the exponential tail of the spectrum and we obtain a negligible contribution to the intensity.

For b_{min}, at high velocities, $v \geq (Z/137)c$, the quantum restriction, $b_{\text{min}} \approx \hbar/2m_e v$, is applicable and can be derived from Heisenberg's uncertainty principle (see *HEA3*). This is the appropriate limit to describe, for example, the X-ray bremsstrahlung of hot intergalactic gas in clusters of galaxies. Thus, for high velocities, $\Lambda = 2m_e v^2/\hbar\omega$. There is, as usual, a cut-off at high frequencies $\omega \geq v/b$.

The Full Answer

Let us compare our result with the full answer which was derived by Bethe and Heitler using a full quantum mechanical treatment of the radiation process. The electron cannot give up more than its total kinetic energy in the radiation process and so no photons are radiated with energies greater than $\varepsilon = \hbar\omega = \frac{1}{2} m_{\mathrm{e}} v^2$.

The intensity of radiation from a single electron of energy $E = \frac{1}{2} m_{\mathrm{e}} v^2$ in the non-relativistic limit is

$$I(\omega) = \frac{8}{3} Z^2 \alpha r_{\mathrm{e}}^2 \frac{m_{\mathrm{e}} c^2}{E} v N \ln \left[\frac{1 + (1 - \varepsilon/E)^{1/2}}{1 - (1 - \varepsilon/E)^{1/2}} \right] \qquad (54)$$

where $\alpha = e^2/4\pi\hbar\varepsilon_0 c \approx 1/137$ is the fine structure constant and $r_{\mathrm{e}} = e^2/4\pi\varepsilon_0 m_{\mathrm{e}} c^2$ is the *classical electron radius*. The constant in front of the logarithm in this expression is exactly the same as that in (53). In addition, in the limit of low energies $\varepsilon \ll E$, the term inside the logarithm reduces to $4E/\varepsilon$.

Thermal Bremsstrahlung

In order to work out the bremsstrahlung, or free-free emission, of a gas at temperature T, we integrate the expression (53) over a Maxwellian distribution of electron velocities

$$N_{\mathrm{e}}(v)\,\mathrm{d}v = 4\pi N_{\mathrm{e}} \left(\frac{m_{\mathrm{e}}}{2\pi kT}\right)^{3/2} v^2 \exp\left(\frac{m_{\mathrm{e}} v^2}{2kT}\right) \mathrm{d}v. \quad (55)$$

The algebra can become somewhat cumbersome at this stage. We can find the correct order-of-magnitude answer if we write $\frac{1}{2} m_{\mathrm{e}} v^2 = \frac{3}{2} kT$ in expression (53). Then, the emissivity of a plasma having electron density N_{e} becomes in the low frequency limit,

$$I(\omega) \approx \frac{Z^2 e^6 N N_{\mathrm{e}}}{12\sqrt{3}\pi^3 \varepsilon_0^3 c^3 m_{\mathrm{e}}^2} \left(\frac{m_{\mathrm{e}}}{kT}\right)^{1/2} g(\omega, T) \quad (56)$$

where $g(\omega, T)$ is a *Gaunt factor*, corresponding to ln Λ, but now integrated over velocity.

At high frequencies, the spectrum of thermal bremsstrahlung cuts off exponentially as $\exp(-\hbar\omega/kT)$, reflecting the population of electrons in the high energy tail of a Maxwellian distribution at energies $\hbar\omega \gg kT$.

Thermal Bremsstrahlung

The total energy loss rate of the plasma may be found by integrating the spectral emissivity over all frequencies. In practice, because of the exponential cut-off, we find the correct functional form by integrating (56) from 0 to $\omega = kT/\hbar$, that is,

$$-(\mathrm{d}E/\mathrm{d}t) = (\text{constant})\, Z^2 T^{1/2} \overline{g} N N_\mathrm{e} \tag{57}$$

where $\overline{g}$ is a frequency averaged Gaunt factor. Detailed calculations give the following answers:

$$\kappa_\nu = \frac{1}{3\pi^2}\left(\frac{\pi}{6}\right)^{1/2} \frac{Z^2 \mathrm{e}^6}{\varepsilon_0^3 c^3 m_\mathrm{e}^2}\left(\frac{m_\mathrm{e}}{kT}\right)^{1/2} g(\nu, T) N N_\mathrm{e} \exp\left(-\frac{h\nu}{kT}\right) \tag{58}$$

$$= 6.8 \times 10^{-51} Z^2 T^{-1/2} N N_\mathrm{e}\, g(\nu, T) \exp(-h\nu/kT)\, \mathrm{W\,m^{-3}Hz^{-1}} \tag{59}$$

where the number densities of electrons N_e and of nuclei N are given in particles per cubic metre. At frequencies $\hbar\omega \ll kT$, the Gaunt factor has only a logarithmic dependence on frequency.

Thermal Bremsstrahlung

A suitable form for X-ray wavelengths is:

$$\text{X-ray} \quad g(\nu, T) = \frac{\sqrt{3}}{\pi} \ln\left(\frac{kT}{h\nu}\right) \, , \tag{60}$$

The functional forms of the logarithmic term can be readily derived from the the above considerations.

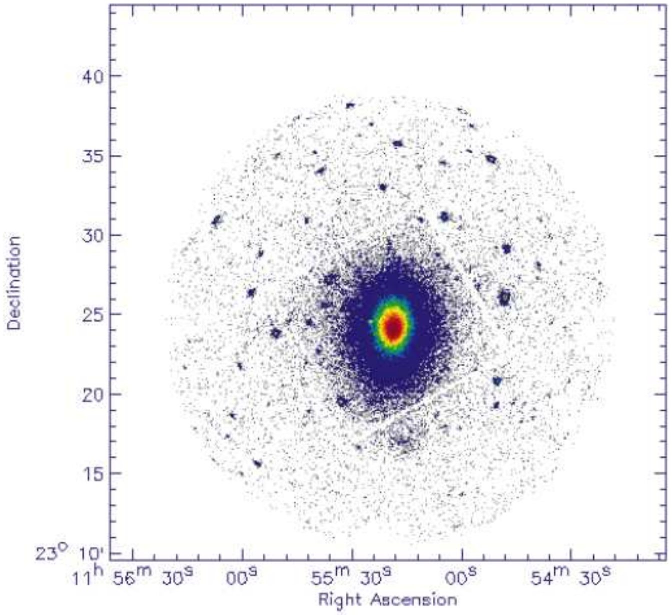
The total loss rate of the plasma is

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{brems}} = 1.435 \times 10^{-40} Z^2 T^{1/2} \bar{g} N N_{\mathrm{e}} \quad \mathrm{W\,m^{-3}}. \tag{61}$$

Detailed calculations show that the frequency averaged Gaunt factor $\bar{g}$ lies in the range $1.1 - 1.5$ and a good approximation is $\bar{g} = 1.2$. A compilation of a large number of useful Gaunt factors for a wide range of physical conditions is given by Karzas and Latter (1961).

Example - X-ray Emission of Clusters of Galaxies

XMM-Newton X-ray Image of the cluster of galaxies Abell 1413 (Pratt and Arnaud 2002).



If p is the pressure of the gas and ϱ its density, both of which vary with position within the cluster, the requirement of hydrostatic equilibrium is

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{GM(\leq r)\varrho}{r^2} \; . \qquad (62)$$

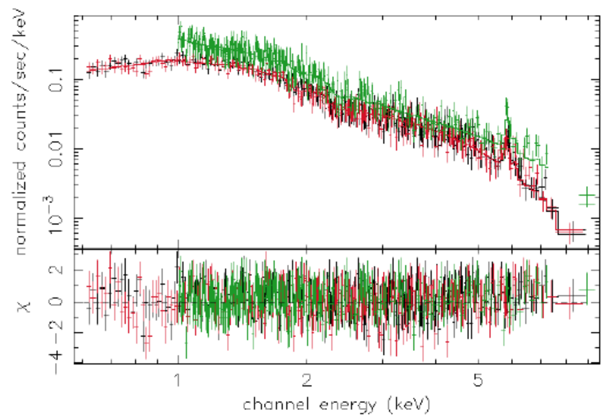
The pressure is related to the local gas density ϱ and temperature T by the perfect gas law

