The private classical capacity and quantum capacity of a quantum channel

I. Devetak∗

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Abstract

A formula for the capacity of a quantum channel for transmitting private classical information is derived. This is shown to be equal to the capacity of the channel for generating a secret key, and neither capacity is enhanced by forward public classical communication. Motivated by the work of Schumacher and Westmoreland on quantum privacy and quantum coherence, parallels between private classical information and quantum information are exploited to obtain an expression for the capacity of a quantum channel for generating pure bipartite entanglement. The latter implies a new proof of the quantum channel coding theorem and a simple proof of the converse. The coherent information plays a role in all of the above mentioned capacities.

Keywords: Cryptography, entanglement, large deviations, quantum channel capacity, wire-tap channels.

1 Introduction

The correspondence between secret classical information and quantum information, after having been part of quantum information folklore for many years, was first explicitly studied by Collins and Popescu [14]. The simplest example of this relationship is the ability to convert a maximally entangled Bell state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$ shared by Alice and Bob into a secret classical key by local bilateral measurements in the $\{|0\rangle, |1\rangle\}$ basis. Since the initial state is pure and hence decoupled from the “environment”, so is the information about the measurement outcomes. The converse direction does not hold in the literal sense: there is no way to recover the entanglement once the measurement has been made. However, given a quantum resource such as a quantum channel, it is conceivable that a secret key generating protocol could be converted into a (pure) entanglement generating protocol by performing all the steps “coherently” [3], e.g. replacing probabilistic mixtures by quantum superpositions. The connection has been exploited in one direction by Shor and Preskill [35] in proving the secrecy of the BB84 [7] quantum key distribution protocol by reduction from the entanglement-based protocol of Lo and Chuah [25] via Calderbank-Shor-Steane (CSS) [13] codes. In a different context, an equivalence has recently been established between the noise thresholds for certain two-way protocols for secret key distillation and entanglement distillation [11] [14].

The motivation for the present work is a paper by Schumacher and Westmoreland [31] in which an information theoretical approach to secret key generation is taken. They invoke standard classical results on wire-tap channels [11] [16] [2] [27] to argue heuristically that the capacity of a noisy quantum channel $\mathcal{N}$ for generating a secret key should be lower bounded by the coherent

∗The author is with the IBM T.J. Watson Research Center, Yorktown Heights, NY 10598, USA. This work was supported in part by the NSA under the US Army Research Office (ARO), grant numbers DAAG55-98-C-0041 and DAAD19-01-1-06.
information $I_c(\rho, \mathcal{N})$ \cite{30} \cite{20} of the channel with respect to an arbitrary input density operator $\rho$. Our first main result is an exact expression for the channel capacity for secret key generation $K(\mathcal{N})$. At the time of writing \cite{31} it was only conjectured that the coherent information played a role in the quantum capacity $Q(\mathcal{N})$. The quantum capacity theorem was originally stated by Lloyd \cite{20}, who also provided heuristic arguments for its validity. Other relevant early works include \cite{30} \cite{3} \cite{4}. It is only recently that a rigorous direct coding theorem has been reported by Shor \cite{34} attaining the coherent information based upper bound of \cite{5} \cite{4}. Our second main result is a new proof of the direct coding theorem via an entanglement generation protocol, in turn, on our secret key generation protocol. Shor’s proof \cite{34} is based on random subspace codes (see also \cite{26}). Our code turns out to be related to a generalization of CSS codes, which is not surprising given its cryptographic origin. In addition we provide a new, simplified proof of the converse theorem of \cite{4}, avoiding difficulties with restricting the encoding to partial isometries.

It is necessary to introduce some notation for dealing with classical-quantum resources \cite{19}. Classical-quantum resources can be static or dynamic. A static bipartite classical-quantum resource, denoted by \{c,q\}, is described by an ensemble $E = \{\rho_x, p(x)\}$. If the indices $x \in \mathcal{X}$ and distribution $p$ are associated with some random variable $X$, and the density operators $\rho^Q_x = \rho_x$ with some quantum system $Q$, the ensemble $E$ may be equated with the classical-quantum system $XQ$. One may similarly have multipartite systems such as $UXQ$ (of the \{c,c,q\} type) and $XQE$ (of the \{c,c,q\} type) with more than one classical or quantum component.

A dynamic bipartite classical-quantum resource, denoted by \{c → q\} is given by a classical-quantum channel $W : x \mapsto \rho_x$, or, alternatively, by the quantum alphabet \{\rho_x\}. Analogous to the static case, the channel $W$ may be equated with the conditional quantum system $Q|X$. Indeed, $\rho^Q_x$ is the state of the quantum system $Q$ conditioned on the classical index being $x$. Dynamic resources are similarly extended to more than two parties.

A useful representation of static classical-quantum systems, which we refer to as the “enlarged Hilbert space” (EHS) representation, is obtained by embedding the classical random variables into quantum systems. For instance, our ensemble $E$ corresponds to the density operator

$$\rho^{A\mathcal{Q}} = \sum_x p(x) |x\rangle \langle x|^A \otimes \rho^Q_x,$$  \hspace{1cm} (1)

where $A$ is a dummy quantum system and $\{|x\rangle : x \in \mathcal{X}\}$ is an orthonormal basis for the Hilbert space $\mathcal{H}_A$ of $A$. A static classical-quantum system may, therefore, be viewed as a special case of a quantum one. The EHS representation is convenient for defining various information theoretical quantities for classical-quantum systems. The von Neumann entropy of a quantum system $A$ with density operator $\rho^A$ is defined as $H(A) = -\text{Tr} \rho^A \log \rho^A$. For a bipartite quantum system $AB$ define the conditional von Neumann entropy

$$H(B|A) = H(AB) - H(A),$$

and quantum mutual information

$$I(A; B) = H(A) + H(B) - H(AB) = H(B) - H(B|A),$$

in formal analogy with the classical definitions. For a tripartite quantum system $ABC$ define the quantum conditional mutual information

$$I(A; B|C) = H(A|C) + H(B|C) - H(AB|C) = H(AC) + H(BC) - H(ABC) - H(C).$$

A commonly used identity is the chain rule

$$I(A; BC) = I(A; B) + I(A; C|B).$$

Notice that for classical-quantum correlations \cite{1} the von Neumann entropy $H(A)$ is just the Shannon entropy $H(X) = -\sum_x p(x) \log p(x)$ of $X$. The conditional entropy $H(Q|X)$ is defined as
receives almost no information about the message. The input $s$ to the composite channel to Bob so that he can identify the message with high probability while at the same time Eve

Finally we need to introduce a classical-quantum analogue of a Markov chain. A classical Markov chain $T \rightarrow X \rightarrow Y$ consists of correlated random variables $T$, $X$ and $Y$ whose probabilities obey

$$\Pr\{Y = y|X = x, T = t\} = \Pr\{Y = y|X = x\},$$

which is to say that $Y$ depends on $T$ only through $X$. Analogously we may define a classical-quantum Markov chain $T \rightarrow X \rightarrow \mathcal{Q}$ associated with an ensemble $\{\rho_{tx}, p(t, x)\}$ for which $\rho_{tx} = \rho_x$. Such an object typically comes about by augmenting the system $X \mathcal{Q}$ by the random variable $T$ (classically) correlated with $X$ via a conditional distribution $Q(t|x) = \Pr\{T = t|X = x\}$. In the EHS representation this corresponds to the state

$$\rho^{X_A \mathcal{Q}} = \sum_x p(x) \sum_u Q(t|x) |t⟩⟨t|^{\mathcal{E}} \otimes |x⟩⟨x|^{\mathcal{A}} \otimes \rho_x^{\mathcal{Q}}. \quad (2)$$

We shall henceforth make liberal use of the concepts defined above and their natural extensions.

The paper is organized as follows. In section 2 we define and find expressions for the private information and key generation capacities $C_p(W)$ and $K(W)$, respectively, of a $\{c \rightarrow qq\}$ type channel $W$. We show that allowing a free forward public channel does not help in either case. In section 3 these findings are applied to a noisy quantum channel $\mathcal{N}$ setting, yielding analogous capacities $C_p(\mathcal{N})$ and $K(\mathcal{N})$. In section 4 we turn to the problem of entanglement generation over the quantum channel $\mathcal{N}$ and find the corresponding capacity $E(\mathcal{N})$. This result is readily translated into an expression for the quantum capacity $Q(\mathcal{N})$ in section 5. We conclude with open problems.

