A General Noncommutative Limit Theorem.

T.C. Dorlas
University of Wales, Swansea
Department of Mathematics
Singleton Park, Swansea SA2 8PP, U.K.
Email: T.C.Dorlas@swansea.ac.uk

and

K. Nemmanee
Department of Mathematics, Faculty of Sciences
Chulalongkorn University
Bangkok, Thailand.
Email: tnattana@netserv.chula.ac.th

Abstract. One of the authors recently proved two non-commutative versions of the central limit theorem. Both of these theorems concerned a pair of creation and annihilation operators. The first theorem is a statement about the limit of the expectation value of a function of the scaled difference of the corresponding number operators with respect to an \( n \)-particle state as \( n \) tends to infinity. Here we extend this theorem to the case of an arbitrary (but fixed) number of creation and annihilation operators. We also formulate a conjecture extending the second theorem about the limit of the trace of a certain exponential of operators over the \( n \)-particle subspace as \( n \) tends to infinity.

1. Definitions and results

In [1] one of the authors proved two non-commutative analogues of the central limit theorem. Both of these theorems concerned a pair of independent bosonic creation and annihilation operators \( a_\pm^* \) and \( a_\pm \). The first of these theorems was introduced as a preliminary to the second, more difficult theorem but is of some interest itself. Here we generalise the first theorem to a system of \( r+1 \) bosonic spin operators \( a_0, \ldots, a_r \) and \( a_0^*, \ldots, a_r^* \). They satisfy the usual commutation relations

\[
[a_i, a_j^*] = \delta_{i,j} \quad (i, j = 0, 1, \ldots, r).
\]

(1.1)
We consider a general transformation of these operators to a new set of creation and annihilation operators $c_0, \ldots, c_r$ given by

$$c_k = \sum_{j=0}^{r} O_{k,j} a_j \quad (k = 0, 1, \ldots, r). \quad (1.2)$$

It is easily seen that these operators again satisfy the usual commutation relations if $O = (O_{i,j})_{i,j=0,1,\ldots,r}$ is an orthogonal matrix. We shall assume in addition that the first row of $O$ is given by

$$O_{0,j} = b_0 = \frac{1}{\sqrt{r + 1}}. \quad (1.3)$$

Next we consider the operators

$$\Delta_k = \sum_{j=0}^{r} O_{k,j} N_j \quad (k = 1, 2, \ldots, r) \quad (1.4)$$

where $N_j$ denotes the number operator corresponding to the creation and annihilation operators $a_j^*$ and $a_j$: $N_j = a_j^* a_j$. We denote its eigenstates by $|n_0, n_1, \ldots, n_r\rangle$ so that $N_j |n_0, n_1, \ldots, n_r\rangle = n_j |n_0, n_1, \ldots, n_r\rangle$ and

$$\Delta_k |n_0, n_1, \ldots, n_r\rangle = \sum_{j=0}^{r} O_{i,j} n_j |n_0, n_1, \ldots, n_r\rangle. \quad (1.5)$$

We also introduce the eigenstates of the number operators $c_k^* c_k$. We write these as $|n; m_1, \ldots, m_r\rangle$, where $n$ is the total number of particles. Thus

$$c_k^* c_k |n; m_1, \ldots, m_r\rangle = m_k |n; m_1, \ldots, m_r\rangle \quad (k = 1, 2, \ldots, r) \quad (1.6)$$

but

$$c_0^* c_0 |n; m_1, \ldots, m_r\rangle = \left( n - \sum_{k=1}^{r} m_k \right) |n; m_1, \ldots, m_r\rangle. \quad (1.7)$$

We have singled out the first of the operators $c^* c$ because, as in [1] we want to take the limit $n \to \infty$ keeping $m_1, \ldots, m_r$ fixed. (This is different from [3] where the number of creation and annihilation operators tends to infinity.) We consider general functions of the operators $\Delta_k$, which are the analogues of the operator $\Delta$ in [1]. Our main theorem now reads:

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Theorem. Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous and bounded function. Then the following limit exists and is given by the right-hand side:

$$
\lim_{n \to \infty} \langle n; m_1, \ldots, m_r | f \left( \frac{\Delta_1}{\sqrt{n}}, \ldots, \frac{\Delta_r}{\sqrt{n}} \right) | n; m_1, \ldots, m_r \rangle
= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left( \prod_{k=1}^{r} \frac{1}{n_k!} H_{n_k}^2(u_k) \right) \exp \left[ -\frac{1}{2} \sum_{k=1}^{r} u_k^2 \right]
\times f(b_0 u_1, \ldots, b_0 u_r) \frac{du_1}{\sqrt{2\pi}} \cdots \frac{du_r}{\sqrt{2\pi}},
$$

where $H_n$ denotes the $n$-th Hermite polynomial:

$$H_n(x) = (-1)^n e^{x^2/2} \left( \frac{d}{dx} \right)^n e^{-x^2/2}.$$

We prove this theorem in §2 by demonstrating that the characteristic functions converge. It is well-known that this suffices: see [2]. As in [1] we can now conjecture results about the restricted trace of exponentials of the form $\exp \left[ -\beta \sum_{k=1}^{r} c_k^* c_k + f(\Delta_1/\sqrt{n}, \ldots, \Delta_r/\sqrt{n}) \right]$ using the Trotter product formula:

$$
\lim_{n \to \infty} \text{Trace}_n \exp \left[ -\beta \sum_{k=1}^{r} c_k^* c_k + f \left( \frac{\Delta_1}{\sqrt{n}}, \ldots, \frac{\Delta_r}{\sqrt{n}} \right) \right]
= \lim_{n \to \infty} \lim_{m \to \infty} \sum_{(n_1, \ldots, n_r)} \prod_{k=1}^{m} \left\{ e^{-\frac{\beta}{m} \sum_{k=1}^{r} c_k^* c_k} e^{f(\Delta_1/\sqrt{n}, \ldots, \Delta_r/\sqrt{n})/m} \right\}^m |n; n_1, \ldots, n_r\rangle
\times \langle n; n_1^{(q+1)}, \ldots, n_r^{(q+1)} | e^{-\frac{\beta}{m} \sum_{k=1}^{r} c_k^* c_k} e^{f(\Delta_1/\sqrt{n}, \ldots, \Delta_r/\sqrt{n})/m} |n; n_1^{(q)}, \ldots, n_r^{(q)}\rangle
= \lim_{m \to \infty} \sum_{(n_1^{(1)}, \ldots, n_r^{(1)}), \ldots, (n_1^{(m)}, \ldots, n_r^{(m)})} \prod_{q=1}^{m} e^{-\frac{\beta}{m} \sum_{k=1}^{r} n_k^{(q)}}
\times \prod_{q=1}^{m} \int_{-\infty}^{\infty} \frac{du_1}{\sqrt{2\pi}} \cdots \int_{-\infty}^{\infty} \frac{du_r}{\sqrt{2\pi}} \left( \prod_{k=1}^{r} \frac{1}{n_k^{(q)}!} H_{n_k^{(q)}}(u_k) \right)
\times \left( \prod_{k=1}^{r} \frac{1}{n_k^{(q+1)}!} H_{n_k^{(q+1)}}(u_k) \right) e^{f(b_0 u_1, \ldots, b_0 u_r)/m} \exp \left[ -\frac{1}{2} \sum_{k=1}^{r} u_k^2 \right].
$$

(1.10)
Notice that we have interchanged the two limits without justification here. Also, we used a straightforward extension of the theorem above. To proceed, we introduce the harmonic oscillator hamiltonians $\mathcal{H}_k$ on $L^2(\mathbb{R}^r)$ in each variable $u_k$:

$$\mathcal{H}_k = -\frac{\partial^2}{\partial u_k^2} + \frac{1}{4} u_k^2 - \frac{1}{2}$$

with complete set of eigenfunctions $\psi_m$ given by

$$\psi_m(u_k) = \frac{1}{\sqrt{m!}} \mathcal{H}_m(u_k) \Omega_0(u_k),$$

where $\Omega_0$ is the ground state:

$$\Omega_0(u_k) = (2\pi)^{-1/4} e^{-\frac{u_k^2}{4}}.$$  

Thus,

$$\mathcal{H}_k \Omega_0 = 0 \quad \text{and} \quad \mathcal{H}_k \psi_m = m \psi_m.$$  

