Introduction to Plasma Physics (PY5012)
Lecture 11: Waves I

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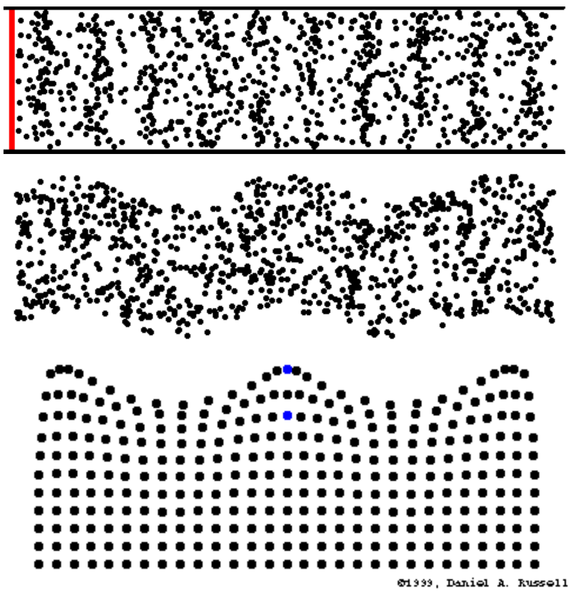
What Defines a Wave?

- Mechanical examples:
  - sound, string, water
- Energy transfer
- Restoring forces:
  - pressure, tension, gravity
- Characteristics:
  - wave speed
  - motion of medium
  - direction of propagation
- Dispersion relation – really important!



Simple Wave Representation

- For plane waves propagating with wave vector **k** = (k_x, k_y, k_z) and angular frequency ω, [N.B. **r** = (x, y, z) is the position vector]

$$U = C_U \exp\left[i\left(\mathbf{k}\cdot\mathbf{r} - \omega t\right)\right]$$

- and for propagation in only the x-direction,

$$U = C_U \exp\left[i\left(kx - \omega t\right)\right]$$

- Constant phase is maintained for a point on the wave when,

$$\frac{d}{dt}\left(kx - \omega t\right) = 0$$

$$\frac{d\left(kx\right)}{dt} - \omega = 0$$

$$\boxed{\frac{dx}{dt} = \frac{\omega}{k} = v_p}$$ <u>phase speed</u>

Wave Group Speed

- Phase speed is not the rate of information (i.e., energy) transfer

- Group speed is similarly defined, but for constant phase on a modulated wave envelope,

$$U \propto \exp\left[i\left(\Delta kx - \Delta\omega t\right)\right]$$

- giving,

$$\frac{d}{dt}\left(\Delta kx - \Delta\omega t\right) = 0$$

$$\frac{dx}{dt} = \frac{\Delta\omega}{\Delta k}$$

$$\lim_{\Delta\omega\to 0}\left(\frac{\Delta\omega}{\Delta k}\right) = \boxed{\frac{d\omega}{dk} = v_g}$$ <u>group speed</u>

Wave Dispersion Relation

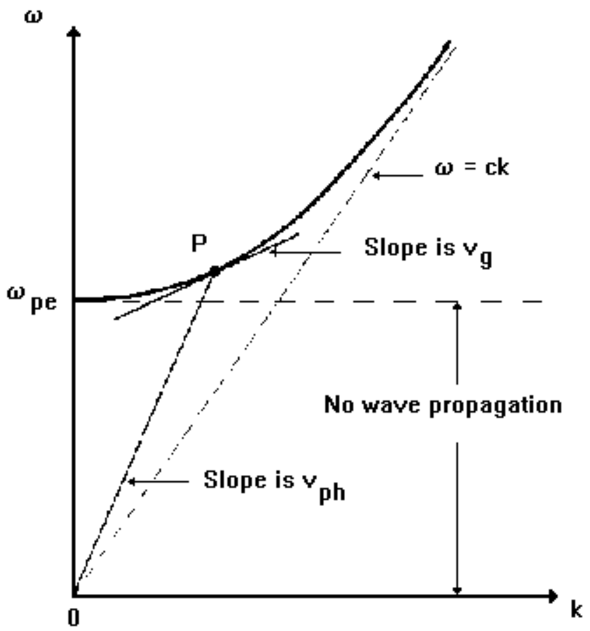
- Everything is contained in <u>dispersion relation</u>,

  $$\omega = \omega(k)$$

- **k** often complex, but waves propagate only for,

  $$\Re(\mathbf{k}) > 0$$

- Dispersion relation indicates cutoffs and resonances



What Makes Plasma Waves Different?

