Introduction to Plasma Physics (PY5012)
Lectures 12 & 13: Waves II

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Linearized MHD Equations

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot \left(\rho_0 \mathbf{v}_1 \right) = 0 \qquad [1]$$

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = -\nabla p_1 + \frac{\left(\nabla \times \mathbf{B}_0 \right)}{\mu_0} \times \mathbf{B}_1 + \frac{\left(\nabla \times \mathbf{B}_1 \right)}{\mu_0} \times \mathbf{B}_0 + \rho_1 \mathbf{g} \qquad [2]$$

$$\frac{\partial p_1}{\partial t} + \left(\mathbf{v}_1 \cdot \nabla \right) p_0 = -\gamma p_0 \nabla \cdot \mathbf{v}_1 \qquad [3]$$

$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times \left(\mathbf{v}_1 \times \mathbf{B}_0 \right) \qquad [4]$$

$$p_1 = \frac{R}{\mu} \rho_1 T_0 + \frac{R}{\mu} \rho_0 T_1 \qquad [5]$$

$$\nabla \cdot \mathbf{B}_1 = 0 \qquad [6]$$

Simple Wave Solutions

- Looking for plane waves of form,

$$U = C_U \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]$$

- with angular frequency ω, wave vector $\mathbf{k} = (k_x, k_y, k_z)$, with position vector $\mathbf{r} = (x, y, z)$. Note, $k = 2\pi / \lambda$.

$$\mathbf{k} \cdot \mathbf{r} = k_x x + k_y y + k_z z$$
$$\mathbf{k} \cdot \mathbf{k} = k^2 = k_x^2 + k_y^2 + k_z^2$$

- Useful solutions for Fourier analysis since,

$$\frac{\partial}{\partial t} \to -i\omega \qquad \frac{\partial^2}{\partial t^2} \to -\omega^2 \qquad \frac{\partial}{\partial x} \to ik_x \qquad \frac{\partial^2}{\partial x^2} \to -k_x^2$$

$$\nabla \to i\mathbf{k} \qquad \nabla\cdot \to i\mathbf{k}\cdot \qquad \nabla\times \to i\mathbf{k}\times$$

Case 1: Acoustic (Pressure) Waves

Acoustic Wave Equations

- Ignore magnetic field and gravity (i.e., $\mathbf{B}_0 = \mathbf{g} = 0$)
  - assume homogeneous medium
  - from equilibrium [A], $\nabla p_0 = 0$ and $p_0 = \text{constant}$
  - for simplicity, $\rho_0 = \text{constant}$

- Linearized equations reduce to:

[1] $$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}_1 = 0 \quad \rightarrow \quad -i\omega\rho_1 + i\rho_0(\mathbf{k}\cdot\mathbf{v}_1) = 0$$ [7]

[2] $$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = -\nabla p_1 \quad \rightarrow \quad -i\omega\rho_0\mathbf{v}_1 = -i\mathbf{k}p_1$$ [8]

[3] $$\frac{\partial p_1}{\partial t} = -\gamma p_0 \nabla \cdot \mathbf{v}_1 \quad \rightarrow \quad -i\omega p_1 = -i\gamma p_0(\mathbf{k}\cdot\mathbf{v}_1)$$ [9]

Acoustic Wave Properties

[8] $$\mathbf{v}_1 = \left(\frac{p_1}{\omega\rho_0}\right)\mathbf{k}$$ [10]

- $\mathbf{v}_1$ parallel to $\mathbf{k}$
- particle motion along propagation direction (<u>longitudinal</u>)

- Also,

[7] $$\frac{\rho_1}{\rho_0} = \frac{(\mathbf{k}\cdot\mathbf{v}_1)}{\omega}$$ [11]

[9] $$p_1 = \gamma p_0 \frac{(\mathbf{k}\cdot\mathbf{v}_1)}{\omega} = \frac{\gamma p_0 \rho_1}{\rho_0}$$