Inserting these functions in (1.10) we obtain

$$\begin{align*}
\lim_{n \to \infty} \text{Trace}_n \exp \left[ -\beta \sum_{k=1}^r c_k^* c_k + f \left( \frac{\Delta_1}{\sqrt{n}}, \ldots, \frac{\Delta_r}{\sqrt{n}} \right) \right] &= \\
= \lim_{m \to \infty} \sum_{(n_1^{(1)}, \ldots, n_r^{(1)}): \sum_{k=1}^r n_k^{(1)} \leq n} \ldots \sum_{(n_1^{(m)}, \ldots, n_r^{(m)}): \sum_{k=1}^r n_k^{(m)} \leq n} \prod_{q=1}^m e^{-\frac{\beta}{m} \sum_{k=1}^r n_k^{(q)}} \\
&\times \prod_{q=1}^m \int \ldots \int \left( \prod_{k=1}^r \psi_{n_k}^{(q)}(u_k) \right) \left( \prod_{k=1}^r \psi_{n_k}^{(q+1)}(u_k) \right) e^{f(b_0 u_1, \ldots, b_0 u_r)/m} \, du_1 \ldots du_r \\
&= \lim_{m \to \infty} \sum_{(n_1^{(1)}, \ldots, n_r^{(1)}): \sum_{k=1}^r n_k^{(1)} \leq n} \ldots \sum_{(n_1^{(m)}, \ldots, n_r^{(m)}): \sum_{k=1}^r n_k^{(m)} \leq n} \\
&\times \prod_{q=1}^m \left( \prod_{k=1}^r \psi_{n_k}^{(q+1)}(u_k) \right) e^{-\frac{\beta}{m} \sum_{k=1}^r \mathcal{H}_k} e^{f(b_0 u_1, \ldots, b_0 u_r)/m} \left| \prod_{k=1}^r \psi_{n_k}^{(q)}(u_k) \right| \\
&= \lim_{m \to \infty} \sum_{(n_1, \ldots, n_r): \sum_{k=1}^r n_k \leq n} \\
&\times \left( \prod_{k=1}^r \psi_{n_k}(u_k) \right) \left( e^{-\frac{\beta}{m} \sum_{k=1}^r \mathcal{H}_k} e^{f(b_0 u_1, \ldots, b_0 u_r)/m} \right)^m \left| \prod_{k=1}^r \psi_{n_k}(u_k) \right| \\
&= \text{Trace} \exp \left[ -\beta \sum_{k=1}^r \mathcal{H}_k + f(b_0 u_1, \ldots, b_0 u_r) \right].
\end{align*}$$

(1.15)
Thus we have the following

**Conjecture.** For any continuous bounded function $f : \mathbf{R}^r \to \mathbf{R}$,

$$
\lim_{{n \to \infty}} \text{Trace}_n e^{-\beta \sum_{k=1}^{r} c_k^* c_k + f(\hat{\mathbf{x}}_1, \ldots, \hat{\mathbf{x}}_n)} = \text{Trace} e^{-\beta \sum_{k=1}^{r} \mathcal{H}_k + f(b_0 u_1, \ldots, b_0 u_r)}.
$$

(1.16)

To prove this result along the lines of [1] is feasible but rather involved. This line of proof is carried out in a separate paper [4]. Another approach might be to use the methods of [5].

**REMARK.** As remarked in [1], the theorem above is a non-commutative analogue of the central limit theorem for Bernoulli random variables. In the latter case there are interesting bounds on the asymptotic behaviour for finite $n$, the so-called Berry-Esseen theorem [6,7] (See also [8]). It is an interesting question whether these bounds extend to the present situation, and in particular how they depend on the number of creation and annihilation operators $r$. 
2. Proof of the main theorem

Given \( m_1, \ldots, m_r \) with \( \sum_{i=1}^r m_i \leq n \), let \( m_0 = n - \sum_{i=1}^r m_i \), and denote 

\[
\mathbf{m} = (m_0, m_1, \ldots, m_r),
\]

\[
|\mathbf{m}| = m_0 + m_1 + \ldots + m_r,
\]

and

\[
\binom{n}{\mathbf{m}} = \frac{n!}{m_0! m_1! \ldots m_r!}.
\]

Then, for arbitrary real numbers \( \alpha_0, \alpha_1, \ldots, \alpha_r \), we compute the following generating function:

\[
\sum_{\mathbf{m}:|\mathbf{m}|=n} \left( \binom{n}{\mathbf{m}} \right)^{1/2} \prod_{p=0}^{r} \alpha_p^{m_p} |n; m_1, \ldots, m_r) = \frac{1}{\sqrt{n!}} \left[ \sum_{k=0}^{r} \alpha_k \sum_{j=0}^{r} O_{k,j} a_j^* \right]^n |0\rangle
\]

\[
= \frac{1}{\sqrt{n!}} \left[ \sum_{j=0}^{r} A_j(\alpha) a_j^* \right]^n |0\rangle
\]

\[
= \sum_{\mathbf{m}:|\mathbf{m}|=n} \left( \binom{n}{\mathbf{m}} \right)^{1/2} \prod_{j=0}^{r} A_j(\alpha)^{n_j} |n_0, n_1, \ldots, n_r),
\]

where we have denoted

\[
A_j(\alpha) = \sum_{k=0}^{r} O_{k,j} \alpha_k.
\]