## 2 Private information transmission and key generation over classical-quantum channels

We begin by defining a general private information transmission protocol for a $\{c \rightarrow qq\}$ channel from Alice to Bob and Eve. The channel is defined by the map $W : x \rightarrow \rho_x^{\mathcal{Q}}$, with $x \in \mathcal{X}$ and the $\rho_x^{\mathcal{Q}}$ defined on a bipartite quantum system $\mathcal{Q}\mathcal{E}$; Bob has access to $\mathcal{Q}$ and Eve has access to $\mathcal{E}$. Alice’s task is to convey, by some large number $n$ uses of the channel $W$ and unlimited use of a public channel (which both Bob and Eve have access to), one of $2^{nR}$ equiprobable messages to Bob so that he can identify the message with high probability while at the same time Eve receives almost no information about the message. The inputs to the composite channel $W^\otimes n$ are classical sequences of the form $x_1 \ldots x_n \in \mathcal{X}^n$, for which we use the shorthand notation $x^n$ (not to be confused with the power operation). The outputs of $W^\otimes n$ are density operators living on some Hilbert space $\mathcal{Q}^n \mathcal{E}^n$. We formally define an $(n, \epsilon)$ private channel code of rate $R$ in the following way. Alice generates a random variable $M$ which she can use for randomization, if necessary. Given the classical message embodied in the random variable $K$ uniformly distributed on the set $[2^nR] := \{1, 2, \ldots, 2^nR\}$, she sends the random variable $X^n = X^n(K, M)$ over the channel $W^\otimes n$ and sends the random variable $S = S(K, M)$ through the public channel. Bob performs a decoding POVM (based on the information contained in $S$) on his system $\mathcal{Q}^n$, yielding the random variable $Y$, and computes his best estimate of Alice’s message $L = L(Y, S)$. We require

$$\Pr\{K \neq L\} \leq \epsilon, \quad (3)$$

$$I(K; S) \leq \epsilon, \quad (4)$$

$$I(K; \mathcal{E}^n|S) \leq \epsilon. \quad (5)$$
The second condition means that the public information $S$ is almost uncorrelated with $K$ and the third implies via the Holevo bound \[22\] that, given the public information, there is no measurement Eve could perform that would reveal more than $\epsilon$ bits of information about $K^1$. We call the rate $R$ achievable if for every $\epsilon, \delta > 0$ and sufficiently large $n$ there exists an $(n, \epsilon)$ code of rate $R - \delta$. The private channel capacity $C_p(W)$ is the supremum of achievable rates $R$.

The above scenario should be contrasted with a secret key generation protocol, where Alice does not care about transmitting a particular message but only about establishing secret classical correlations with Bob, about which Eve has arbitrarily little information. The definition of an $(n, \epsilon)$ key generation code is almost the same as that of a private channel code, with the difference that now $K$ itself is a function of $M$. The secret key capacity $K(W)$ is similarly given by the supremum of achievable rates $R$.

**Theorem 1**

\[
C_p(W) = K(W) = \lim_{l \to \infty} \frac{1}{n} \max_{T, X^n} \{I(T; Q^n) - I(T; E^n)\},
\]

where $Q^n|X$ is given by $W$ and $T \to X^n \to Q^n E^n$ is a Markov chain.

Note that the limit in equation (6) indeed exists, by standard arguments (see e.g. \[5\], Appendix A). It should be noted that the above formula does not quite attain the ultimate goal of being effectively computable, due to the $l \to \infty$ limit. This seems to be a ubiquitous problem in quantum information theory, and we shall encounter it two more times in this paper, namely, in theorem \[5\] and proposition \[4\].

Proving that the right hand side of (6) is achievable is called the direct coding theorem, whereas showing that it is an upper bound is called the converse. It is obvious from our definition that $K(W) \geq C_p(W)$, since any private channel can be used for generating a secret key. Hence it suffices to prove the converse for $K(W)$ and achievability for $C_p(W)$.

**Proof of Theorem 1 (converse)** We shall prove that, for any $\delta, \epsilon > 0$ and sufficiently large $n$, if an $(n, \epsilon)$ secret key generation code has rate $R$ then

\[
R - \delta \leq \frac{1}{n} \max_{T, X^n} \{I(T; Q^n) - I(T; E^n)\}.
\]

The proof parallels the classical one from \[2\]. Fano’s inequality \[13\] says

\[
H(K|L) \leq 1 + \Pr\{K \neq L\} nR.
\]

Hence

\[
nR = H(K) = I(K; L) + H(K|L) \leq I(K; L) + 1 + n\epsilon \log |X|.
\]

The last inequality follows from condition \[9\]. Furthermore,

\[
I(K; L) \leq I(K; S Q^n) = I(K; S) + I(K; Q^n|S) \leq I(K; Q^n|S) - I(K; E^n|S) + 2\epsilon = I(T; Q^n|S) - I(T; E^n|S) + 2\epsilon,
\]

\[^1\]As we shall see, $\epsilon$ can be made to decrease exponentially in $n$. 

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where \( T = KS \). The first inequality is a consequence of the Holevo bound \[22\] and the second inequality follows from conditions \([4]\) and \([5]\). Since, without loss of generality, \( \epsilon \leq \frac{\delta}{6 \log |X|} \) and \( n \geq \frac{2}{3} \),

\[
\frac{1}{n} [I(T; Q^n|S) - I(T; E^n|S)] \geq R - \delta,
\]

with \( S \rightarrow T \rightarrow X^n \rightarrow Q^nE^n \) a Markov chain. Since the left hand side can be written as the average of

\[
\frac{1}{n} [I(T_s; Q^n_s) - I(T_s; E^n_s)]
\]

with respect to the distribution of \( S \), and the Markov condition \( T_s \rightarrow X^n_s \rightarrow Q^n_sE^n_s \) holds for each \( s \), choosing the particular value of \( s \) that maximizes \( \[\] \) proves the claim. 

For the direct coding theorem we shall need two lemmas. The first is a quantum version of the Chernoff bound from \([3]\).

**Lemma 2 (Ahlswede, Winter)** Let \( \xi_1, \ldots, \xi_\mu \) be independent identically distributed (i.i.d.) random variables with values in the algebra \( B(\mathcal{H}) \) of bounded linear operators on some Hilbert space \( \mathcal{H} \), which are bounded between 0 and the identity operator \( I \). Assume that the expectation value \( \mathbb{E} \xi_m = \theta \geq tI \). Then for every \( 0 < \eta < 1/2 \)

\[
\Pr \left\{ \frac{1}{\mu} \sum_{m=1}^{\mu} \xi_m \notin [(1 \pm \eta)\theta] \right\} \leq 2 \dim \mathcal{H} \exp \left( -\frac{\mu \eta^2 t}{2 \ln 2} \right),
\]

where \( [(1 \pm \eta)\theta] = [(1 - \eta)\theta; (1 + \eta)\theta] \) is an interval in the operator order: \( [A; B] = \{ \xi \in B(\mathcal{H}) : A \leq \xi \leq B \} \).

The second lemma is Winter’s “gentle operator” lemma \([37]\). It says that a POVM element that succeeds on a state with high probability does not disturb it much.

**Lemma 3 (Winter)** For a state \( \rho \) and operator \( 0 \leq A \leq I \), if \( \text{Tr} (\rho A) \geq 1 - \lambda \), then

\[
\| \rho - \sqrt{\lambda} \rho \sqrt{\lambda} \|_1 \leq \sqrt{8\lambda}.
\]

The same holds if \( \rho \) is only a subnormalized density operator. 

In the above, \( \| A \|_1 = \text{Tr} \sqrt{AA^\dagger} \) denotes the trace norm of some operator \( A \). It is a norm in the sense that the trace distance between two operators \( A \) and \( B \), \( \| A - B \|_1 \), satisfies the triangle inequality

\[
\| A - C \|_1 \leq \| A - B \|_1 + \| B - C \|_1.
\]

**Proof of Theorem 1** (direct coding theorem) We shall construct a private channel code that achieves the expression \([6]\) without making use of the public channel from Alice to Bob. Consequently, the public channel cannot increase \( C_p(W) \) or \( K(W) \). Fixing the random variable \( X \) with distribution \( p \), our goal is first to show that a private information rate of \( I(X; Q) - I(X; E) \) is achievable. We shall draw heavily on ideas from Winter’s POVM compression paper \([38]\). Define \( \sigma_x = \text{Tr}_Q (\rho_x^{QE}) \) and \( \omega_x = \text{Tr}_E (\rho_x^{QE}) \), the local alphabets for Eve and Bob, respectively, and let

\[
\omega = \sum_x p_x \omega_x.
\]