- Defining the sound speed, $$c_s^2 = \frac{\gamma p_0}{\rho_0}$$

$$p_1 = c_s^2 \rho_1$$ [12]

  - for $\mathbf{k}\cdot\mathbf{v}_1 \neq 0$, then ρ_1 and $p_1 \neq 0$ (<u>compressive</u>)

Acoustic Dispersion Relation

- Taking scalar product with **k**,

[10] $$\mathbf{k}\cdot\mathbf{v}_1 = \left(\frac{p_1}{\omega\rho_0}\right)\mathbf{k}\cdot\mathbf{k} = \left(\frac{p_1}{\omega\rho_0}\right)k^2$$

- Rearranging,

[11] $$(\mathbf{k}\cdot\mathbf{v}_1) = \frac{\omega\rho_1}{\rho_0}$$

[12] $$\rho_1 = \frac{p_1}{c_s^2}$$

- Substitute, $$(\mathbf{k}\cdot\mathbf{v}_1) = \frac{\omega p_1}{c_s^2\rho_0}$$

- Equating $\mathbf{k}\cdot\mathbf{v}_1$,

$$\left(\frac{p_1}{\omega\rho_0}\right)k^2 = \frac{\omega p_1}{c_s^2\rho_0}$$

$$\boxed{\omega^2 = k^2 c_s^2}$$ (dispersion relation) [13]

Acoustic Phase and Group Speeds

- Phase speed:

$$\frac{\omega}{k} = \pm c_s$$

[13] $$\boxed{\mathbf{v}_p = v_p\mathbf{k}' = \pm c_s\mathbf{k}'}$$

- Group velocity: $$v_g = \frac{\partial\omega}{\partial\mathbf{k}} = \left(\frac{\partial\omega}{\partial k_x},\frac{\partial\omega}{\partial k_y},\frac{\partial\omega}{\partial k_z}\right)$$

[13] $$\omega^2 = c_s^2\left(k_x^2 + k_y^2 + k_z^2\right)$$

- Differentiating, $$2\omega\frac{\partial\omega}{\partial\mathbf{k}} = c_s^2\left(2k_x, 2k_y, 2k_z\right)$$

$$\frac{\partial\omega}{\partial\mathbf{k}} = c_s^2\frac{\left(k_x, k_y, k_z\right)}{\omega}$$

