A Combinatorial Identity

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Abstract

We prove an interesting combinatorial identity, which we came across in counting contributions from forest graphs, but may be of more general interest.

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1 The main result.

The aim of this note is to prove the following combinatorial identity, which we came across in evaluating contributions from forest graphs to a cluster expansion for classical gas correlation functions (see [1]).

Theorem 1 Given \( m, p \in \mathbb{N} \) with \( 1 \leq p \leq m \), and a collection \( (x_i)_{i=1}^m \) of (complex) numbers \( x_i \in \mathbb{C} \), the following identity holds

\[
\sum_{\{I_1, \ldots, I_p\} \in \Pi_p(\{1, \ldots, m\})} \prod_{j=1}^p \left( \sum_{i \in I_j} x_i \right)^{|I_j|-1} = \left( \frac{m-1}{p-1} \right) \left( \sum_{i=1}^m x_i \right)^{m-p},
\]

(1)

where the sum is over the set \( \Pi_p(\{1, \ldots, m\}) \) of all partitions \( (I_j)_{j=1}^p \) of \( \{1, \ldots, m\} \) into \( p \) non-empty subsets.
2 Proof.

The proof is done by induction on $m$ and $p$. For $m = p$ both sides are equal to 1, and for $p = 1$ both sides are equal to $(\sum_{i=1}^{m} x_i)^{m-1}$. Now assume that (1) holds true for a given $m \geq 1$ and all $p \leq m$. Let $\mathcal{L}_m, \mathcal{R}_m$ denote the left- and right-hand side of (1) respectively. We may assume that $1 \in I_1$ and expand the factor $(\sum_{i \in I_1} x_i)^{|I_1|^{-1}}$ in powers of $l_1$:

$$
\left( \sum_{i \in I_1} x_i \right)^{\frac{|I_1|}{1-1}} = \sum_{n=0}^{\frac{|I_1|-1}{n}} \left( \sum_{i \in I_1} x_i \right)^{\frac{|I_1|}{1-1-n}}.
$$

Inserting this into $\mathcal{L}_{m+1}$ and denoting $\tilde{I}_1 = I_1 \setminus \{1\}$, we have

$$
\mathcal{L}_{m+1} = \sum_{n=0}^{m+1-p} l_1^n \sum_{\{\tilde{I}_1, \ldots, I_p\} \in \Pi_p(\{1, \ldots, m+1\})} \left( \sum_{i \in \tilde{I}_1} x_i \right)^{\frac{|\tilde{I}_1|}{1-n}} \prod_{j=2}^{p} \left( \sum_{i \in I_j} x_i \right)^{\frac{|I_j|}{1-n}}.
$$

If $n = 0$, we separate out the term $\tilde{I}_1 = \emptyset$, for which $\{I_2, \ldots, I_p\} \in \Pi_{p-1}(\{2, \ldots, m+1\})$. If $\tilde{I}_1 \neq \emptyset$ then $\{\tilde{I}_1, I_2, \ldots, I_p\} \in \Pi_p(\{2, \ldots, m+1\})$. Conversely, given a partition $\{\tilde{I}_1, I_2, \ldots, I_p\} \in \Pi_p(\{2, \ldots, m+1\})$ we obtain a unique partition of $\{1, \ldots, m+1\}$ by adding 1 to any of the sets $\tilde{I}_j$ with $j \in \{1, \ldots, p\}$. We can therefore write

$$
\mathcal{L}_{m+1} = \sum_{\{I_2, \ldots, I_p\} \in \Pi_{p-1}(\{2, \ldots, m+1\})} \prod_{j=2}^{p} \left( \sum_{i \in I_j} x_i \right)^{\frac{|I_j|}{1-n}}
$$

$$+ \sum_{n=0}^{m+1-p} l_1^n \sum_{j_1=1}^{p} \sum_{\{I_1, \ldots, I_p\} \in \Pi_p(\{2, \ldots, m+1\})} \left( \sum_{i \in I_j} x_i \right)^{\frac{|I_j|}{1-n}} \prod_{j=2}^{p} \left( \sum_{i \in I_j} x_i \right)^{\frac{|I_j|}{1-n}}
$$