In what follows we shall assume familiarity with the notions of typical sets \( T_{X,\delta}^n \), typical subspaces \( \Pi_{E,\delta}^n \), and conditionally typical subspaces \( \Pi_{E|X,\delta}^n \). These are defined in Appendix A for convenience.
Fixing $\delta > 0$, we have the following properties (for $x^n \in T^n_{X,\delta}$, where applicable) [37 97]:

$$s := \Pr \{ X^n \in T^n_{X,\delta} \} \geq 1 - \epsilon \quad (11)$$

$$\Tr \sigma_{x^n} \Pi^n_{\|X,\delta}(x^n) \geq 1 - \epsilon \quad (12)$$

$$\Tr \sigma_{x^n} \Pi^n_{\|X,\delta}(x^n) \geq 1 - \epsilon \quad (13)$$

$$\Tr \omega_{x^n} \Pi^n_{\|X,\delta}(x^n) \geq 1 - \epsilon \quad (14)$$

$$\Tr \omega_{x^n} \Pi^n_{\|X,\delta}(x^n) \geq 1 - \epsilon \quad (15)$$

$$\Tr \Pi^2_{\|X,\delta}(x^n) \leq \alpha^{-1} \quad (16)$$

$$\Pi^n_{\|X,\delta}(x^n) \sigma_{x^n} \Pi^n_{\|X,\delta}(x^n) \leq \beta \Pi^n_{\|X,\delta}(x^n) \quad (17)$$

$$\Tr \Pi^n_{\|X,\delta}(x^n) \leq \beta^{-1} \quad (18)$$

$$\Pi^n_{\|X,\delta}(x^n) \leq \alpha \Pi^n_{\|X,\delta}(x^n) \quad (19)$$

Here $\alpha = 2^{-n[H(\mathcal{E})+\epsilon \delta]}$, $\beta = 2^{-n[H(\mathcal{E}(X)-\epsilon \delta)]}$, $\alpha = 2^{-n[H(\mathcal{Q})+\epsilon \delta]}$, $\beta = 2^{-n[H(\mathcal{Q}(X)+\epsilon \delta)]}$ for some constant $c$ and $\epsilon = 2^{-n c \epsilon^2}$ for some constant $c'$. Define, for $x^n \in T^n_{X,\delta}$,

$$\xi^n_x = \Pi^n_{\|X,\delta}(x^n) \sigma_{x^n} \Pi^n_{\|X,\delta}(x^n),$$

$$\xi^n_x = \Pi^n_{\|X,\delta}(x^n) \Pi^n_{\|X,\delta}(x^n).$$

Since $\sigma_{x^n}$ commutes with $\Pi^n_{\|X,\delta}(x^n)$, $\xi^n_x \leq \sigma_{x^n}$. From this, [12] and [13]

$$\Tr \xi^n_x = \Tr \xi^n_x - \Tr (\mathcal{I} - \Pi^n_{\|X,\delta}(X,\delta+1)) \xi^n_x$$

$$\geq 1 - 2\epsilon.$$

Let $p'$ be the **pruned** distribution $p^{\otimes n}$ with respect to the set $T^n_{X,\delta}$, namely

$$p'(x^n) = \begin{cases} \frac{p(x^n)}{s} & x^n \in T^n_{X,\delta} \\ 0 & \text{otherwise,} \end{cases}$$

where $s$ is as defined in [11]. Then $\Tr \theta' \geq 1 - 2\epsilon$, for

$$\theta' = \sum_{x^n \in T^n_{X,\delta}} p'(x^n) \xi^n_x.$$  

Let $\Pi$ be the projector onto the subspace spanned by the eigenvectors of $\theta'$ with eigenvalue $\geq \epsilon \alpha$. By [11], the support of $\theta'$ has dimension $\leq \alpha^{-1}$, so eigenvalues smaller than $\epsilon \alpha$ contribute at most $\epsilon$ to $\Tr \theta'$. Hence, $\Tr \theta \geq 1 - 3\epsilon$ for $\theta = \Pi \theta' \Pi$. Also let $\xi^n_x = \Pi^n_{\|X,\delta}, \mu' = 2^{n(1-2\epsilon)\delta}, k' = 2^{n(1-2\epsilon)\delta-1}$ and define $\mu' \nu' \nu' \nu' \nu' \nu'$ i.d. random variables $U_{km}$, $m \in [\mu'], k \in [k']$, each distributed according to $p'$. Observe that $\theta = \mathcal{E} \xi^n_{U_km}$, where $\mathcal{E}$ denotes taking expectation values with respect to the distribution $p'$. Define the event

$$\nu_k = \left\{ \frac{1}{\mu'} \sum_{m=1}^{\mu'} \xi^n_{U_km} \in [(1 \pm \epsilon)\theta] \right\}.$$

According to lemma [2]

$$\Pr \{ \text{not } \nu_k \} \leq 2\Tr \Pi \exp \left( -\mu' \frac{\epsilon^2 \alpha}{2\beta \ln 2} \right), \forall k.$$  

The right hand side, being a double exponential in $n$, can be made $\leq \epsilon \nu'^{-1}$ for all $k$ and sufficiently large $n$. Now we shall argue that $\{U_{km} \}$ is a good code for the $\{c \to q\}$ channel $\mathcal{Q}(X)$ with high
probability. It is a random code of size \(2^n(I(X;Q)−2(\varepsilon+c\delta))\), and each codeword is chosen according to the pruned distribution \(p'\). The proof of the Holevo-Schumacher-Westmoreland theorem [23] involves choosing codewords according to the distribution \(p^n\) and can be easily modified to work for \(p'\) (see Appendix B). Consequently, the expectation of the average probability of error can be made to decay exponentially with \(n\):

\[
\mathbb{E} p_e(\{U_{km}\}) \leq 10\epsilon.
\]

Define the event

\[
i_0 = \{p_e(\{U_{km}\}) \leq \sqrt{\epsilon}\}.
\]

(21)

Then Markov’s lemma (from standard probability theory), according to which for any random variable \(X\) and constant \(\gamma > 0\)

\[
\Pr\{X \geq \gamma \mathbb{E}X\} \leq \gamma^{-1},
\]

implies

\[
\Pr\{\text{not } i_0\} \leq 10\sqrt{\epsilon}.
\]

By construction,

\[
\Pr\{\text{not } i_0 \& i_1 \ldots \& i_{\kappa'}\} \leq \sum_{k=0}^{\kappa'} \Pr\{\text{not } i_k\} \leq \epsilon + 10\sqrt{\epsilon},
\]

(22)

so there exists a particular value \(\{u_{km}\}\) of \(\{U_{km}\}\), for which \(i_k\) holds for all \(k = 0, \ldots, \kappa'\) (in fact, we have shown this holds with probability \(\geq 1 - \epsilon - 10\sqrt{\epsilon}\) for a randomly chosen \(\{u_{km}\}\)). For all \(x^n \in T_{X,\delta}\) we have, by lemma [8]

\[
\|\sigma_{x^n} - \xi_{x^n}\|_1 \leq \|\sigma_{x^n} - \xi_{x^n}\|_1 + \|\xi_{x^n} - \xi_{x^n}\|_1 \leq \epsilon + \sqrt{16\epsilon}.
\]

(23)

Now, \(i_k\) implies

\[
\text{Tr} \left[\frac{1}{\mu'} \sum_{m} \xi_{u_{km}}\right] \geq 1 - 4\epsilon
\]

and hence, by [24] and lemma [8]

\[
\left\|\frac{1}{\mu'} \sum_{m=1}^{\mu'} \sigma_{u_{km}} - \theta\right\|_1 \leq \frac{1}{\mu'} \sum_{m} \|\sigma_{u_{km}} - \xi_{u_{km}}\|_1 + \frac{1}{\mu'} \sum_{m} \|\xi_{u_{km}} - \xi_{u_{km}}\|_1 + \frac{1}{\mu'} \sum_{m=1}^{\mu'} \xi_{u_{km}} - \theta\right\|_1 \leq (\epsilon + \sqrt{16\epsilon}) + \sqrt{32\epsilon} + \epsilon = 2\epsilon + 4\sqrt{\epsilon} + 4\sqrt{2\epsilon}.
\]

(24)

So far we only have a bound on the average error probability for the channel code. We would like each individual codeword to have low error probability. By [21], at most a fraction \(\sqrt{\epsilon}\) of the codewords \(u_{km}\) have error probability \(\geq \sqrt{\epsilon}\). Moreover, at most a fraction \(\sqrt{\epsilon}\) of the values of \(k\) are such that a fraction \(\geq \sqrt{\epsilon}\) of the \(u_{km}\) for that particular \(k\) have error probability \(\geq \sqrt{\epsilon}\). We shall expurgate these values of \(k\) from the code, without loss of generality retaining the set \([\kappa]\) with \(\kappa = (1 - \sqrt{\epsilon})\kappa'\). For each \(k \in [\kappa]\), order the \(u_{km}\) according to increasing error probability, and retain only the first \(\mu = (1 - \sqrt{\epsilon})\mu'\) of them. This slight reduction in \(\kappa'\) and \(\mu'\) now ensures that each codeword has error probability \(\leq \sqrt{\epsilon}\). Since

\[
\left\|\frac{1}{\mu'} \sum_{m=1}^{\mu'} \sigma_{u_{km}} - \frac{1}{\mu} \sum_{m=1}^{\mu} \sigma_{u_{km}}\right\|_1 \leq 2\sqrt{\epsilon},
\]

(25)
we now have
\[ \left\| \frac{1}{\mu} \sum_{m=1}^{\mu} \sigma_{ukm} - \theta \right\|_1 \leq 2\epsilon \sqrt{\epsilon} + 4\sqrt{2\epsilon} + 2\sqrt{\epsilon} =: \epsilon'. \tag{26} \]

Note that this expurgation ensures that all the \( u_{km} \) are distinct; if they were not, the probability of error for a repeated codeword would be \( \geq \frac{1}{2} \), a contradiction for sufficiently large \( n \). Defining
\[ \sigma_k = \frac{1}{\mu} \sum_{m=1}^{\mu} \sigma_{ukm} \]
and
\[ \sigma = \frac{1}{\kappa} \sum_{k=1}^{\kappa} \sigma_k \tag{27} \]
we have
\[ \|\sigma_k - \sigma\|_1 \leq 2\epsilon', \quad \forall k. \tag{28} \]

