Lecture 9 - Angular Momentum Transport

- Topics in today's lecture:
  - Angular momentum transport
    - Magnetic braking
    - Stellar winds
    - Viscous Disk

Angular Momentum Transport

- Three processes have been proposed:

  1. Magnetic braking
  2. Stellar wind
  3. Gas viscosity

- Each theory attempts to explain why the Sun now contains the majority of matter in the Solar System, and little of its AM.

1. Magnetic braking

- Material in the inner part of the disk must be ionized. Fields are frozen if *the plasma beta value* is large:

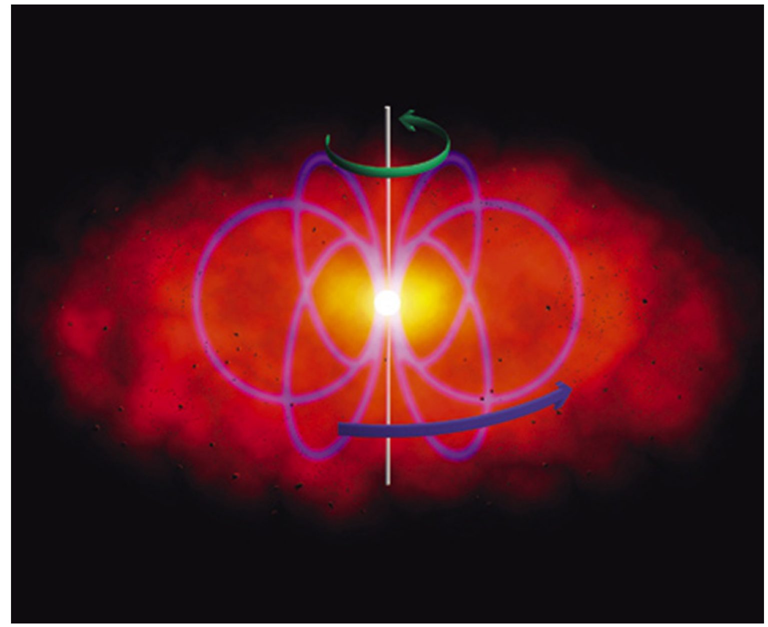
$$\beta = \frac{P_G}{P_B}$$

where $P_G = nkT + \frac{1}{2} \rho v^2$ and $P_B = B^2 / 8\pi$.

- If $P_G > P_B \Rightarrow \beta > 1$ (high beta plasma; fields frozen into plasma).

- If $P_G < P_B \Rightarrow \beta < 1$ (low beta plasma; fields slip through plasma).

- For $\beta > 1$, the magnetic field of the Sun will experience resistance as it drags through the plasma, which will *slow down* the rotation of the Sun.



1. Magnetic braking (cont.)

- Star's magnetic field co-rotates the wind out to the *Alfven Radius*.
- We know that the gas pressure of the wind is

$$P_G = nkT + \tfrac{1}{2}\rho v^2 = nkT + 1/2\ nm\ v^2$$

- For $v > 500\ km\ s^{-1}$, $1/2\ nmv^2 > nkT \Rightarrow P_G \sim 1/2\ nm\ v^2$.
- Equating the gas and magnetic pressures: $\frac{B^2}{8\pi} = 1/2nmv^2$
- As $B = M_B / r^3$ (M_B is the magnetic dipole moment) gives

$$\frac{M_B^2}{8\pi r^6} = 1/2nmv^2 \qquad \textit{Eqn. 1}$$

1. Magnetic Braking (cont.)

- Now from conservation of mass $$\frac{dM}{dt} = \dot{M} = 4\pi r^2 \rho v = 4\pi r^2 nmv$$

$$=> nm = \frac{\dot{M}}{4\pi r^2 v}$$

- Substituting this into *Eqn. 1* and rearranging gives

$$\boxed{r_A = \left(\frac{M_B^2}{\dot{M}v}\right)^{1/4}}$$ *Alfven Radius*

- Solar wind co-roates with the magnetic field for $r<r_A$. For the Sun, $r_A \sim 5\text{-}50R_{sun}$.