- Plasma properties:
  - gas-like
  - charged
  - magnetic field (complicates everything)

- <u>Fluid equations</u>
  - mass continuity
  - equation of motion
  - energy equation
  - ideal gas law
- <u>EM equations</u>
  - Maxwell's equations
  - induction equation
  - Ohm's law

Fluid Equations I

- Mass Continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

  - matter is neither created nor destroyed
  - but, $\nabla \cdot (a\mathbf{B}) = (\mathbf{B} \cdot \nabla) a + a \nabla \cdot \mathbf{B}$

$$\boxed{\frac{\partial \rho}{\partial t} + (\mathbf{v} \cdot \nabla)\rho + \rho \nabla \cdot \mathbf{v} = 0}$$

- Equation of Motion

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \mathbf{F}$$

  - Newton's 2nd law; **F** represents all external forces
  - $\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$ is the convective time derivative

Expanded Equation of Motion

- External forces

  - magnetic force: $\mathbf{J} \times \mathbf{B}$ (the Lorentz force)
  - gravitational force: $\rho \mathbf{g}$
  - viscous force (ignored)

- New Equation of Motion

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \mathbf{J} \times \mathbf{B} + \rho \mathbf{g}$$

  - Expand *D*/*Dt*,

$$\boxed{\rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \mathbf{J} \times \mathbf{B} + \rho \mathbf{g}}$$

Fluid Equations II

- Ideal Gas Law

$$p = \frac{R}{\mu}\rho T$$

  - μ is mean atomic weight

- Energy Equation

$$\frac{D}{Dt}\left(\frac{p}{\rho^{\gamma}}\right) = -L$$

  - L represents all energy losses
  - γ is the ratio of specific heats (c_p/c_v; constant pressure to constant volume), normally taken as 5/3
  - only consider adiabatic case (i.e., at constant energy) $L = 0$

Modified Energy Equation

- Apply change rule,

$$\frac{D}{Dt}\left(\frac{p}{\rho^{\gamma}}\right) = \frac{1}{\rho^{\gamma}}\frac{Dp}{Dt} - \frac{\gamma p}{\rho^{\gamma+1}}\frac{D\rho}{Dt} = 0$$

- Expand $\frac{D}{Dt}$,

$$\frac{1}{\rho^{\gamma}}\frac{\partial p}{\partial t} + \frac{1}{\rho^{\gamma}}(\mathbf{v}\cdot\nabla)p - \frac{\gamma p}{\rho^{\gamma+1}}\frac{\partial \rho}{\partial t} - \frac{\gamma p}{\rho^{\gamma+1}}(\mathbf{v}\cdot\nabla)\rho = 0$$

$$\frac{\partial p}{\partial t} + (\mathbf{v}\cdot\nabla)p - \frac{\gamma p}{\rho}\left(\frac{\partial \rho}{\partial t} + (\mathbf{v}\cdot\nabla)\rho\right) = 0$$

- But, $-\left(\frac{\partial \rho}{\partial t} + (\mathbf{v}\cdot\nabla)\rho\right) = \rho\nabla\cdot\mathbf{v}$ from mass continuity equation,

$$\frac{\partial p}{\partial t} + (\mathbf{v}\cdot\nabla)p + \frac{\gamma p}{\rho}(\rho\nabla\cdot\mathbf{v}) = 0$$

$$\frac{\partial p}{\partial t} + (\mathbf{v}\cdot\nabla)p = -\gamma p\nabla\cdot\mathbf{v}$$

EM Equations I

- Ampère's Law

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

  - gradients in magnetic fields create electric currents
  - in MHD approx. displacement current (RHS 2nd term) is neglected