$$\boxed{\mathbf{v}_g = c_s^2\frac{k}{\omega}\mathbf{k}' = \pm c_s\mathbf{k}'}$$

Acoustic Wave Complications

- Hydrostatic equilibrium [A], $\nabla p_0 = -\rho_0 \mathbf{g}$

$$p_0(z) = p_0(0)\exp(-z/H)$$
$$\rho_0(z) = \rho_0(0)\exp(-z/H)$$

- where H is the pressure scale height,

$$H = \frac{p_0}{\rho_0 g} = \frac{RT}{g}$$

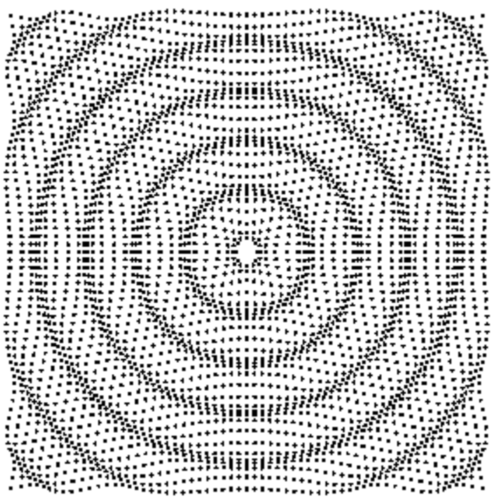
- Pressure variations follow,

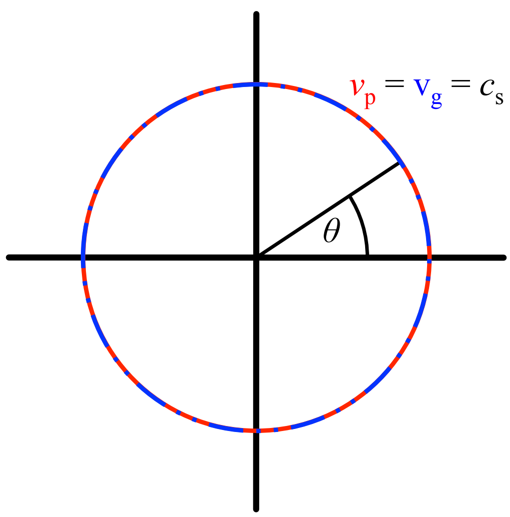
$$\frac{\partial^2 Q}{\partial t^2} - c_s^2(z)\frac{\partial^2 Q}{\partial z^2} + \Omega_s^2(z)Q = 0$$
$$\omega^2 = k_z^2 c_s^2 + \Omega_s^2$$

- Real solutions (propagation) for $k_z > 0$,

$$\omega > \Omega_s = \omega_{ac} = \frac{c_s}{2H}$$

Acoustic Wave Summary



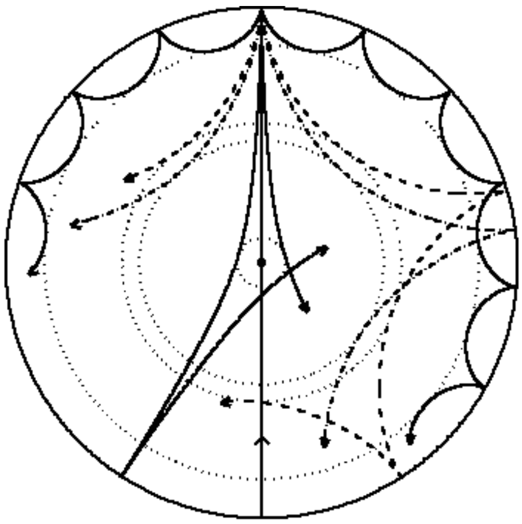


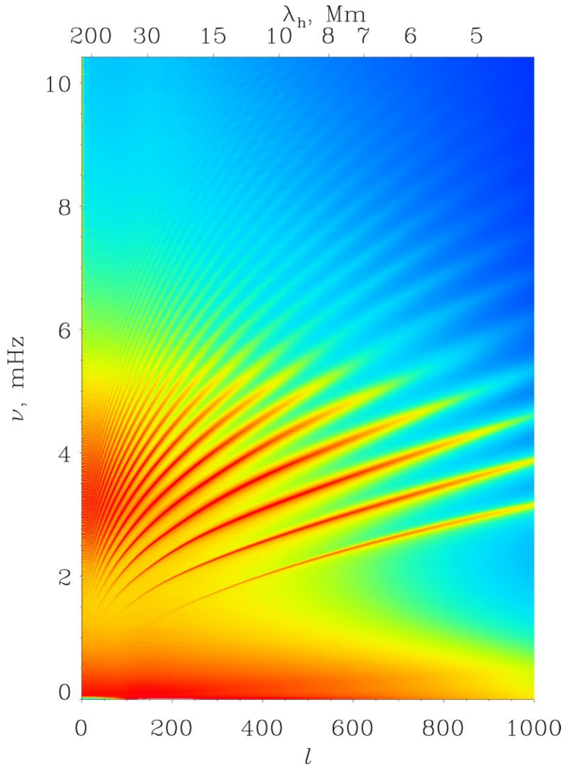
- Restoring force: pressure
- Directionality: isotropic
- Transmission: $\omega > \omega_{ac}$