2. Stellar wind braking

- Stellar wind literally *blows* the angular momentum problem *away*.

- The total AM is $L = M_{sun}\, vr$

  as $v = r\omega$ (ω is the angular velocity) => $L = M_{sun}\, r^2\omega$

- The rate of loss of AM is $\boxed{\dfrac{dL}{dt} = \dfrac{dM_{sun}}{dt} r^2\omega}$ *Eqn. 2*

- Particles lost from the star also carry away angular momentum. Given an initial mass, rotation rate, and radius, we can thus calculate the rate of AM loss.

- We know that solar-type star have strong *T-Tauri* winds in their youth, which ultimately clear away residual gas and dust in the disk.

- This theory also explains why more massive stars rotate faster: they do not have proper winds like the Sun, and therefore are not as good at losing angular momentum.

2. Stellar wind braking (cont.)

- *Eqn. 2* only applies to equatorial regions. The correct global form for distance beyond the Alfven radius is:

$$\frac{dL}{dt} = \frac{2}{3}\dot{M}r_A^2\omega \qquad \textit{Eqn. 3}$$

  where the 2/3 accounts for the fact that higher altitudes contribute less to AM losses.

- Setting $dL/dt \sim L/\tau_D$ gives $$\frac{L}{\tau_D} = \frac{2}{3}\dot{M}r_A^2\omega$$

- As the AM of a spherically symmetric star is $L = 2/5\ Mr^2\omega$ we can then find an estimate for the spin-down time

$$\boxed{\tau_D = \frac{3}{5}\frac{Mr^2}{\dot{M}r_A^2}}$$

- This gives the spin-down time in terms of the mass loss rate and the Alfven radius. This predicts a spin down time of $<10^{10}$ years for the Sun.

2. Stellar wind braking (cont.)

- But how does the angular velocity change with time depend on the mass loss rate?

- We know $L = 2/5Mr^2\omega$. This gives

$$\frac{dL}{dt} = 2/5r^2\left(\omega\frac{dM}{dt} + M\frac{d\omega}{dt}\right)$$

- Equating this with *Eqn. 3*:

$$2/3\frac{dM}{dt}r_A^2\omega = 2/5r^2\left(\omega\frac{dM}{dt} + m\frac{d\omega}{dt}\right)$$

- Rearranging gives

$$3/5r^2M\frac{d\omega}{dt} = (r_A^2 - 3/5r^2)\omega\frac{dM}{dt}$$

- Since $3/5r^2 << r_A^2$, we can write:

$$r_A^2\omega\frac{dM}{dt} = 3/5Mr^2\frac{d\omega}{dt}$$

2. Stellar wind braking (cont.)

- Integrating then gives:

$$\int_{\omega_0}^{\omega} \frac{d\omega}{\omega} = \frac{5r_A^2}{3r^2} \int_{M_0}^{M} \frac{dM}{M}$$

$$\ln\left(\frac{\omega}{\omega_0}\right) = \frac{5r_A^2}{3r^2} \ln\left(\frac{M}{M_0}\right)$$

- Therefore,

$$\boxed{\omega = \omega_0 \left(\frac{M}{M_0}\right)^{5r_A^2/3r^2}}$$

where M and M_0 are the final and initial masses, and ω and ω_0 are the final and initial AM.