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J}}$$

- Solenoidal Constraint

$$\boxed{\nabla \cdot \mathbf{B} = 0}$$

  - no magnetic monopoles

- Faraday's Law

$$\boxed{\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}}$$

  - spatially varying electric field induces a magnetic field

EM Equations II

- Gauss' Law

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}}$$

  - charge conservation

- Ohm's Law

$$\boxed{\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B})}$$

  - generalized form
  - couples EM equations to plasma equations through **v**

- Induction Equation

$$\boxed{\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})}$$

  - combination of Ohm's, Ampère's, and Faraday's laws
  - diffusivity term ignored

Wave Assumptions

- Wave amplitudes are small
  - allows for linearization of MHD equations

- Basic state is a static equilibrium

<u>Eqn. of Motion</u>: $$0 = -\nabla p_0 + \mathbf{J}_0 \times \mathbf{B}_0 + \rho_0 \mathbf{g}$$ [A]

<u>Solenoidal Constraint</u>: $$\nabla \cdot \mathbf{B}_0 = 0$$ [B]

<u>Ideal Gas Law</u>: $$p_0 = \frac{R}{\mu} \rho_0 T_0$$ [C]

  - quantities $\mathbf{X}_0$ and X_0 are the initial equilibrium state
  - not necessary to be static (just simplified proof)

Wave Perturbations

- After wave initiation,

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1(\mathbf{r}, t)$$
$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1(\mathbf{r}, t)$$
$$\rho = \rho_0 + \rho_1(\mathbf{r}, t)$$
$$p = p_0 + p_1(\mathbf{r}, t)$$
$$T = T_0 + T_1(\mathbf{r}, t)$$

  - $\mathbf{X}$ and X are the perturbed quantities
  - $\mathbf{X}_1$ and X_1 are the applied perturbations (<< $\mathbf{X}_0$ and X_0 quantities)

- Static initial condition:
  - $\mathbf{v}_0 = 0$ $\quad\quad$ $\mathbf{v} = \mathbf{v}_1(\mathbf{r}, t)$
  - initial quantities are time independent $\quad \dfrac{\partial \mathbf{X}_0}{\partial t} = 0 \quad \dfrac{\partial X_0}{\partial t} = 0$

MHD Linearization (continuity eqn.)

- Put perturbed quantities into MHD equations and neglect products of small terms (i.e., $X_1 Y_1$)

- <u>Mass Continuity</u>:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\cancel{\frac{\partial \rho_0}{\partial t}} + \frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 \mathbf{v}_1) + \nabla \cdot \cancel{(\rho_1 \mathbf{v}_1)} = 0$$

  - but with $\frac{\partial \rho_0}{\partial t} = 0$ and dropping $X_1 Y_1$ terms,

$$\boxed{\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 \mathbf{v}_1) = 0}$$

MHD Linearization (eqn. of motion)

- <u>Eqn. of Motion</u>:

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \mathbf{J} \times \mathbf{B} + \rho \mathbf{g}$$

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} + \cancel{\rho_1 \frac{\partial \mathbf{v}_1}{\partial t}} + \cancel{\rho_0(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_1} + \cancel{\rho_1(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_1} =$$
$$-\nabla p_0 - \nabla p_1 + \mathbf{J}_0 \times \mathbf{B}_0 + \mathbf{J}_0 \times \mathbf{B}_1 + \mathbf{J}_1 \times \mathbf{B}_0 + \cancel{\mathbf{J}_1 \times \mathbf{B}_1} + \rho_0 \mathbf{g} + \rho_1 \mathbf{g}$$

  - Neglecting $X_1 Y_1$ terms and substituting for $\mathbf{J}$,

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = -\nabla p_1 + \frac{(\nabla \times \mathbf{B}_0)}{\mu_0} \times \mathbf{B}_1 + \frac{(\nabla \times \mathbf{B}_1)}{\mu_0} \times \mathbf{B}_0 + \rho_1 \mathbf{g} + \cancel{(-\nabla p_0 + \mathbf{J}_0 \times \mathbf{B}_0 + \rho_0 \mathbf{g})}$$

  - but, $-\nabla p_0 + \mathbf{J}_0 \times \mathbf{B}_0 + \rho_0 \mathbf{g} = 0$,

$$\boxed{\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = -\nabla p_1 + \frac{(\nabla \times \mathbf{B}_0)}{\mu_0} \times \mathbf{B}_1 + \frac{(\nabla \times \mathbf{B}_1)}{\mu_0} \times \mathbf{B}_0 + \rho_1 \mathbf{g}}$$

MHD Linearization (energy/induction eqns.)