By Fannes’ inequality (see e.g. [28]) we can estimate
\[ I(K; \mathcal{E}^n) = \frac{1}{\kappa} \sum_{k=1}^{\kappa} \left[ H(\sigma) - H(\sigma_k) \right] \]
\[ \leq \eta(2\epsilon') + 2n\epsilon' \log \dim \mathcal{H}_E, \tag{29} \]

when \( 2\epsilon' \leq \frac{1}{e} \) and where \( \eta(x) = -x \log x \). We are now in a position to describe our private channel code. The random variable \( M \) is uniformly distributed on \([\mu]\), the message \( K \) is uniformly distributed on \([\kappa]\) and the channel input is \( X(K, M) = u_{KM} \). By construction, Bob can perform a measurement that correctly identifies the pair \((k, m)\), and hence \( k \), with probability \( \geq 1 - 4\sqrt{\epsilon'} \). The rate of the code is bounded as
\[ R \geq I(K; \mathcal{E}_n) - I(T; \mathcal{Q}) - 6(c' + c''\delta), \]
for sufficiently large \( n \). Equation (29) and \( \epsilon = 2^{-c'n} \) ensures that \( I(K; \mathcal{E}^n) \) can be made arbitrarily small (indeed exponentially small in \( n \)) for sufficiently large \( n \). Notice that by simulating some channel \( X|T \) in her lab, Alice can effectively produce the \( QE|T \) channel for \( T \rightarrow X \rightarrow QE \); thus \( I(T; \mathcal{Q}) - I(T; \mathcal{E}) \) is also achievable.

The multi-letter formula (6) follows from applying the above to the super-channel \( W^{\otimes l} \).

**Remark** The classical analogue of theorem [1] namely the capacity \( C_p(W) \) of a \( \{c \rightarrow cc\} \) channel \( W = YZ|X \) was first discovered in [11] for a weaker notion of secrecy, and later strengthened in [27]. Our result implies a new proof of the classical direct coding theorem, using large deviation techniques instead of hashing/extractors as in the work of Maurer and collaborators [27].

### 3 Private information transmission and key generation over quantum channels

Now we shall apply these results to the setting where Alice and Bob are connected via a noisy quantum channel \( \mathcal{N} : B(\mathcal{H}_P) \rightarrow B(\mathcal{H}_Q) \). Here \( B(\mathcal{H}_P) \) denotes the space of bounded linear operators on \( \mathcal{H}_P \), the Hilbert space of the quantum system \( P \). The channel \( \mathcal{N} \) is (non-uniquely) defined in terms of the operation elements \( \{A_i\} \), \( \sum_i A_i A_i^\dagger = I \), as
\[ \mathcal{N}(\rho) = \sum_i A_i \rho A_i^\dagger. \]

This representation is exploited in Shor’s proof of the quantum channel capacity theorem [34]. Here we take a different approach to noisy channels, propagated by Schumacher and collaborators.
The channel is physically realized by an isometry \( U_N : B(\mathcal{H}_P) \to B(\mathcal{H}_Q) \), called an isometric extension of \( N \), which explicitly includes the unobserved environment \( E \).

We shall assume that the environment \( E \) is completely under the control of the eavesdropper Eve, and the quantum system \( Q \) is under Bob’s control. Suppose Alice’s initial density operator is given by \( \rho^P \). Defining \( \omega^Q = N(\rho^P) = \text{Tr}_E U_N(\rho^P) \) and \( \sigma^E = \text{Tr}_Q U_N(\rho^P) \), the coherent information is defined as

\[
I_c(\rho^P, N) = H(\omega^Q) - H(\sigma^E).
\]

Note that, although there is an infinite family of \( U_N \) corresponding to a given \( N \), the coherent information is independent of this choice [30]. Since we are interested in transmitting private classical information, the most general protocol requires Alice to prepend a \( \{c \to q\} \) channel \( PE \) to \( j \) instances of \( N \), for arbitrarily large \( j \). This induces a \( \{c \to qq\} \) channel \( Q^jE^j|X \), and we may now apply the results of the previous section. Combining the \( l \to \infty \) limit from equation (6) with the \( j \to \infty \) one and absorbing \( T \) into \( X \) gives

\[
C_p(N) = K(N) = \lim_{l \to \infty} \frac{1}{l} \max_{I \in P} \left\{ I(X; Q^l) - I(X; E^l) \right\}.
\]

It is easily verified (see also section 4) that this may be rewritten as

\[
C_p(N) = K(N) = \lim_{l \to \infty} \frac{1}{l} \max_{\rho \in H_P} I_p(\rho, N^{\otimes l}),
\]

where we introduce the private information \( I_p \):

\[
I_p(\rho, N) = I_c(\rho, N) - \min_{\{p(x), \rho_x\}} \left\{ \sum_x p(x) I_c(\rho_x, N) : \sum_x p(x) \rho_x = \rho \right\}.
\]

The above expression for \( K(N) \) is almost implicit in [31], albeit without proof. Note that \( I_p(\rho, N) \geq I_c(\rho, N) \) since any decomposition of \( \rho \) into pure states sets the expression minimized in [31] to zero. It is codes corresponding to \( I_p(\rho, N) = I_c(\rho, N) \) that will be relevant for entanglement generation.

In Appendix C we give an example illustrating the possibility of \( I_p > I_c > 0 \).

4 Entanglement generation over quantum channels

In this section we apply the above results to the more difficult problem of entanglement generation over quantum channels. The objective is for Alice and Bob to share a nearly maximally entangled state on a \( 2^nR \times 2^nR \) dimensional Hilbert space, by using a large number \( n \) instances of the noisy quantum channel \( N \). Before getting into details we should recall some facts about fidelities and purifications (mostly taken from [28]). The fidelity of two density operators with respect to each other can be defined as

\[
F(\rho, \sigma) = \|\sqrt{\rho} \sqrt{\sigma}\|_1^2.
\]

For two pure states \( |\chi\rangle, |\zeta\rangle \) this amounts to

\[
F(|\chi\rangle, |\zeta\rangle) = |\langle \chi | \zeta \rangle|^2.
\]

\footnote{The standard formulation of [28] refers to a channel \( N : B(\mathcal{H}_Q) \to B(\mathcal{H}_Q) \) with the same input and output Hilbert space. The channel is physically realized by appending an environment system \( E \), wlog initially in a pure state, applying a unitary operation \( U \otimes \mathbb{I}_E \) on the joint system, and tracing out \( E \). Here we adapt the slightly more general approach of [28] in which the input and output Hilbert space of the channel may differ.
}

\footnote{Our definition of fidelity is the square of the quantity defined in [28].}
Lemma 4 Consider two collections of orthonormal states \( (|\chi_j\rangle)_{j\in[N]} \) and \( (|\zeta_j\rangle)_{j\in[N]} \) such that \( \langle \chi_j | \zeta_j \rangle \geq 1 - \epsilon \) for all \( j \). There exist phases \( \gamma_j \) and \( \delta_j \) such that
\[
\langle \hat{\chi} | \hat{\zeta} \rangle \geq 1 - \epsilon,
\]
where
\[
|\hat{\chi}\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{i\gamma_j} |\chi_j\rangle,
\]
\[
|\hat{\zeta}\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{i\delta_j} |\zeta_j\rangle.
\]

Proof Define the Fourier transformed states
\[
|\hat{\chi}_s\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{2\pi ij s/N} |\chi_j\rangle,
\]
and similarly define \( |\hat{\zeta}_s\rangle \). It is easy to see that
\[
\frac{1}{N} \sum_{s=1}^{N} \langle \hat{\chi}_s | \hat{\zeta}_s \rangle = \frac{1}{N} \sum_{j=1}^{N} \langle \chi_j | \zeta_j \rangle \geq 1 - \epsilon,
\]
hence at least one value of \( s \) obeys
\[
e^{i\theta_s} \langle \hat{\chi}_s | \hat{\zeta}_s \rangle \geq 1 - \epsilon,
\]
for some phase \( \theta_s \). Setting \( \gamma_j = 2\pi j s/N \) and \( \delta_j = \gamma_j + \theta_s \) satisfies the statement of the lemma.
Moreover, a fraction \( 1 - \sqrt{\epsilon} \) of the values of \( s \) satisfy
\[
e^{i\theta_s} \langle \hat{\chi}_s | \hat{\zeta}_s \rangle \geq 1 - \sqrt{\epsilon},
\]
a fact we shall use in Appendix D.