Using the identity (2.1) we have

\[
\sum_{\mathbf{m}:|\mathbf{m}|=n} \sum_{\mathbf{m}':|\mathbf{m}'|=n} \left( \binom{n}{\mathbf{m}} \right)^{1/2} \left( \binom{n}{\mathbf{m}'} \right)^{1/2} \prod_{k=0}^{r} \alpha_k^{m_k} \beta_k^{m_k'}
\]

\[
\times \langle n; m_1', \ldots, m_r' | \exp \left[ i \sum_{k=1}^{r} t_k \Delta_k \right] | n; m_1, \ldots, m_r \rangle = \sum_{\mathbf{m}':|\mathbf{m}'|=n} \prod_{j=0}^{r} A_j(\alpha)^{n_j} A_j(\beta)^{n_j} \exp \left[ i \sum_{k=1}^{r} t_k \sum_{j=0}^{r} O_{k,j} a_j \right]
\]

\[
= \left\{ \sum_{j=0}^{r} A_j(\alpha) A_j(\beta) \exp \left[ i \sum_{k=1}^{r} t_k O_{k,j} \right] \right\}^n.
\]
We now use the following two lemmas:

**Lemma 1** Let \( A_j(\alpha) = \sum_{k=0}^r O_{k,j} \alpha_k \) then the coefficient of \( \prod_{k=0}^r \alpha_k^{n_k} \beta_k^{n_k} \) in the expansion of

\[
\left\{ \sum_{j=0}^r A_j(\alpha) A_j(\beta) \exp \left[ i \sum_{k=1}^r t_k O_{k,j} \right] \right\}^n
\]

is:

\[
\sum_{\{n(p,q)\}_{p,q=1}^{r} \star} \frac{n!}{\prod_{q=1}^r \left( n_q - \sum_{p=1}^r n(p,q) \right)!} \prod_{p=1}^r D_{p,q}(t)^{n(p,q)} \\
\times \prod_{p=1}^r \frac{1}{\Pi_{p=1}^r n(p,q)!} \prod_{p=1}^r \frac{1}{n(p,q)!} \prod_{p=1}^r D_{p}(t)^{2n_p - \sum_{q=1}^r (n(p,q) + n(q,p))},
\]

(2.4)

where the star denotes the following restrictions on the sum:

\[
\sum_{q=1}^r n(p,q) \leq n_p \text{ and } \sum_{p=1}^r n(p,q) \leq n_q,
\]

and where

\[
D_{p,q}(t) = \sum_{j=0}^r O_{p,j} O_{q,j} \exp \left[ i \sum_{k=1}^r O_{k,j} t_k \right]
\]

(2.5)

and \( D_{p}(t) = D_{p,0}(t) \).

**Lemma 2.** The following hold:
1. \( \lim_{n \to \infty} D_p \left( \frac{t}{\sqrt{n}} \right) = 0 \) (\( p = 1, 2, \ldots, r \)), and \( \lim_{n \to \infty} D_0 \left( \frac{t}{\sqrt{n}} \right) = 1 \);
2. \( \lim_{n \to \infty} \sqrt{n} D_p \left( \frac{t}{\sqrt{n}} \right) = i t_p b_0 \) (\( p = 1, 2, \ldots, r \));
3. \( \lim_{n \to \infty} D_{p,p} \left( \frac{t}{\sqrt{n}} \right) = 1 \) and \( \lim_{n \to \infty} D_{p,q} \left( \frac{t}{\sqrt{n}} \right) = 0 \), (\( 1 \leq p, q \leq r \));
4. \( \lim_{n \to \infty} D_0 \left( \frac{t}{\sqrt{n}} \right)^n = \exp \left[ -\frac{1}{2} b_0^2 \sum_{k=1}^r t_k^2 \right] \).
By Lemma 1,

\[
\lim_{n \to \infty} \langle n; n_1, \ldots, n_r \rangle \exp \left[ \frac{i}{\sqrt{n}} \sum_{k=1}^{r} t_k \Delta_k \right] |n; n_1, \ldots, n_r\rangle =
\]