- Adiabatic Energy Eqn.: $\frac{\partial p}{\partial t} + (\mathbf{v}\cdot\nabla)p = -\gamma p \nabla\cdot\mathbf{v}$

$$\cancel{\frac{\partial p_0}{\partial t}} + \frac{\partial p_1}{\partial t} + (\mathbf{v}_1\cdot\nabla)p_0 + \cancel{(\mathbf{v}_1\cdot\nabla)p_1} = -\gamma p_0 \nabla\cdot\mathbf{v}_1 - \cancel{\gamma p_1 \nabla\cdot\mathbf{v}_1}$$

  - but with $\frac{\partial p_0}{\partial t} = 0$ and dropping $X_1 Y_1$ terms,

$$\boxed{\frac{\partial p_1}{\partial t} + (\mathbf{v}_1\cdot\nabla)p_0 = -\gamma p_0 \nabla\cdot\mathbf{v}_1}$$

- Induction Eqn.: $\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B})$

$$\cancel{\frac{\partial \mathbf{B}_0}{\partial t}} + \frac{\partial \mathbf{B}_1}{\partial t} = \nabla\times(\mathbf{v}_1\times\mathbf{B}_0) + \cancel{\nabla\times(\mathbf{v}_1\times\mathbf{B}_1)}$$

  - but with $\frac{\partial \mathbf{B}_0}{\partial t} = 0$ and dropping $X_1 Y_1$ terms,

$$\boxed{\frac{\partial \mathbf{B}_1}{\partial t} = \nabla\times(\mathbf{v}_1\times\mathbf{B}_0)}$$

MHD Linearization (gas law/solenoidal constraint)

- Ideal Gas Law:

$$p = \frac{R}{\mu}\rho T$$

$$\cancel{p_0} + p_1 = \cancel{\frac{R}{\mu}\rho_0 T_0} + \frac{R}{\mu}\rho_1 T_0 + \frac{R}{\mu}\rho_0 T_1 + \cancel{\frac{R}{\mu}\rho_1 T_1}$$

  - but $p_0 = \frac{R}{\mu}\rho_0 T_0$ and dropping $X_1 Y_1$ terms,

$$\boxed{p_1 = \frac{R}{\mu}\rho_1 T_0 + \frac{R}{\mu}\rho_0 T_1}$$

- Solenoidal Constraint:

$$\nabla\cdot\mathbf{B} = 0$$

$$\cancel{\nabla\cdot\mathbf{B}_0} + \nabla\cdot\mathbf{B}_1 = 0$$

  - but with $\nabla\cdot\mathbf{B}_0 = 0$,

$$\boxed{\nabla\cdot\mathbf{B}_1 = 0}$$

Summary of Linearized MHD Equations

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot \left(\rho_0 \mathbf{v}_1\right) = 0 \qquad [1]$$

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = -\nabla p_1 + \frac{\left(\nabla \times \mathbf{B}_0\right)}{\mu_0} \times \mathbf{B}_1 + \frac{\left(\nabla \times \mathbf{B}_1\right)}{\mu_0} \times \mathbf{B}_0 + \rho_1 \mathbf{g} \qquad [2]$$

$$\frac{\partial p_1}{\partial t} + \left(\mathbf{v}_1 \cdot \nabla\right) p_0 = -\gamma p_0 \nabla \cdot \mathbf{v}_1 \qquad [3]$$

$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times \left(\mathbf{v}_1 \times \mathbf{B}_0\right) \qquad [4]$$

$$p_1 = \frac{R}{\mu} \rho_1 T_0 + \frac{R}{\mu} \rho_0 T_1 \qquad [5]$$

$$\nabla \cdot \mathbf{B}_1 = 0 \qquad [6]$$