Expanding the quantity $\mathcal{R}_{m+1}$ on the right-hand side of (1) in powers of
it follows that it suffices to prove the identities

\[
\sum_{\{I_2, \ldots, I_p\} \in \Pi_{p-1}(\{2, \ldots, m+1\})} \prod_{j=2}^{p} \left( \sum_{i \in I_j} |I_j|^{-1} \right) \\
+ \sum_{j_1=1}^{p} \sum_{\{I_1, \ldots, I_p\} \in \Pi_p(\{2, \ldots, m+1\}) \setminus \{I_1\} \geq n} \left( \sum_{i \in I_1} |I_1|^{-1} \right) \prod_{j=1}^{p} \left( \sum_{i \in I_j} |I_j|^{-1} \right) \\
= \left( \frac{m}{p-1} \right) \left( \sum_{i=2}^{m+1} x_i \right)^{m+1-p}
\]

for \(n = 0\), and

\[
\sum_{j_1=1}^{p} \sum_{\{I_1, \ldots, I_p\} \in \Pi_p(\{1, \ldots, m\}) \setminus \{I_1\} \geq n} \left( |I_1|^{-1} \right) \left( \sum_{i \in I_1} |I_1|^{-1} \right) \prod_{j=1}^{p} \left( \sum_{i \in I_j} |I_j|^{-1} \right) \\
= \left( \frac{m}{p-1} \right) \left( \sum_{i=1}^{m} x_i \right)^{m+1-p-n}
\]

for all \(1 \leq n \leq m+1 - p\). Replacing \(\{2, \ldots, m+1\}\) by \(\{1, \ldots, m\}\) we prove the equivalent

\[
\sum_{\{I_1, \ldots, I_{p-1}\} \in \Pi_{p-1}(\{1, \ldots, m\})} \prod_{j=1}^{p-1} \left( \sum_{i \in I_j} |I_j|^{-1} \right) \\
+ \sum_{j_1=1}^{p} \sum_{\{I_1, \ldots, I_p\} \in \Pi_p(\{1, \ldots, m\}) \setminus \{I_1\} \geq n} \left( \sum_{i \in I_1} |I_1|^{-1} \right) \prod_{j=1}^{p} \left( \sum_{i \in I_j} |I_j|^{-1} \right) \\
= \left( \frac{m}{p-1} \right) \left( \sum_{i=1}^{m} x_i \right)^{m+1-p}
\]

for \(n = 0\), and

\[
\sum_{j_1=1}^{p} \sum_{\{I_1, \ldots, I_p\} \in \Pi_p(\{1, \ldots, m\}) \setminus \{I_1\} \geq n} \left( |I_1|^{-1} \right) \left( \sum_{i \in I_1} |I_1|^{-1} \right) \prod_{j=1}^{p} \left( \sum_{i \in I_j} |I_j|^{-1} \right) \\
= \left( \frac{m}{p-1} \right) \left( \sum_{i=1}^{m} x_i \right)^{m+1-p-n}
\]
for all $1 \leq n \leq m + 1 - p$.

We start with the case of $n = 0$. By the induction hypothesis, the first term equals
\[
\sum_{\{I_1, \ldots, I_{p-1}\} \in \Pi_{p-1}(\{1, \ldots, m\})} \prod_{j=1}^{p-1} \left( \sum_{i \in I_j} x_i \right)^{|I_j|-1} = (m - 1) \left( \sum_{i=1}^{m} x_i \right)^{m+1-p}.
\]

Then second term can be written as
\[
\sum_{j_1=1}^{p} \sum_{\{I_1, \ldots, I_{p}\} \in \Pi_{p}(\{1, \ldots, m\})} \prod_{j=1}^{p} \left( \sum_{i \in I_j} x_i \right)^{|I_j|-1} = \sum_{j_1=1}^{p} \sum_{\{I_1, \ldots, I_{p}\} \in \Pi_{p}(\{1, \ldots, m\})} \prod_{j \neq j_1}^{p} \left( \sum_{i \in I_j} x_i \right)^{|I_j|-1} = (m - 1) \left( \sum_{i=1}^{m} x_i \right)^{m+1-p}
\]
again by the induction hypothesis. Adding the two contributions for $n = 0$ yields (2).