- Phase speed: c_s
- Group speed: c_s

Acoustic Wave Example

- Helioseismology





Case 2: Magnetic (Alfvén) Waves

Alfvén Wave Equations

- Ignore pressure and gravity (i.e., $p_0 = \mathbf{g} = 0$)
  - from equilibrium [A], $0 = \mu_0(\mathbf{J}_0 \times \mathbf{B}_0) = (\nabla \times \mathbf{B}_0) \times \mathbf{B}_0$
  - assume no pressure variations, $p_1 = \rho_1 = 0$
  - assume uniform equilibrium field distribution, $\mathbf{B}_0 = B_0 \mathbf{z}'$

- Linearized equations reduce to:

[1] $\nabla \cdot \mathbf{v}_1 = 0 \quad \rightarrow \quad i(\mathbf{k} \cdot \mathbf{v}_1) = 0$ [14]

[2] $\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = \frac{(\nabla \times \mathbf{B}_1)}{\mu_0} \times \mathbf{B}_0 \quad \rightarrow \quad -i\omega\rho_0 \mathbf{v}_1 = \frac{(i\mathbf{k} \times \mathbf{B}_1)}{\mu_0} \times \mathbf{B}_0$ [15]

[4] $\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0) \quad \rightarrow \quad -i\omega \mathbf{B}_1 = i\mathbf{k} \times (\mathbf{v}_1 \times \mathbf{B}_0)$ [16]

[6] $\nabla \cdot \mathbf{B}_1 = 0 \quad \rightarrow \quad i\mathbf{k} \cdot \mathbf{B}_1 = 0$ [17]

Alfvén Wave Properties I

[1] $\nabla \cdot \mathbf{v}_1 = 0$

- no divergent/convergent motions (incompressible)

[14] $\mathbf{k} \cdot \mathbf{v}_1 \equiv k v_1 \cos\theta_{kv_1} = 0$

$\therefore \theta_{\mathbf{kv}_1} = 90°$

- $\mathbf{v}_1$ at right angles to $\mathbf{k}$ (transverse)

- Taking scalar product with $\mathbf{B}_0$,

[15] $-\omega\rho_0 \mathbf{v}_1 \cdot \mathbf{B}_0 = \frac{(\mathbf{k} \times \mathbf{B}_1)}{\mu_0} \times \mathbf{B}_0 \cdot \mathbf{B}_0 = 0$

$\mathbf{v}_1 \cdot \mathbf{B}_0 \equiv v_1 B_0 \cos\theta_{\mathbf{v}_1\mathbf{B}} = 0$

$\therefore \theta_{\mathbf{v}_1\mathbf{B}} = 90°$ [18]

- $\mathbf{v}_1$ at right angles to $\mathbf{B}_0$ (perpendicular)

Alfvén Wave Properties II

- Expand [16] using standard vector identity,

$$-\omega\mathbf{B}_1 = \mathbf{k}\times(\mathbf{v}_1\times\mathbf{B}_0)$$
$$= (\mathbf{k}\cdot\mathbf{B}_0)\mathbf{v}_1 - (\mathbf{k}\cdot\mathbf{v}_1)\mathbf{B}_0$$

- But $(\mathbf{k}\cdot\mathbf{v}_1)=0$ from [14],

$$-\omega\mathbf{B}_1 = (\mathbf{k}\cdot\mathbf{B}_0)\mathbf{v}_1 \qquad [19]$$

- Taking scalar product with $\mathbf{B}_0$,

$$-\omega\mathbf{B}_1\cdot\mathbf{B}_0 = (\mathbf{k}\cdot\mathbf{B}_0)(\mathbf{v}_1\cdot\mathbf{B}_0)$$

- But $(\mathbf{v}_1\cdot\mathbf{B}_0)=0$ from [18],

$$\mathbf{B}_1\cdot\mathbf{B}_0 \equiv B_1 B_0 \cos\theta_{B_0B_1} = 0 \qquad [20]$$
$$\therefore \theta_{\mathbf{B}_0\mathbf{B}_1} = 90°$$

  - $\mathbf{B}_1$ at right angles to $\mathbf{B}_0$ (perpendicular)