$$p = \frac{\varrho kT}{\mu m_{\mathrm{H}}} \; , \qquad (63)$$

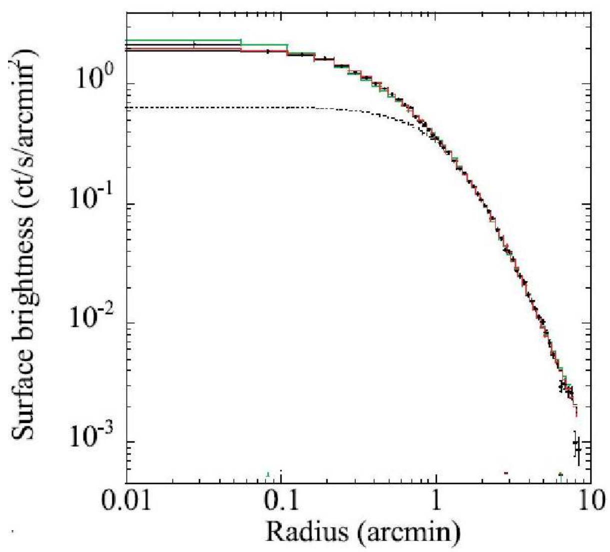
where m_{H} is the mass of the hydrogen atom and μ is the mean molecular weight of the gas. For a fully ionised gas with the standard cosmic abundance of the elements, a suitable value is $\mu = 0.6$.

Example - X-ray Emission of Clusters of Galaxies

X-ray spectrum in fifth annulus



Average X-ray emissivity as a function of radius



Differentiating with respect to r and substituting, we find

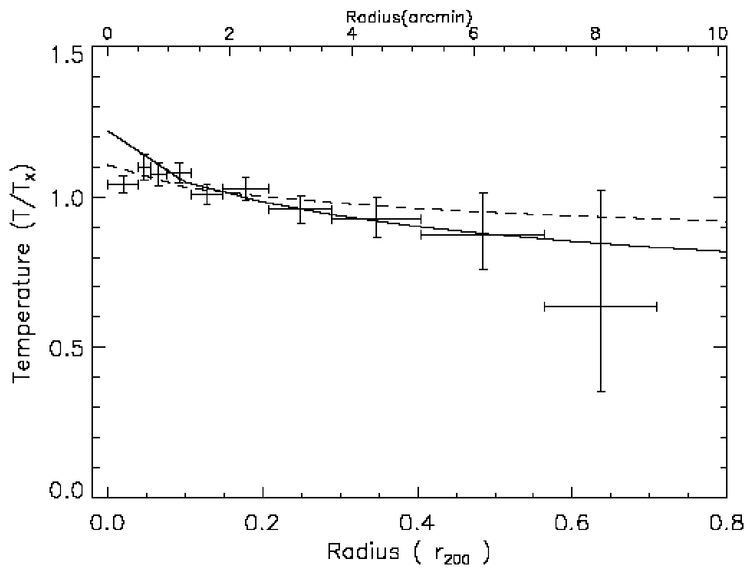
$$\frac{\varrho k T}{\mu m_{\rm H}}\left(\frac{1}{\varrho}\frac{{\rm d}\varrho}{{\rm d}r}+\frac{1}{T}\frac{{\rm d}T}{{\rm d}r}\right)=-\frac{GM(\leq r)\varrho}{r^2}. \quad (64)$$