For the case $n = 1$, we have similarly,
\[
\sum_{j_1=1}^{p} \sum_{\{I_1, \ldots, I_{p}\} \in \Pi_{p}(\{1, \ldots, m\})} |\tilde{I}_{j_1}| \left( \sum_{i \in \tilde{I}_{j_1}} x_i \right)^{|\tilde{I}_{j_1}|-1} \prod_{j \neq j_1}^{p} \left( \sum_{i \in I_j} x_i \right)^{|I_j|-1} = \sum_{j_1=1}^{p} |\tilde{I}_{j_1}| \sum_{\{I_1, \ldots, I_{p}\} \in \Pi_{p}(\{1, \ldots, m\})} \prod_{j \neq j_1}^{p} \left( \sum_{i \in I_j} x_i \right)^{|I_j|-1} = m \left( \sum_{i=1}^{m} x_i \right)^{m-p} = m \left( \sum_{i=1}^{m} x_i \right)^{m-p}.
\]

Next we prove the case where $2 \leq n \leq m + 1 - p$. The key idea is to apply the derivation operator $\sum_{k=1}^{m} \frac{\partial^{n-1}}{\partial x_k^{n-1}}$ to the l.h.s. of (1). This gives
\[
\sum_{k=1}^{m} \frac{\partial^{n-1}}{\partial x_k^{n-1}} \sum_{\{I_1, \ldots, I_{p}\} \in \Pi_{p}(\{1, \ldots, m\})} \prod_{j=1}^{p} \left( \sum_{i \in I_j} x_i \right)^{|I_j|-1} = \sum_{j'=1}^{p} \sum_{\{I_1, \ldots, I_{p}\} \in \Pi_{p}(\{1, \ldots, m\})} \prod_{j \neq j'}^{p} \left( \sum_{i \in I_j} x_i \right)^{|I_j|-1} \frac{\partial^{n-1}}{\partial x_k^{n-1}} \sum_{k \in I_{j'}} \left( \sum_{i \in I_{j'}} x_i \right)^{|I_{j'}|-1}.
\]
The \( j' \) term on the right-hand side is obviously equal to zero unless \(|I_{j'}| \geq n\). In that case,

\[
\sum_{k \in I_{j'}} \frac{\partial^{n-1}}{\partial x_k^{n-1}} \left( \sum_{i \in I_{j'}} x_i \right)^{|I_{j'}|-1} = |I_{j'}| \prod_{r=1}^{n-1} (|I_{j'}| - r) \left( \sum_{i \in I_{j'}} x_i \right)^{|I_{j'}|-n},
\]

independently of \( k \in I_{j'} \). Hence we obtain

\[
\sum_{k=1}^{m} \frac{\partial^{n-1}}{\partial x_k^{n-1}} \sum_{\{I_1, \ldots, I_p\} \in \Pi_p(\{1, \ldots, m\})} \prod_{j=1}^{p} \left( \sum_{i \in I_j} x_i \right)^{|I_j|-1}
\]

\[
= \sum_{j'=1}^{p} \sum_{\{I_1, \ldots, I_p\} \in \Pi_p(\{1, \ldots, m\}) \atop |I_{j'}| \geq n} \prod_{r=0}^{n-1} (|I_{j'}| - r) \left( \sum_{i \in I_{j'}} x_i \right)^{|I_{j'}|-n} \prod_{j=1}^{p} \left( \sum_{i \in I_j} x_i \right)^{|I_j|-1}
\]

\[
= n! \left[ \sum_{j'=1}^{p} \sum_{\{I_1, \ldots, I_p\} \in \Pi_p(\{1, \ldots, m\}) \atop |I_{j'}| \geq n} \left( \sum_{i \in I_{j'}} x_i \right)^{|I_{j'}|-n} \prod_{j=1}^{p} \left( \sum_{i \in I_j} x_i \right)^{|I_j|-1} \right].
\]

Note that the quantity between square brackets is nothing but the left-hand side of (3). On the other hand, applying the derivation operator \( \sum_{k=1}^{m} \frac{\partial^{n-1}}{\partial x_k^{n-1}} \) to the right-hand side of (1) yields

\[
\binom{m-1}{p-1} \sum_{k=1}^{m} \frac{\partial^{n-1}}{\partial x_k^{n-1}} \left( \sum_{i=1}^{m} x_i \right)^{m-p}
\]

\[
= m \binom{m-1}{p-1} \prod_{r=0}^{n-2} (m - p - r) \left( \sum_{i=1}^{m} x_i \right)^{m+1-p-n}
\]

\[
= n! \binom{m}{p} \binom{m+1-p-n}{n} \left( \sum_{i=1}^{m} x_i \right)^{m+1-p-n},
\]

which is just \( n! \) times the right-hand side of (3). This concludes the proof of (3), and hence also the proof of Theorem 1.

\[\blacksquare\]

**References**