Alfvén Dispersion Relation I

- Multiply [16] by ω and substitute for $\mathbf{v}_1$ from [15],

$$\omega^2\mathbf{B}_1 = \frac{1}{\mu_0\rho_0}\mathbf{k}\times\left\{\left[(\mathbf{k}\times\mathbf{B}_1)\times\mathbf{B}_0\right]\times\mathbf{B}_0\right\} \qquad [21]$$

- Expanding inner triple vector product,

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = (\mathbf{C}\cdot\mathbf{A})\mathbf{B} - (\mathbf{C}\cdot\mathbf{B})\mathbf{A}$$
$$(\mathbf{k}\times\mathbf{B}_1)\times\mathbf{B}_0 = (\mathbf{B}_0\cdot\mathbf{k})\mathbf{B}_1 - (\mathbf{B}_0\cdot\mathbf{B}_1)\mathbf{k}$$

- But $(\mathbf{B}_0\cdot\mathbf{B}_1)=0$ from [20],

$$\mathbf{k}\times\left\{\left[(\mathbf{k}\times\mathbf{B}_1)\times\mathbf{B}_0\right]\times\mathbf{B}_0\right\} = \mathbf{k}\times\left\{\left[(\mathbf{B}_0\cdot\mathbf{k})\mathbf{B}_1\right]\times\mathbf{B}_0\right\}$$
$$= (\mathbf{k}\cdot\mathbf{B}_0)\left[(\mathbf{B}_0\cdot\mathbf{k})\mathbf{B}_1\right] - \left(\mathbf{k}\cdot\left[(\mathbf{B}_0\cdot\mathbf{k})\mathbf{B}_1\right]\right)\mathbf{B}_0$$
$$= (\mathbf{k}\cdot\mathbf{B}_0)^2\mathbf{B}_1 - (\mathbf{k}\cdot\mathbf{B}_1)(\mathbf{B}_0\cdot\mathbf{k})\mathbf{B}_0$$

- And $(\mathbf{k}\cdot\mathbf{B}_1)=0$ from [17],

$$\mathbf{k}\times\left\{\left[(\mathbf{k}\times\mathbf{B}_1)\times\mathbf{B}_0\right]\times\mathbf{B}_0\right\} = (\mathbf{k}\cdot\mathbf{B}_0)^2\mathbf{B}_1$$

Alfvén Dispersion Relation II

[19] $$\omega^2 \mathbf{B}_1 = \frac{(\mathbf{k}\cdot\mathbf{B}_0)^2}{\mu_0\rho_0}\mathbf{B}_1$$

$$\omega^2 = \frac{(\mathbf{k}\cdot\mathbf{B}_0)^2}{\mu_0\rho_0} \quad [22]$$

- Recall that $\mathbf{B}_0 = B_0\mathbf{z}'$ and $(\mathbf{k}\cdot\mathbf{z}') = k_z = k\cos\theta_{\mathbf{kB}_0}$,

$$\omega^2 = \frac{(\mathbf{k}\cdot\mathbf{z}')^2 B_0^2}{\mu_0\rho_0} = \frac{(k\cos\theta_{\mathbf{kB}_0})^2 B_0^2}{\mu_0\rho_0}$$

- Defining the Alfvén speed,

$$v_A^2 = \frac{B_0^2}{\mu_0\rho_0}$$

$$\boxed{\omega^2 = (k\cos\theta_{\mathbf{kB}_0})^2 v_A^2} \quad \text{(dispersion relation)} \quad [23]$$

Alfvén Phase and Group Speeds

- Alfvén waves are anisotropic
  - $(\mathbf{k}\cdot\mathbf{B}_0)$ term in [22], the generalized dispersion relation
- Phase speed:

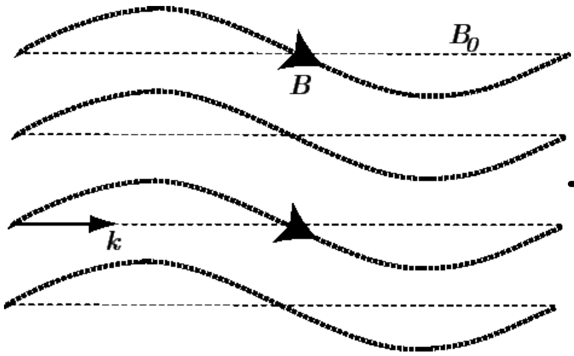
[23] $$\boxed{\frac{\omega}{k} = \pm v_A\cos\theta_{\mathbf{kB}_0} = v_p}$$

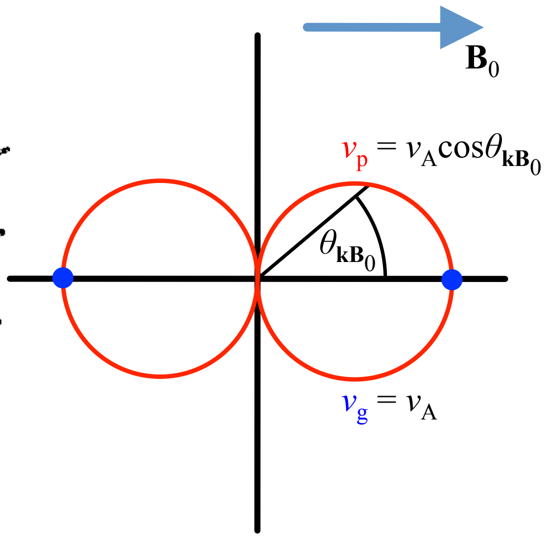
- Group velocity: $$\mathbf{v}_g = \frac{\partial\omega}{\partial\mathbf{k}} = \left(\frac{\partial\omega}{\partial k_x}, \frac{\partial\omega}{\partial k_y}, \frac{\partial\omega}{\partial k_z}\right)$$

[23] $$\omega^2 = \pm v_A k\cos\theta_{\mathbf{kB}_0} = \pm v_A k_z$$

- Differentiating, $$\boxed{\frac{\partial\omega}{\partial\mathbf{k}} = \pm v_A\mathbf{z}' = \mathbf{v}_g}$$

Alfvén Wave Summary

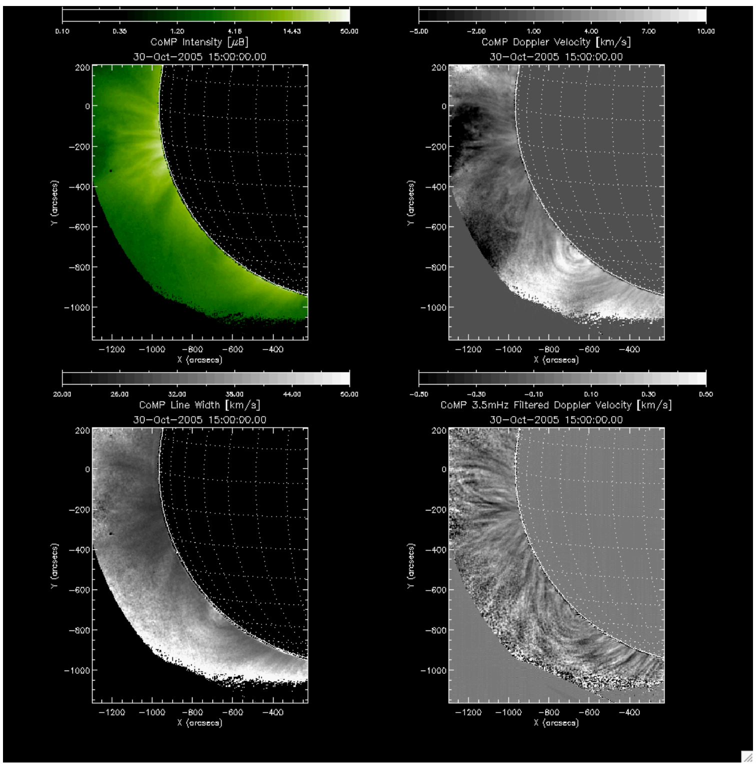




- Restoring force: **B**–field tension
- Directionality: anisotropic
- Transmission: all ω
- Phase speed: $v_A \cos\theta_{\mathbf{kB}_0}$
- Group speed: v_A