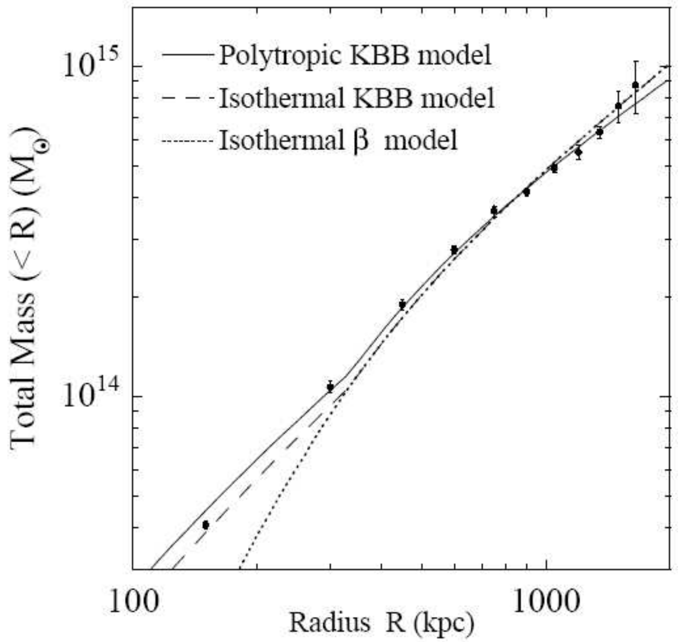
Reorganising,

$$M(\leq r)=-\frac{kTr^2}{G\mu m_{\rm H}}\left[\frac{{\rm d}(\log\varrho)}{{\rm d}r}+\frac{{\rm d}(\log T)}{{\rm d}r}\right]. \quad (65)$$

Thus, by measuring the temperature of the gas as a function of radius and the bremsstrahlung emissivity of the gas, the mass within radius r can be found.

The Mass Distribution in Abell 1413





Both the temperature distribution and number density of electrons need to be deprojected to find the spatial distribution as a function of radius. This is carried through the pair of Abel integrals,

$$I_\nu(a) = \frac{1}{2\pi}\int_a^\infty \frac{\kappa_\nu(r) r}{(r^2 - a^2)^{1/2}}\,\mathrm{d}r \qquad \kappa_\nu(r) = \frac{4}{r}\frac{\mathrm{d}}{\mathrm{d}r}\int_r^\infty \frac{I_\nu(a) a}{(a^2 - r^2)^{1/2}}\,\mathrm{d}a\,. \qquad (66)$$

Non-relativistic Bremsstrahlung Losses

To find the energy loss rate of a single high energy electron, we integrate (52) over all frequencies. In practice, this means integrating from 0 to ω_{max} where ω_{max} corresponds to the cut-off, $b_{\mathrm{min}} \approx \hbar/2m_{\mathrm{e}}v$. This angular frequency is approximately

$$\omega_{\mathrm{max}} = 2\pi/\tau \sim 2\pi v/b_{\mathrm{min}} \approx 4\pi m_{\mathrm{e}} v^2/\hbar, \tag{67}$$

that is, to order of magnitude $\hbar\omega \approx \frac{1}{2} m_{\mathrm{e}} v^2$. This is just the kinetic energy of the electron and is obviously the maximum amount of energy which can be lost in a single encounter with the nucleus. We should therefore integrate (52) from $\omega = 0$ to $\omega_{\mathrm{max}} \approx m_{\mathrm{e}} \nu v^2/2\hbar$ and thus,

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{brems}} \approx \int_0^{\omega_{\mathrm{max}}} \frac{Z^2 e^6 N}{12\pi^3 \epsilon_0^3 c^3 m_{\mathrm{e}}^2} \frac{1}{v} \ln\Lambda \, \mathrm{d}\omega \approx \frac{Z^2 e^6 N v}{24\pi^3 \epsilon_0^3 c^3 m_{\mathrm{e}} \hbar} \ln\Lambda \tag{68}$$

Note that the total energy loss rate of the electron is proportional to v, that is, to the square root of the kinetic energy E: $-\mathrm{d}E/\mathrm{d}t \propto E^{1/2}$.

In practical applications of this formula, it is necessary to integrate over the energy distribution of the particles.