The following relation between fidelity and the trace distance will be needed:
\[
1 - \sqrt{F(\rho, \sigma)} \leq \frac{1}{2} \| \rho - \sigma \|_1 \leq \sqrt{1 - F(\rho, \sigma)},
\]
the second inequality becoming an equality for pure states. A purification \( |\Phi_\rho\rangle^{RQ} \) of a density operator \( \rho^Q \) is some pure state living in an augmented quantum system \( RQ \) such that \( \text{Tr}_R(|\Phi_\rho\rangle\langle\Phi_\rho|^{RQ}) = \rho^Q \). Any two purifications \( |\Phi_\rho\rangle^{RQ} \) and \( |\Phi'_\rho\rangle^{RQ} \) of \( \rho^Q \) are related by some local unitary \( U \) on the reference system \( R \)
\[
|\Phi'_\rho\rangle^{RQ} = (U^R \otimes I^Q)|\Phi_\rho\rangle^{RQ}.
\]
A theorem by Uhlmann states that, for a fixed purification \( \Phi_\sigma \) of \( \sigma \),
\[
F(\rho, \sigma) = \max_{\Phi_\rho} F(|\Phi_\rho\rangle, |\Phi_\sigma\rangle).
\]
A corollary of this theorem is the monotonicity property of fidelity
\[
F(\rho^RQ, \sigma^RQ) \leq F(\rho^Q, \sigma^Q),
\]
where \( \rho^Q = \text{Tr}_R \rho^RQ \) and \( \sigma^Q = \text{Tr}_R \sigma^RQ \).
Returning to the problem of entanglement generation, an \((n, \epsilon)\) code is defined as follows. Alice prepares, without loss of generality, a pure bipartite state \(|\Upsilon\rangle^A_P^n\) in her lab, defined on \(H_A \otimes H_P^n\), \(\dim H_A = \kappa\), and sends the \(P^n\) portion of it through the channel. Bob performs a general decoding quantum operation on the channel output \(D: B(H_Q^n) \rightarrow B(H_B), \dim H_B = \kappa\), yielding the state
\[
\Omega^{AB} = [I^A \otimes (D \circ N^{\otimes^p})(|\Upsilon\rangle\langle\Upsilon|)^{A^p^n}].
\]
(35)
The rate of the code is \(R = \frac{1}{n} \log \kappa\). We require
\[
F(|\Phi^\kappa\rangle^A^B, \Omega^{AB}) \geq 1 - \epsilon,
\]
where
\[
|\Phi^\kappa\rangle^{AB} = \sqrt{\frac{\kappa}{\kappa}} \sum_{k=1}^{\kappa} |k\rangle^A |k\rangle^B
\]
is the standard maximally entangled state shared by Alice and Bob. We shall call a rate \(R\) achievable if for every \(\epsilon, \delta > 0\) and sufficiently large \(n\) there exists an \((n, \epsilon)\) code of rate \(R - \delta\). The entanglement generating capacity \(E(N)\) is given by the supremum of achievable \(R\).

**Theorem 5** Given the channel \(N\),
\[
E(N) = \lim_{l \rightarrow \infty} \frac{1}{l} \max_{\rho \in H^{\otimes l}_P} I_c(\rho, N^{\otimes l}).
\]
(36)

**Remark** Note that \(K(N) \geq E(N)\) is obvious since any pure entanglement can be converted into a secret key by performing a measurement in the \(|k\rangle\) basis. It is not clear from formulas (30) and (36) whether the inequality can be made strict. We return to this issue in the final section.

The converse theorem makes use of the following simple lemma [4].

**Lemma 6** For two states \(\rho^{RQ}\) and \(\sigma^{RQ}\) of a quantum system \(RQ\) of dimension \(d\) with fidelity \(f = F(\rho^{RQ}, \sigma^{RQ})\),
\[
|\Delta H(\rho^{RQ}) - \Delta H(\sigma^{RQ})| \leq \frac{2}{e} + 4 \log d \sqrt{1 - f},
\]
where
\[
\Delta H(\rho^{RQ}) = H(\rho^Q) - H(\rho^{RQ}).
\]

**Proof** By the monotonicity of fidelity, \(F(\rho^B, \sigma^B) \geq f\). The lemma follows from a double application of Fannes’ inequality [28] and [30].

**Proof of Theorem 5** (converse) We shall prove that, for any \(\delta, \epsilon > 0\) and sufficiently large \(n\), if an \((n, \epsilon)\) code has rate \(R\) then \(R - \delta \leq \frac{1}{n} I_c(\rho, N^{\otimes n})\), where \(\rho\) the restriction of \(|\Upsilon\rangle\) to \(H_P^n\). Evidently, it suffices to prove this for \(\epsilon \leq \frac{1}{16 \log \dim H_P}\) and \(n \geq \frac{4}{\epsilon^3}\). The converse relies on the quantum data processing inequality, which says that quantum post-processing cannot increase the coherent information [30].

\[
I_c(\rho, N^{\otimes n}) \geq I_c(\rho, \mathcal{D} \circ N^{\otimes n}) = \Delta H(\Omega) \geq \Delta H(|\Phi^\kappa\rangle\langle\Phi^\kappa|) - \frac{2}{e} - 8nR\sqrt{\epsilon} \geq nR - \frac{2}{e} - 8n \log \dim H_P \sqrt{\epsilon},
\]
from which the claim follows. The first inequality is the data processing inequality and the second inequality is an application of lemma [10].
Proof of Theorem 5 (direct coding theorem) It suffices to demonstrate that a rate of 
\( I_c(\rho, N) \) is achievable for any \( \rho \in \mathcal{H}_F \). The regularized formula \( I_c(\rho, N) \) is obtained by additional blocking. Following [31], consider the eigen-decomposition of \( \rho \) into the orthonormal pure state ensemble \( \{ p(x), |\phi_x\rangle \} \),
\[
\sum_x p(x) |\phi_x\rangle \langle \phi_x| = \rho.
\] (37)
The distribution \( p \) defines a random variable \( X \). Let \( U_N : \mathcal{B}(|\phi_x\rangle) \to \mathcal{B}(\mathcal{H}_F) \) be an isometric extension of \( N \). Define the channel \( W : x \mapsto U_N |\phi_x\rangle^N = |\phi'_x\rangle^{Q_E} \). Define the local output density matrices seen by Bob and Eve by \( \omega_x^Q = \text{Tr}_E (|\phi'_x\rangle \langle \phi'_x|^{Q_E}) \) and \( \sigma_x^E = \text{Tr}_Q (|\phi'_x\rangle \langle \phi'_x|^{Q_E}) \), respectively, and the averages over \( x \) by \( \omega^Q \) and \( \sigma^E \), respectively. In section 2 we showed that for any \( \delta \) there exists an \((n, \epsilon)\) code, defined by \( \{ u_{km} : k \in [k], m \in [\mu] \} \), of rate \( \frac{1}{n} \log \kappa = I_c(\rho, N) - \delta \). Indeed [31]
\[
I(X; Q) - I(X; E) = H(\omega^Q) - \sum_x p(x) H(\omega_x^Q) - H(\sigma^E) + \sum_x p(x) H(\sigma_x^E)
\]
\[
= H(\omega^Q) - H(\sigma^E) = I_c(\rho, N),
\]
since \( H(\omega_x^Q) = H(\sigma_x^E) \) for all \( x \).

In what follows we shall be dealing with blocks of length \( n \) and use the abbreviated notation \( Q \) for \( Q^n \), etc. Consider sending the state \( |\phi_{km}\rangle^N := |\phi_{u_{km}}\rangle^P \) through the isometric extension channel \( U_N^{\otimes n} \)
\[
|\phi'_{km}\rangle^{Q_E} = U_N^{\otimes n} |\phi_{km}\rangle^P.
\]
Define \( \sigma_x^E = \text{Tr}_Q (|\phi'_{km}\rangle \langle \phi'_{km}|^{Q_E}) \) and
\[
\sigma_k^E = \frac{1}{\mu} \sum_m \sigma_{km}^E.
\]
As shown in section 2, there exists a \( \theta^E \) such that, for all \( k \),
\[
||\sigma_k^E - \theta^E||_1 \leq \epsilon. \quad (38)
\]
In addition, there is a measurement Bob can perform on \( Q \) that with probability \( \geq 1 - \epsilon \) correctly identifies the index \( km \). Since any measurement can be written as a unitary operation on a larger Hilbert space (including some ancilla initially in a pure state) followed by a von Neumann measurement on the ancilla, there exists a unitary \( V^{QB'B'} \) such that
\[
(I^{E} \otimes V^{QB'B'}) |\phi'_{km}\rangle^{Q_E} |0\rangle^{B'} |0\rangle^{B'} = |\psi_{km}\rangle^{QB'B'}
\]
and
\[
F(\rho_{km}^{QB'}, |k\rangle^{B'} |m\rangle^{B'}) \geq 1 - \epsilon,
\]
for \( \rho_{km}^{QB'} = \text{Tr}_Q (|\psi_{km}\rangle \langle \psi_{km}|^{QB'B'}) \). By Uhlmann’s theorem, for the purification \( |\psi_{km}\rangle^{QB'B'} \) of \( \rho_{km}^{QB'} \) there exists a “purification” \( |\chi_{km}\rangle^{QB'E} |k\rangle^{B'} |m\rangle^{B'} \) of \( |k\rangle^{B'} |m\rangle^{B'} \) such that
\[
\langle \psi_{km}|^{QB'B'} |\chi_{km}\rangle^{QB'E} |k\rangle^{B'} |m\rangle^{B'} \geq 1 - \epsilon. \quad (39)
\]
Let \( \tilde{\sigma}_{km}^E = \text{Tr}_Q (|\chi_{km}\rangle \langle \chi_{km}|^{QB'E}) \). Since \( |\psi_{km}\rangle^{QB'B'} \) is also a purification of \( \sigma_{km}^E \), we have, by [31], the monotonicity of fidelity and [31]:
\[
||\sigma_{km}^E - \tilde{\sigma}_{km}^E||_1 \leq 2 \sqrt{1 - F(\sigma_{km}^E, \tilde{\sigma}_{km}^E)} \leq 2 \sqrt{1 - F(|\psi_{km}\rangle^{QB'B'}, |\chi_{km}\rangle^{QB'E} |k\rangle^{B'} |m\rangle^{B'})} \leq 2 \epsilon.
\]
Define
\[ \tilde{\sigma}_k^E = \frac{1}{\mu} \sum_m \tilde{\sigma}_{km}^E. \] (40)