Alfvén Wave Example

- Nothing *convincing* yet
- Primary issue:
  - plasma anisotropic where important or measureable
- Several claims in (solar) literature
  - image plane loops
  - LOS velocity
  - transverse waves
    - $\nu \approx 3.5$ mHz
    - $\nu_{ac} \approx 5$ mHz at photosphere



Case 3: Magnetic Pressure (Magnetoacoustic) Waves

Magnetoacoustic Wave Equations

- Ignore gravity (i.e., $\mathbf{g} = 0$)
  - assume uniform equilibrium field distribution, $\mathbf{B}_0 = B_0\mathbf{z}'$
- Linearized equations reduce to,

$$\frac{\omega^2\mathbf{v}_1}{v_A^2} = k^2\cos^2\left(\theta_{\mathbf{kB}_0}\right)\mathbf{v}_1 - (\mathbf{k}\cdot\mathbf{v}_1)k\cos\left(\theta_{\mathbf{kB}_0}\right)\hat{\mathbf{B}}_0 + \left[\left(1+\frac{c_s^2}{v_A^2}\right)(\mathbf{k}\cdot\mathbf{v}_1) - k\cos\left(\theta_{\mathbf{kB}_0}\right)\left(\hat{\mathbf{B}}_0\cdot\mathbf{v}_1\right)\right]\mathbf{k}$$

- with resulting dispersion relation,

$$\boxed{\omega^4 - \omega^2k^2\left(c_s^2+v_A^2\right) + c_s^2v_A^2k^4\cos^2\theta_{\mathbf{kB}_0} = 0}$$

Magnetoacoustic Wave Properties

- Phase velocities:

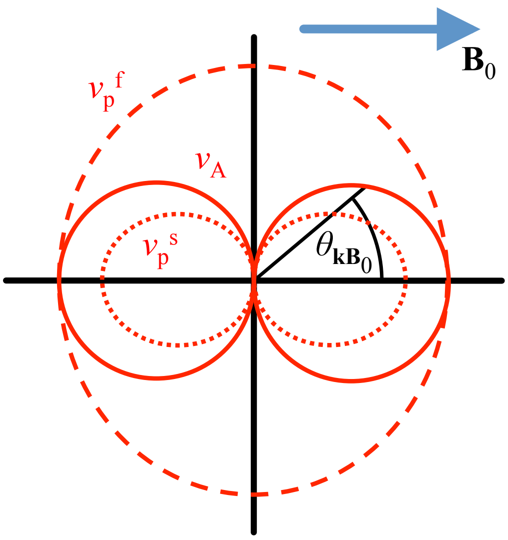
$$\frac{\omega}{k} = v_{\mathrm{p}}^{\mathrm{f}} = \sqrt{\frac{1}{2}\left(c_{\mathrm{s}}^2 + v_{\mathrm{A}}^2\right) + \frac{1}{2}\sqrt{c_{\mathrm{s}}^4 + v_{\mathrm{A}}^4 - 2c_{\mathrm{s}}^2 v_{\mathrm{A}}^2 \cos 2\theta_{\mathbf{kB}_0}}}$$

$$v_{\mathrm{p}}^{\mathrm{s}} = \sqrt{\frac{1}{2}\left(c_{\mathrm{s}}^2 + v_{\mathrm{A}}^2\right) - \frac{1}{2}\sqrt{c_{\mathrm{s}}^4 + v_{\mathrm{A}}^4 - 2c_{\mathrm{s}}^2 v_{\mathrm{A}}^2 \cos 2\theta_{\mathbf{kB}_0}}}$$