Relativistic Bremsstrahlung Losses

The formulae we have derived are correct in the rest frame of the electron, namely,

$$I(\omega') = \int_{b'_{\min}}^{b'\max} 2\pi b'(\gamma N) v K' \, \mathrm{d}b' \tag{69}$$

where we have written the number density of nuclei γN because of length contraction. Since the collision parameters b' are perpendicular to the direction of motion, it follows that, since $y = y'$, the same collision parameters appropriate for the laboratory frame of reference can be used. I have given a discussion of the relevant collision parameters in *HEA3* and I will not repeat that discussion here. It suffices to note that we can write the emission spectrum in the frame of the electron

$$I(\omega') = \frac{Z^2 e^6 N\gamma}{12\pi^3 \varepsilon_0^3 c^3 m_e^2} \frac{1}{v} \ln\Lambda. \tag{70}$$

Notice that there is at best a very weak dependence upon frequency ω and so we again obtain the characteristic flat bremsstrahlung intensity spectrum.

Relativistic Bremsstrahlung Losses

On transforming this spectrum to the laboratory frame of reference, we note that the bandwidth changes as $\Delta\omega = \gamma\Delta\omega'$ and so the spectrum becomes

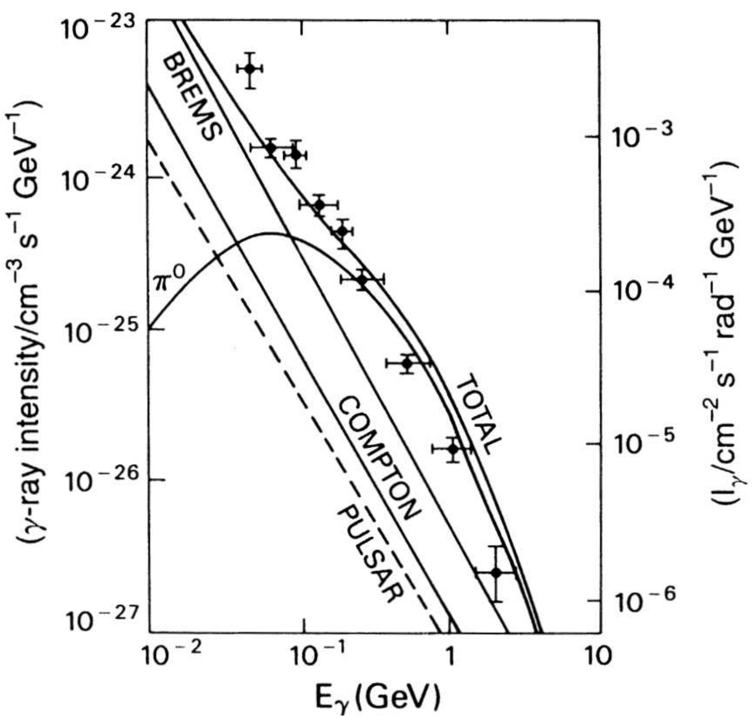
$$I(\omega) = \frac{Z^2 e^6 N}{12\pi^3 \varepsilon_0^3 c^3 m_\mathrm{e}^2} \frac{1}{v} \ln \Lambda \tag{71}$$

where the integral extends up to energies $E = \hbar\omega = \gamma m_\mathrm{e} c^2$, where $\gamma \gg 1$. Thus, the rate of loss of energy of the relativistic electron is

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_\mathrm{rel} = \int_0^{E/\hbar} I(\omega)\,\mathrm{d}\omega = \frac{Z^2 e^6 N \bar{g}}{12\pi^3 \varepsilon_0^3 c^4 \hbar} E. \tag{72}$$

Notice that the dependence of the energy loss rate changes from $E^{1/2}$ to E between the non-relativistic and relativistic cases.

The Galactic Gamma-ray Emission



The diagram shows the γ-ray spectrum of our Galaxy as well as theoretical estimates of the emission by Stecker (1977). At energies $\varepsilon > 70$ MeV, the dominant emission mechanism is the decay of neutral pions created in collisions between cosmic rays and the nuclei of atoms and molecules of the interstellar gas. This spectrum peaks at about 70 MeV and so there must be another mechanism which contributes at the lower energies. Relativistic bremsstrahlung may be the dominant source of emission at these energies. The spectrum labelled 'brems' is derived from an extrapolation of the relativistic electron spectrum in our Galaxy to energies $1 < E < 1000$ MeV.