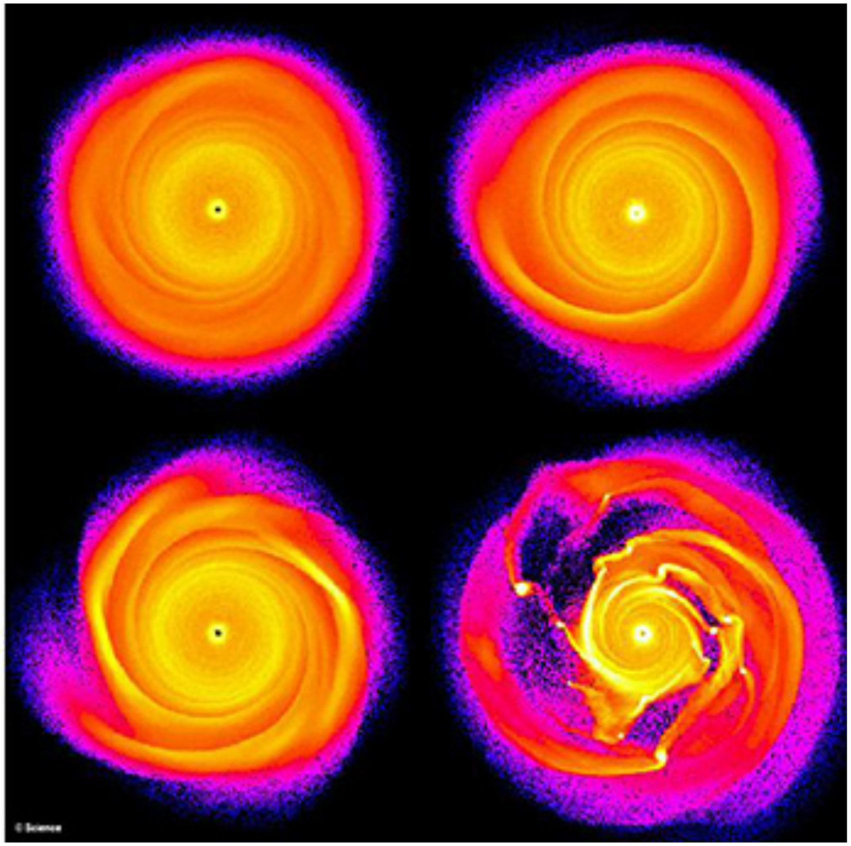
- For a particular mass loss rate, $M = M_0 - \frac{dM}{dt}\tau_M$, where $\tau\omega_m$ is the mass duration of mass loss.

3. Gas viscosity

- As well as the AM *transfer* of angular momentum from star to the disk, must also transport AM *within* the disk.

- One way we can do this is by convection: numerical models shows turbulent eddies can develop, which can transport AM.

- Gravitational interactions between clumps of material in the disk can cause angular momentum to be transported *outwards*, while gas and dust move *inwards*.



3. AM Transport in disk?

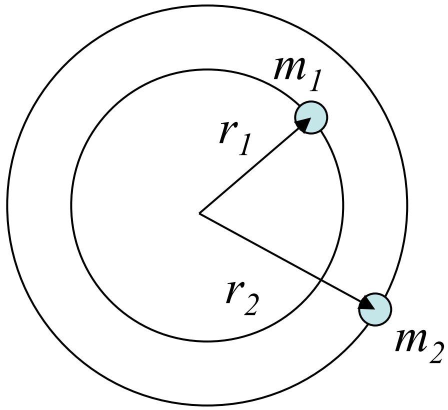
- The total energy associated with an orbit is: $E = KE + PE = 1/2\ mv^2 - GMm/r$.

- For a circular orbit, $mv^2/r = GMm/r^2 \Rightarrow v^2 = GM/r$.