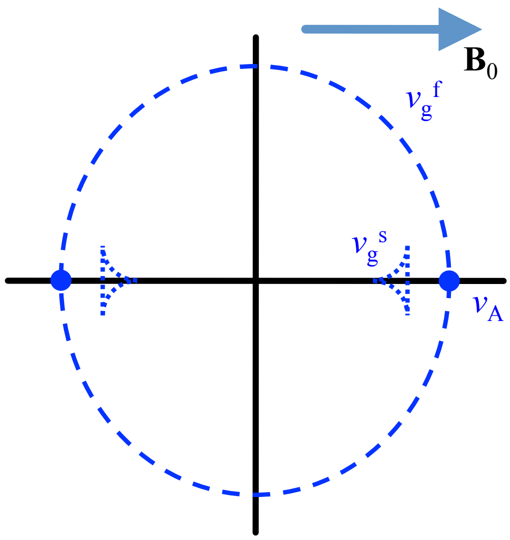
Wave Mode	Propagation	Low-Beta	High-Beta
Alfvén	along $\mathbf{B}_0$	magnetic tension	
Fast	isotropic	magnetic pressure	gas pressure
Slow	roughly along $\mathbf{B}_0$	gas pressure	magnetic tension

Magnetoacoustic Wave Summary

- Phase speed:

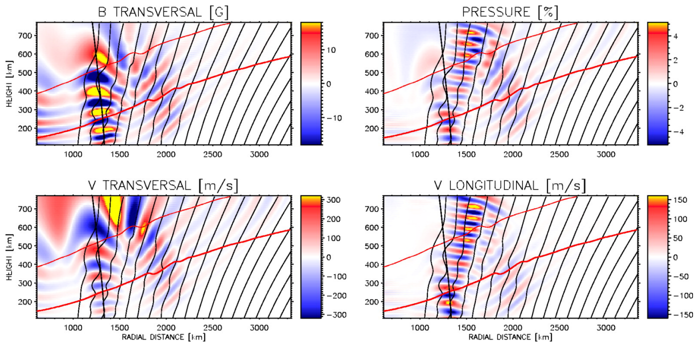


- Group speed:

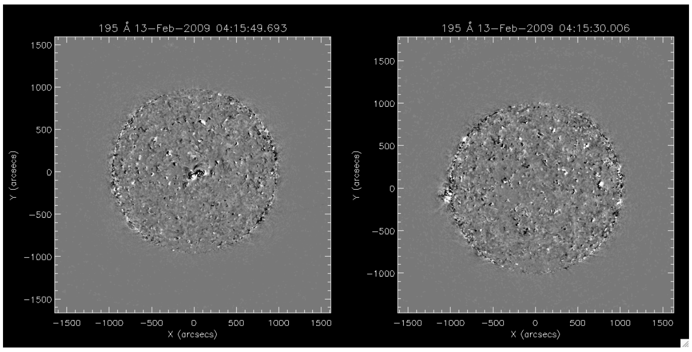


Magnetoacoustic Wave Examples

- Sunspot waves
  - primarily guided
  - slow mode?



- Coronal "EIT wave" fronts
  - need to propagate across B-field
  - fast mode?



Wave Summary

- Acoustic waves
  - particle motion along **k** direction (longitudinal)
  - phase and group speeds are c_s in all directions (isotropic)

- (Sheer) Alfvén waves
  - particle motion at right angles to **k** direction (transverse)
  - **B** perturbation at right angles to **k** direction (perpendicular)
  - phase speed varies as $v_A \cos\theta_{\mathbf{kB}_0}$ (anisotropic)
  - group speed is v_A along **B** direction (anisotropic)

- Magnetoacoustic waves
  - Alfvén – as above
  - Fast – gas and **B** pressure in phase, also isotropic
  - Slow – gas and **B** pressure out of phase, also anisotropic