Then
\[ \| \tilde{\sigma}_k^E - \theta^E \|_1 \leq \frac{1}{\mu} \sum_m \| \sigma_{km}^E - \tilde{\sigma}_{km}^E \|_1 + \| \sigma_k^E - \theta^E \|_1 \] (41)
\[ \leq 2\sqrt{\epsilon} + \epsilon. \] (42)

By (41)
\[ F(\tilde{\sigma}_k^E, \theta^E) \geq 1 - 2\sqrt{\epsilon} - \epsilon. \] (43)

Consider the set of quantum codewords \( \{ |\phi_k\rangle \} \):
\[ |\phi_k\rangle^P = \sqrt{\frac{1}{\mu}} \sum_m e^{i\gamma_{km}} |\phi_{km}\rangle^P, \] (44)
with the phases \( \gamma_{km} \) to be specified. After transmission through \( U^\otimes_n \), adding the ancilla \( |0\rangle^B |0\rangle^{B'} \) and applying \( V^{QB'B'} \), \( |\phi_k\rangle^P \) becomes
\[ |\psi_k\rangle^{QEB'B'} = \sqrt{\frac{1}{\mu}} \sum_m e^{i\gamma_{km}} |\psi_{km}\rangle^{QEB'B'}. \] (45)

By (43) and lemma 1, we can choose the phases \( \gamma_{km} \) and \( \delta_{km} \) so that
\[ \langle \psi_k |^{QEB'B'} |k\rangle^B |\phi_k\rangle^{QEB'} \geq 1 - \epsilon, \] (46)
where
\[ |\phi_k\rangle^{QB'} = \sqrt{\frac{1}{\mu}} \sum_m e^{i\delta_{km}} |\chi_{km}\rangle^Q |m\rangle^{B'}. \]

Note that \( \tilde{\sigma}_k^E = \text{Tr}_{QB'} (|\phi_k\rangle \langle \phi_k|^{QEB}) \), as defined in (40). Hence, fixing a purification \( |\Phi_\theta\rangle^{QEB'} \) of \( \theta^E \), for all \( k \in [\kappa] \) there exists a unitary \( U_k^{QB'} \) such that (cf. [32])
\[ \langle \Phi_\theta |^{QEB'} [U_k^{QB'} \otimes I^E] |\phi_k\rangle^{QEB'} \geq 1 - 2\sqrt{\epsilon} - \epsilon, \]
by applying Uhlmann’s theorem to (43). Introducing the “controlled” unitary
\[ U^{QB'} = \sum_k |k\rangle^B \langle k| \otimes U_k^{QB'}, \]
the above may be rewritten as
\[ \langle \Phi_\theta |^{QEB'} [U^{QB'} \otimes I^E] |\psi_k\rangle^{QEB'B'} \geq 1 - 4\sqrt{\epsilon} - 4\epsilon. \]

Combining this with (43) gives
\[ \langle \Phi_\theta |^{QEB'} [U^{QB'B'} \otimes I^E] |\psi_k\rangle^{QEB'B'} \geq 1 - 4\sqrt{\epsilon} - 4\epsilon. \] (47)

We can now define our entanglement generating code. Alice prepares the state
\[ |\Upsilon\rangle^{AP} = \sqrt{\frac{1}{\kappa}} \sum_k |k\rangle^A |\phi_k\rangle^P, \] (48)
keeps the system $A$ and sends the system $P$ through the channel. Bob subsequently applies the decoding operator
\[ D : \omega \mapsto \text{Tr}_{QB'} \left[ U^{BQ'B'} V^{Q'B'} (\omega \otimes |0\rangle \langle 0|^P \otimes |0\rangle \langle 0|^B') V^{Q'B'} U^{BQ'B'} \right], \]
resulting in some state $\Omega^{AB}$, which by (47) and the monotonicity of fidelity obeys
\[ F(\Omega^{AB}, |\Phi^\kappa\rangle^{AB}) \geq 1 - 4\sqrt{\epsilon} - 4\epsilon. \]  
(50)

This concludes the proof of the direct coding theorem. \[ \blacksquare \]

Remark Transforming a private channel code into an entanglement generating one appears to work only for pure state decompositions of $\rho$. Otherwise, the pure states $|\phi\rangle$ become effectively shared by Alice, Bob and Eve. The decoding operation $D$ would then involve performing joint operations on spatially separated quantum systems belonging to Bob and Alice, which cannot be accomplished in general without additional quantum resources.

Remark Note the similarity between (44) and CSS codes [13]. Indeed, here we have a coset-like decomposition of a $\{c \rightarrow q\}$ “error correction” code of size $\kappa \mu$ into $\kappa \{c \rightarrow q\}$ “privacy amplification” codes of size $\mu$ (see [28] for a nice exposition of these concepts in the context of the Shor-Preskill result [35]). The differences lie in that CSS codes have an additional algebraic structure and are composed of purely classical rather than classical-quantum codes.

5 Quantum information transmission over quantum channels.

Finally we arrive at our destination: recovering the formula for the quantum capacity $Q(N)$ of a quantum channel $N$. This quantity has been rigorously defined in [4] and we briefly review it here. An $(n, \epsilon)$ code is defined by an encoding operation $E : B(\mathcal{H}) \rightarrow B(\mathcal{H}_P^\otimes n)$ and a decoding operation $D : B(\mathcal{H}_Q^\otimes n) \rightarrow B(\mathcal{H})$, such that
\[ \min_{|\phi\rangle \in \mathcal{H}} F(|\phi\rangle, (D \circ N^\otimes n \circ E)(|\phi\rangle \langle \phi|)) \geq 1 - \epsilon. \]  
(51)

The rate of the code is given by $R = \frac{1}{n} \log \dim \mathcal{H}$. The quantum capacity of the channel $Q(N)$ is the supremum of all achievable $R$.

There is an alternative definition, in which the condition (51) and the definition of $R$ are replaced by
\[ R = \max_{\rho \in \mathcal{H}} \{ H(\rho) : F_c(\rho, D \circ N^\otimes n \circ E) \geq 1 - \epsilon \}. \]  
(52)

Here $F_c$ is the entanglement fidelity [30, 28]
\[ F_c(\rho, \mathcal{O}) = F(|\Psi\rangle, (I \otimes \mathcal{O})(|\Psi\rangle \langle \Psi|)), \]
where $|\Psi\rangle$ is some purification of $\rho$ ($F_c$ is independent of the particular choice of $|\Psi\rangle$). We denote the corresponding capacity by $\tilde{Q}(N)$. In [4], $Q(N)$ and $\tilde{Q}(N)$ were called the subspace transmission and entanglement transmission capacities of the channel, respectively, and were shown to be equal.

It comes as no surprise that entanglement generation and entanglement transmission are closely related. This intuition is made rigorous by the following proposition.

Proposition 7 Given the channel $N$,
\[ Q(N) = \tilde{Q}(N) = E(N) = \lim_{l \to \infty} \frac{1}{l} \max_{\rho \in \mathcal{H}_P^\otimes l} I_c(\rho, N^\otimes l). \]  
(53)
They may perform a bilateral twirling operation \([10]\) to transform \(\Omega\) into a Werner state

\[
\tilde{E}^2 = \int dU (U \otimes U^\dagger) \Omega (U \otimes U^\dagger)^\dagger = \frac{1}{\kappa - 1} (I - |\Phi^\kappa\rangle \langle \Phi^\kappa|),
\]

which is now interpreted as being in the state \(|\Phi^\kappa\rangle\) with probability \(f\). Since teleporting a state \(\rho\) living in a \(\kappa\)-dimensional Hilbert space \(\mathcal{H}\) via the maximally entangled \(|\Phi^\kappa\rangle\) yields an entanglement fidelity of 1, using \(T(\Omega)\) instead will give an entanglement fidelity of at least \(f \geq 1 - \epsilon\). Actually, one need not perform the full twirling operation. The twirl is equivalent to applying some bilateral \(U \otimes U^*\) chosen at random. Thus there exists a particular value of \(U\) for which the entanglement fidelity is \(\geq f\). Furthermore, the \(U \otimes U^*\) easily absorbed into \(|\Upsilon\rangle\) and \(D\) of the entanglement generating protocol. Choosing \(\rho\) to be maximally entropic proves the claim. Thus \(\tilde{Q}(\mathcal{N}) = E(\mathcal{N})\).

\[\blacksquare\]

**Remark** It is possible to modify the proof of the direct coding part of theorem\([4]\) to lower bound \(\tilde{Q}(\mathcal{N})\) directly rather than via \(E(\mathcal{N})\). This is done in Appendix D, where we also show the existence of random entanglement transmission codes of large blocklength \(n\) with rate arbitrarily close to \(I_c(\rho, \mathcal{N})\) and the nice property that the average density operator of the codewords is arbitrarily close to \(\rho^\otimes n\).