\[
= \lim_{n \to \infty} \sum_{\{p(q)\}^{r}_{q=1}} \frac{(n - \sum_{p=1}^{r} n_p)!}{(n + \sum_{p=1}^{r} n_p - 2 \sum_{p=1}^{r} n_p)!} \prod_{p=1}^{r} n_p!
\]

\[
\times D_0 \left( \frac{t}{\sqrt{n}} \right) \prod_{p=1}^{r} \left( \frac{n + \sum_{p=1}^{r} n(p,q) - 2 \sum_{p=1}^{r} n_p}{\sqrt{n}} \right) D_{p,q} \left( \frac{t}{\sqrt{n}} \right) \prod_{p,q=1}^{r} \frac{1}{D_{p,q} \left( \frac{t}{\sqrt{n}} \right)}^{2n_p - \sum_{q=1}^{r} (n(p,q) + n(q,p))}. \tag{2.6}
\]

Now, because of Lemma 2.2 we must compensate each factor \( D_p \left( \frac{t}{\sqrt{n}} \right) \) by a factor \( \sqrt{n} \). But,

\[
\frac{(n - \sum_{p=1}^{r} n_p)!}{(n + \sum_{p=1}^{r} n_p - 2 \sum_{p=1}^{r} n_p)!} \sim n \sum_{p=1}^{r} (n_p - \sum_{q=1}^{r} n(p,q)), \tag{2.7}
\]

which is just the number of factors \( \sqrt{n} \) needed to cover all the factors \( D_p \left( \frac{t}{\sqrt{n}} \right) \).

By Lemma 2.3 this means that there cannot be any factors \( D_{p,q} \left( \frac{t}{\sqrt{n}} \right) \) with \( p \neq q \), i.e. \( n(p,q) = 0 \) for \( p \neq q, p, q \geq 1 \). It now follows that

\[
\lim_{n \to \infty} \langle n; n_1, \ldots, n_r \rangle \exp \left[ \frac{i}{\sqrt{n}} \sum_{k=1}^{r} t_k \Delta_k \right] |n; n_1, \ldots, n_r\rangle =
\]

\[
= \sum_{\{n(p)\}^{r}_{p=1}} \prod_{p=1}^{r} \left( \frac{n_p}{n(p,p)} \right) \frac{1}{\prod_{p=1}^{r} (n_p - n(p,p))!}
\]

\[
\times \exp \left[ \frac{1}{2} \sum_{p=1}^{r} t_k^2 \right] \prod_{p=1}^{r} (ib_0 t_p)^{2(m_p - n(p,p))} \tag{2.8}
\]

\[
= \sum_{m_1=0}^{n_1} \cdots \sum_{m_r=0}^{n_r} \prod_{p=1}^{r} \left( \frac{n_p}{m_p} \right) \left( -1 \right)^{m_p} (ib_0 t_p)^{2m_r} e^{-\frac{1}{2} m_p t_p^2}
\]

\[
= \prod_{p=1}^{r} \left\{ \frac{1}{n_p} \int_{-\infty}^{\infty} H_{n_p}(x) e^{ib_0 t_p x_p - \frac{1}{2} x_p^2} \frac{dx_p}{\sqrt{2\pi}} \right\}. \tag*{8}
\]
The last identity was proved in [1]. We have thus proved the convergence of the characteristic function. It is well-known that this implies the convergence for arbitrary continuous functions $f$ as in the theorem.

3. Proof of Lemma 1.

First observe that

$$
\begin{align*}
\sum_{j=0}^{r} A_j(\alpha) A_j(\beta) \exp \left[ i \sum_{k=1}^{r} O_{k,j} t_k \right] &= \\
= &\sum_{j=0}^{r} \sum_{p=0}^{r} O_{p,j} \alpha_p \sum_{q=0}^{r} O_{q,j} \beta_q \exp \left[ i \sum_{k=1}^{r} O_{k,j} t_k \right] \\
= &\sum_{j=0}^{r} O_{j,j}^2 \exp \left[ i \sum_{k=1}^{r} O_{k,j} t_k \right] \alpha_0 \beta_0 \\
&+ \sum_{p=1}^{r} \sum_{j=0}^{r} O_{0,j} O_{p,j} \exp \left[ i \sum_{k=1}^{r} O_{k,j} t_k \right] (\alpha_p \beta_0 + \alpha_0 \beta_p) \\
&+ \sum_{p=1}^{r} \sum_{q=1}^{r} O_{p,j} O_{q,j} \exp \left[ i \sum_{k=1}^{r} O_{k,j} t_k \right] \alpha_p \beta_q \\
= &D_0(t) \alpha_0 \beta_0 + \sum_{p=1}^{r} D_p(t) (\alpha_p \beta_0 + \alpha_0 \beta_p) + \sum_{p,q=1}^{r} D_{p,q}(t) \alpha_p \beta_q.
\end{align*}
$$