- Therefore

$$E = 1/2m\frac{GM}{r} - \frac{GMm}{r}$$

$$\Rightarrow E = -\frac{GMm}{2r}$$



- The angular momentum is given by $L = mvr = m\sqrt{\frac{GM}{r}}r$

$$\Rightarrow L = m\sqrt{GM}\frac{r}{\sqrt{r}}$$

- Lynden-Bell and Pringle (1974) considered how AM could be transported in a disk. Take two bodies in circular orbits with masses m_1 and m_2. The total energy and AM of the system are:

$$E = -\frac{GM}{2}\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right) \qquad \textit{Eqn. 4}$$

$$L = \sqrt{GM}\,(m_1\frac{r_1}{\sqrt{r_1}} + m_2\frac{r_2}{\sqrt{r_2}}) \qquad \textit{Eqn. 5}$$

3. AM transport in disk

- Now suppose the orbits are perturbed slightly, while conserving *L, Eqns.4* and *5* become:

$$\Delta E = \frac{GM}{2}\left(\frac{m_1}{r_1^2}\delta r_1 + \frac{m_2}{r_2^2}\delta r_2\right) \qquad \textit{Eqn.6}$$

$$\Delta L = \sqrt{GM}\,(m_1\sqrt{\delta r_1} + m_2\sqrt{\delta r_2}) = 0 \qquad \textit{Eqn.7}$$

- *Eqn. 7* implies that

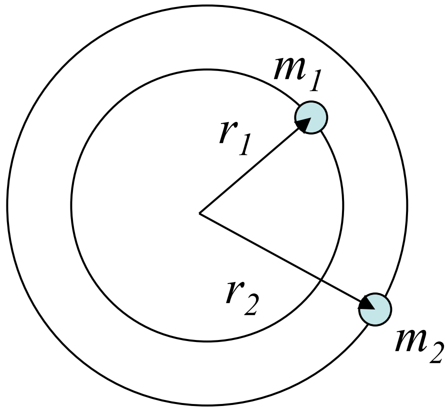
$$\delta r_2 = -\frac{m_1}{m_2}\sqrt{\frac{r_2}{r_1}}\delta r_1$$

- Substituting this into *Eqn. 6* gives

$$\Delta E = \frac{GM}{2}\left(\frac{m_1}{r_1^2}\delta r_1 + \frac{m_2}{r_2^2}\left(-\frac{m_1}{m_2}\sqrt{\frac{r_2}{r_1}}\delta r_1\right)\right)$$

$$= \frac{GMm_1\delta r_1}{2r_1^2}\left(1 - \frac{r_1^2}{r_2^2}\sqrt{\frac{r_2}{r_1}}\right)$$

$$\boxed{=> \Delta E = -\frac{GMm_1\delta r_1}{2r_1^2}\left(\left(\frac{r_1}{r_2}\right)^{3/2} - 1\right)}$$



- If $r_1 < r_2$ we can reduce the energy of the system if $\Delta r_1 < 0$ i.e. moving m_1 closer to the center. Thus the energy can be reduced while conserving AM by moving the initially closer body in and moving the initially outer body further out => *all the mass is concentrated near the inner parts of the solar system, while the AM occupies the outer parts.* See Planetary Science by Cole and Woolfson for further details.

Properties of the Solar System

1. All the planets' orbits lie roughly in the ecliptic plane.
2. The Sun's rotational equator lies nearly in this plane.
3. Planetary orbits are slightly elliptical, very nearly circular.
4. The planets revolve in a west-to-east direction. The Sun rotates in the same west-to-east direction.
5. The planets differ in composition. Their composition varies roughly with distance from the Sun: dense, metal-rich planets are in the inner part and giant, hydrogen-rich planets are in the outer part.
6. Meteorites differ in chemical and geologic properties from the planets and the Moon.
7. The Sun and most of the planets rotate in the same west-to-east direction. Their obliquity (the tilt of their rotation axes with respect to their orbits) are small. Uranus and Venus are exceptions.
8. The rotation rates of the planets and asteroids are similar-5 to 15 hours.
9. The planet distances from the Sun obey Bode's law.
10. Planet-satellite systems resemble the solar system.
11. The Oort Cloud and Kuiper Belt of comets.
12. The planets contain about 99% of the solar system's angular momentum but the Sun contains over 99% of the solar system's mass.