## 6 Discussion

We have defined and found expressions for the private information transmission \(C_p\) and secret key generation \(K\) capacities for classical-quantum wire-tap channels \(W\) and quantum channels \(\mathcal{N}\). A subclass of the corresponding protocols was made “coherent” to yield entanglement generation and quantum information transmission protocols achieving the respective capacities \(E(\mathcal{N})\) and \(Q(\mathcal{N})\). Thus we have established a very important operational connection between quantum privacy and quantum coherence \([31]\).

Our results show that \(C_p(W) = K(W), C_p(\mathcal{N}) = K(\mathcal{N})\) and \(E(\mathcal{N}) = Q(\mathcal{N})\). On the other hand, it is obvious operationally, as well as from \(I_p(\rho, \mathcal{N}) \geq I_c(\rho, \mathcal{N})\), that \(K(\mathcal{N}) \geq E(\mathcal{N})\). Although it is easy to find examples of strict inequality between \(I_c(\rho, \mathcal{N})\) and \(I_p(\rho, \mathcal{N})\) for particular \((\rho, \mathcal{N})\) pairs (see Appendix C), it is not clear whether this still holds when optimized over \(\rho\) and in the asymptotic sense of \([30]\) and \([36]\). In particular, we would like to know whether there exist quantum channels which cannot be used for transmitting quantum information, yet may be used to establish a secret key. The “static” analogue of this question was answered recently in \([23]\) by providing an example of a bipartite state with non-zero distillable secret key but zero \((\text{two-way!})\) distillable entanglement. It is natural to expect that a channel related to this state would demonstrate a separation between \(K\) and \(E\) \([36]\).

Another open problem is whether the formula for \(C_p(W)\) may be single-letterized (as in the purely classical case \([16]\)) for general channels or at least certain classes of channels. The same
question is open for $C_p(\mathcal{N})$, whereas counterexamples are known for $Q(\mathcal{N})$ [20]. A non-trivial class of channels exists (so-called degradable channels) for which $Q(\mathcal{N})$ is efficiently computable [18], and it can be shown that this property extends to $C_p(W)$ (we do not know this to be true for $C_p(\mathcal{N})$). More generally, we would like to be able to say something about the convergence rate of the limits in equations (30) and (36).

A natural extension of the present work would be to allow two-way public/classical communication for key/entanglement generation. This enhanced auxiliary resource is known to improve the capacities in both cases [4, 27, 10], but it seems unlikely that reasonable information-theoretical formulas exist in general.

The results of section 2 were independently obtained by Cai and Yeung [12], see [39, 40].

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# A Definitions of typical sequences and subspaces

We shall list definitions and properties of typical sequences and subspaces [17, 29, 37]. Consider some general classical-quantum system $UXQ$ in the state defined by the ensemble $(p(u,x), \rho_{ux})$. $X$ is defined on the set $\mathcal{X}$ and $U$ on the set $\mathcal{U}$. Denote by $p(x)$ and $P(x|u)$ the distribution of $X$ and conditional distribution of $X|U$, respectively.

For the probability distribution $p$ on the set $\mathcal{X}$ define the set of typical sequences (with $\delta > 0$)

$$T_{p,\delta}^n = \{x^n : \forall x \ |N(x|x^n) - np(x)| \leq n\delta\},$$

where $N(x|x^n)$ counts the number of occurrences of $x$ in the word $x^n = x_1 \ldots x_n$ of length $n$. When the distribution $p$ is associated with some random variable $X$ we may use the notation $T^n_{p,\delta}$.

For the stochastic map $P : \mathcal{U} \rightarrow \mathcal{X}$ and $u^n \in \mathcal{U}^n$ define the set of conditionally typical sequences (with $\delta > 0$) by

$$T_{p,\delta}^n(u^n) = \{x^n : \forall u, x \ |N((u,x)|(u^n,x^n)) - P(x|u)N(u|u^n)| \leq n\delta\}.$$

When the stochastic map $P$ is associated with some conditional random variable $X|U$ we may use the notation $T^n_{X|U,\delta}(u^n)$.

For a density operator $\rho$ on a $d$-dimensional Hilbert space $\mathcal{H}$, with eigen-decomposition $\rho = \sum_{k=1}^d \lambda_k |k\rangle \langle k|$ define (for $\delta > 0$) the typical projector as

$$\Pi_{\rho,\delta}^n = \sum_{k^n \in T_{\rho,\delta}^n} |k^n\rangle \langle k^n|.$$

When the density operator $\rho$ is associated with some quantum system $Q$ we may use the notation $\Pi_{Q\delta}^n$.

For a collection of states $\rho_u, u \in \mathcal{U}$, and $u^n \in \mathcal{U}^n$ define the conditionally typical projector as

$$\Pi_{(\rho_u),\delta}^n(u^n) = \bigotimes_{u} \Pi_{\rho_u,\delta}^I(u),$$

where $I_u = \{i : u_i = u\}$ and $\Pi_{\rho_u,\delta}^I$ denotes the typical projector of the density operator $\rho_u$ in the positions given by the set $I_u$ in the tensor product of $n$ factors. When the $\{\rho_u\}$ are associated with some conditional classical-quantum system $Q|U$ we may use the notation $\Pi_{Q|U,\delta}^n(u^n)$.
B  The modified HSW theorem

Define \( \nu = 2^{n[I(X;Q)-2(c+c'\delta)]} \) i.i.d. random variables \( \{U_s\}, s \in [\nu] \), according to the pruned distribution \( p' \). We shall show that this random set can be made into a HSW code for the \( \{c \rightarrow q\} \) channel \( Q|X \) with low probability of error. More precisely, we shall construct a decoding POVM \( \{Y_s\} \) such that

\[
\mathbb{E}[p_e(\{U_s\})] = \mathbb{E}[\text{Tr}(\omega_{U_s}(I - Y_s))] \leq 10\epsilon.
\]

We shall need the following lemma due to Hayashi and Nagaoka [21]: For any operators \( 0 \leq S \leq I \) and \( T \geq 0 \)

\[
I - (S + T)^{-\frac{1}{2}}S(S + T)^{-\frac{1}{2}} \leq 2(I - S + 4T).
\]

(54)

The \( \{Y_s\} \) are constructed as follows

\[
Y_s = (\sum_t \Lambda_{U_t})^{\frac{1}{2}} \Lambda_{Y_s} (\sum_t \Lambda_{U_t})^{\frac{1}{2}}
\]

with

\[
\Lambda_{x^n} = \Pi_{Q,\delta(|X|+1)}^n \Pi_{Q|X,\delta}(x^n) \Pi_{Q,\delta(|X|+1)}^n.
\]

Then, by (54),

\[
p_e(\{U_s\}) \leq 2(1 - \text{Tr} \omega_{U_s} \Lambda_{U_s}) + 4 \sum_{t \neq s} \text{Tr} \omega_{U_s} \Lambda_{U_t}.
\]

(55)

It is not hard to verify (cf. lemma 6 of [21]) that for \( x^n \in T_{N,\delta}^n \)

\[
\text{Tr} \omega_{x^n} \Lambda_{x^n} \geq 1 - 3\epsilon
\]

follows from (14) and (15). By (18)

\[
\text{Tr} \Lambda_{x^n} \leq \tilde{\beta}^{-1}.
\]

Also note that

\[
\mathbb{E}[\omega_{U_s}] = \sum_{x^n} p'(x^n) \omega_{x^n} \leq (1 - \epsilon)^{-1} \omega \otimes^n,
\]

so that, by (19),

\[
\Pi_{Q,\delta(|X|+1)}^n \mathbb{E}[\omega_{U_s}] \Pi_{Q,\delta(|X|+1)}^n \leq (1 - \epsilon)^{-1} \tilde{\alpha} \Pi_{Q,\delta(|X|+1)}^n.
\]

Putting everything together, taking the expectation of (55) and noting that \( U_s \) and \( U_t \) are independent for \( s \neq t \), we have

\[
\mathbb{E}[p_e(\{U_s\})] \leq 6\epsilon + 4(\nu - 1)\text{Tr}(\mathbb{E}[\omega_{U_s}], \mathbb{E}[\Lambda_{U_t}])
\]

\[
\leq 6\epsilon + 4\nu(1 - \epsilon)^{-1} \tilde{\alpha} \tilde{\beta}^{-1} \leq 10\epsilon,
\]

the last inequality coming from our choice of \( \nu \).