(3.1)

It now follows that

$$
\left\{ \sum_{j=0}^{r} A_j(\alpha) A_j(\beta) \exp \left[ i \sum_{k=1}^{r} O_{k,j} t_k \right] \right\}^n = \\
= \sum_{(n(p,q))_{p,q=0}^{r}} \sum_{\sum_{p,q=0}^{r} n(p,q)=n} \frac{n!}{\prod_{p,q=0}^{r} n(p,q)!} \prod_{p,q=0}^{r} D_{p,q}(t)^{n(p,q)} (\alpha_p \beta_q)^{n(p,q)}.
$$

(3.2)

In order to extract the terms with $\prod_{p=0}^{r} (\alpha_p \beta_p)^{n_p}$, that is, $\sum_{p=0}^{r} n(p,q) = n_p$ and $\sum_{p=0}^{r} n(p,q) = n_q$, notice that these are $2(r+1)$ conditions on $(r+1)^2$ variables, but with one redundancy: if we sum the first set of relations over $p$ or the second set over $q$ we must obtain the same number $\sum_{p=0}^{r} n_p = n$. This leaves $r^2$ free
variables which we take to be \( n(p, q) \) with \( 1 \leq p, q \leq r \). Then

\[
\begin{align*}
n(0, p) &= n_p - \sum_{q=1}^{r} n(q, p), \\
n(p, 0) &= n_p - \sum_{q=1}^{r} n(p, q), \\
n(0, 0) &= n + \sum_{p, q=1}^{r} n(p, q) - 2 \sum_{p=1}^{r} n_p.
\end{align*}
\] (3.3)

This yields the expression quoted in the lemma.


Notice first that

\[
\exp \left[ i \frac{1}{\sqrt{n}} \sum_{k=1}^{r} O_{k,j} t_k \right] \to 1 \text{ as } n \to \infty. \tag{4.1}
\]

This shows that

\[
D_0 \left( \frac{t}{\sqrt{n}} \right) = b_0^2 \sum_{j=0}^{r} \exp \left[ i \frac{1}{\sqrt{n}} \sum_{k=1}^{r} O_{k,j} t_k \right] \to b_0^2 (r + 1) = 1. \tag{4.2}
\]

and more generally,

\[
D_{p,q} \left( \frac{t}{\sqrt{n}} \right) \to \sum_{j=0}^{r} O_{p,j} O_{q,j} = \delta_{p,q}. \tag{4.3}
\]

Multiplying by \( \sqrt{n} \) we have for \( p \geq 1 \),

\[
\lim_{n \to \infty} \sqrt{n} D_p \left( \frac{t}{\sqrt{n}} \right) = \\
= \lim_{n \to \infty} \sum_{j=0}^{r} O_{0,j} O_{p,j} \sqrt{n} \left( \exp \left[ i \frac{1}{\sqrt{n}} \sum_{k=1}^{r} O_{k,j} t_k \right] - 1 \right) \tag{4.4}
\]

\[
= \sum_{j=0}^{r} b_0 O_{p,j} \left\{ i \sum_{k=1}^{r} O_{k,j} t_k \right\} = i b_0 t_p.
\]
Finally,

\[
\lim_{n \to \infty} D_0 \left( \frac{t}{\sqrt{n}} \right)^n = \\
= \lim_{n \to \infty} \left\{ \sum_{j=0}^{r} b_0^2 \exp \left[ i \frac{1}{\sqrt{n}} \sum_{k=1}^{r} O_{k,j} t_k \right] \right\}^n \\
= \lim_{n \to \infty} \left\{ 1 - \frac{1}{2} b_0^2 \sum_{j=0}^{r} \left( \sum_{k=1}^{r} O_{k,j} t_k \right)^2 + {\mathcal O} \left( \frac{1}{n^{1/2}} \right) \right\}^n \\
= \exp \left[ -\frac{1}{2} b_0^2 \sum_{j=0}^{r} \left( \sum_{k=1}^{r} O_{k,j} t_k \right)^2 \right] = \exp \left[ -\frac{1}{2} b_0^2 \sum_{k=1}^{r} t_k^2 \right].
\]

(4.5)

as stated.

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