C  An example of \( I_p(\rho,N) > I_c(\rho,N) \)

Consider a four dimensional Hilbert space \( \mathcal{H}_4 \), with orthonormal basis \( \{|1), |2), |3\rangle, |4\rangle\}. \) Let \( \Pi_{12} \) be the projector onto the space spanned by \( |1) \) and \( |2) \), and let \( \Pi_{34} \) be the projector onto the space spanned by \( |3\rangle \) and \( |4\rangle \). Our channel\(^4\) \( N : B(\mathcal{H}_4) \rightarrow B(\mathcal{H}_4) \) is given by

\[
N(\rho) = \Pi_{12} \rho \Pi_{12} + \mathcal{D}_{34}(\Pi_{34} \rho \Pi_{34}),
\]

\(^4\)This channel was suggested to us by J. A. Smolin.
where $\mathcal{D}_{34}$ is the completely depolarizing channel on the two-dimensional subspace spanned by $|3\rangle$ and $|4\rangle$. Defining $\pi_{12} = \frac{1}{2} \Pi_{12}$ and $\pi_{34} = \frac{1}{2} \Pi_{34}$, it is easily verified that

\[
I_c(\pi_{12}, N) = 1 \\
I_c(\pi_{34}, N) = -1. 
\]  

(56)

Define, for some small positive $\epsilon$,

\[
\pi'_{12} = (1 - \epsilon)\pi_{12} + \epsilon \pi_{34} \\
\pi'_{34} = (1 - \epsilon)\pi_{34} + \epsilon \pi_{12}.  
\]

By continuity, for sufficiently small $\epsilon$

\[
I_c(\pi'_{12}, N) > 0 \\
I_c(\pi'_{34}, N) < 0. 
\]  

(57)

Since

\[
\pi'_{12} = \epsilon \pi'_{34} + \frac{1 - \epsilon - \epsilon^2}{2} |1\rangle\langle 1| + \frac{1 - \epsilon - \epsilon^2}{2} |2\rangle\langle 2| + \frac{\epsilon^2}{2} |3\rangle\langle 3| + \frac{\epsilon^2}{2} |4\rangle\langle 4|, 
\]

is a valid decomposition of $\pi'_{12}$, it is readily seen that

\[
I_p(\pi'_{12}, N) > I_c(\pi'_{12}, N) > 0. 
\]

D  The average density operator of random quantum codes

In this section we show how to convert a subclass of the entanglement generation codes described in section 4 into entanglement transmission ones of the same rate $I_c(\rho, N) - \delta$. Then we construct random entanglement transmission codes of the same rate such that the average density operator of the codewords becomes arbitrarily close to $\rho \otimes n$ for large enough blocklength $n$.

Alice is given the system $A'$, entangled with some reference system $A$ she has no access to, in some general state with Schmidt decomposition

\[
|\Psi\rangle_{AA'} = \sum_k \alpha_k |k\rangle_A |\phi_k\rangle_{A'}. 
\]

Her goal is transfer the entanglement with $A$ from her system $A'$ to Bob’s $B$. Notice that the states $|\phi_{km}\rangle^P$ (and hence $|\phi_k\rangle^P$) are mutually orthogonal. Consequently, there is an isometric encoding $E$ defined by $|k\rangle_A \mapsto |\phi_k\rangle^P$ which maps $|\Psi\rangle_{AA'}$ to

\[
|\Upsilon\rangle^A = \sum_k \alpha_k |k\rangle_A |\phi_k\rangle^P, 
\]

(58)

now bearing a strong resemblance to (48). By following through the remaining steps of the proof of theorem 6 it is easily seen that after applying the decoding operation $D$ given by (49) one arrives at (c.f. (50)):

\[
F(|\Psi\rangle, (I \otimes (\mathcal{D} \circ N)^\otimes n \circ E))(|\Psi\rangle\langle \Psi|)) \geq 1 - 4\sqrt{\epsilon} - 4\epsilon. 
\]  

(59)

Choosing $|\Psi\rangle$ to be maximally entangled implies, via (52), an achievable entropy rate of $\frac{1}{n} \log \kappa = I_c(\rho, N) - \delta$.

The set $\mathcal{S} = \{|\phi_k\rangle\}$ is sometimes referred to as the quantum code. A natural quantity to define is the quantum code density operator

\[
\rho(\mathcal{S}) = \frac{1}{\kappa} \sum_{k=1}^{\kappa} |\phi_k\rangle\langle \phi_k|.
\]
i.e. the input to the channel $\mathcal{N}^{\otimes n}$ as seen by someone ignorant of the encoded state. Little can be said about $\rho(S)$ for any particular quantum code $S$ given by our construction. However if we consider random codes, a probabilistic mixture of deterministic codes given by an ensemble $\{p_{\beta}, S_{\beta}\}$, we can make the average code density operator

$$\overline{\rho} = \sum_{\beta} p_{\beta} S_{\beta}$$

be arbitrarily close to $\rho^{\otimes n}$. We shall show this via a double randomization of our original protocol.

1. First recall that for fixed $k$ and fixed set $\{ |\phi_{km}\rangle \}_{m \in [\mu]}$, the $k$th quantum codeword $|\phi_k\rangle$ was chosen from one of $\mu$ Fourier states. If they were all “$\epsilon$-good” quantum codewords, in the sense of equation (40), then picking $|\phi_k\rangle$ at random according to the uniform distribution on the set of Fourier states, for each $k$, would result in a random code with average code density operator

$$\overline{\rho} = \frac{1}{\kappa \mu} \sum_{k,m} |\phi_{km}\rangle \langle \phi_{km}|,$$

According to the proof of lemma 4, a fraction $1 - \sqrt{\epsilon}$ of the Fourier states are $\sqrt{\epsilon}$-good codewords. The random code in which each $|\phi_k\rangle$ is uniformly distributed over these $\sqrt{\epsilon}$-good codewords has an average code density operator $\overline{\rho}$ for which

$$\| \overline{\rho} - \frac{1}{\kappa \mu} \sum_{(k,m) \in [\kappa] \times [\mu]} |\phi_{km}\rangle \langle \phi_{km}| \| \leq 2\sqrt{\epsilon}. \quad (60)$$

At the same time, equation (60) must be modified to account for the codes being $\sqrt{\epsilon}$-good instead of $\epsilon$-good:

$$F(|\Psi\rangle, (I \otimes (D \circ \mathcal{N}^{\otimes n} \circ \mathcal{E}))(|\Psi\rangle \langle \Psi|)) \geq 1 - 6\sqrt{\epsilon} - 2\epsilon. \quad (61)$$

This concludes the first layer of randomization.

2. Backtracking to section 2, by equation (22), choosing the $\{u_{km}\}_{k \in [\kappa'], m \in [\mu']}$ at random according to $p'$, followed by expurgation, results in failure with probability $\epsilon + 10\sqrt{\epsilon}$, thus modifying the fidelity estimate (61) to $1 - 10\sqrt{\epsilon} - 6\sqrt{\epsilon} - 3\epsilon$. This is the second layer of randomization. The average code density matrix before the expurgation is (denoting by $E$ the expectation value over the $\{u_{km}\}$)

$$E \left[ \frac{1}{\kappa' \mu'} \sum_{(k,m) \in [\kappa'] \times [\mu']} |\phi_{km}\rangle \langle \phi_{km}| \right] = \sum_{x^n} p'(x^n) |\phi_{x^n}\rangle \langle \phi_{x^n}|,$$

for which

$$\| \sum_{x^n} p'(x^n) |\phi_{x^n}\rangle \langle \phi_{x^n}| - \rho^{\otimes n} \| \leq 2\epsilon.$$

The expurgation itself has a small effect on the average code density operator:

$$\left\| \frac{1}{\kappa \mu} \sum_{(k,m) \in [\kappa] \times [\mu]} |\phi_{km}\rangle \langle \phi_{km}| - \frac{1}{\kappa' \mu'} \sum_{(k,m) \in [\kappa'] \times [\mu']} |\phi_{km}\rangle \langle \phi_{km}| \right\| \leq 4\sqrt{\epsilon},$$

implying

$$\left\| E \left[ \frac{1}{\kappa \mu} \sum_{(k,m) \in [\kappa] \times [\mu]} |\phi_{km}\rangle \langle \phi_{km}| \right] - E \left[ \frac{1}{\kappa' \mu'} \sum_{(k,m) \in [\kappa'] \times [\mu']} |\phi_{km}\rangle \langle \phi_{km}| \right] \right\| \leq 4\sqrt{\epsilon},$$
and hence
\[ \left\| E \left[ \frac{1}{\kappa \mu} \sum_{(k,m) \in [\kappa] \times [\mu]} |\phi_{km}\rangle \langle \phi_{km}| \right] - \rho^\otimes n \right\| \leq 4 \sqrt{\varepsilon} + 2 \epsilon. \]

By (60),
\[ \left\| \overline{\rho} - E \left[ \frac{1}{\kappa \mu} \sum_{(k,m) \in [\kappa] \times [\mu]} |\phi_{km}\rangle \langle \phi_{km}| \right] \right\| \leq 2 \sqrt{\varepsilon}. \]

where \( \overline{\rho} := E \rho \) is the average code density matrix of our doubly randomized protocol. Hence
\[ \| \overline{\rho} - \rho^\otimes n \| \leq 4 \sqrt{\varepsilon} + 2 \sqrt{\varepsilon} + 2 \epsilon, \]

which concludes the argument.

References


Biography I. Devetak received his Ph.D. in Electrical Engineering from Cornell University in 2002. Since then he has been a post-doctoral researcher at the IBM T.J. Watson Research Center, Yorktown Heights, NY. He will be joining the Electrical Engineering faculty at the University of Southern California in January 2005.