Introduction to Plasma Physics (PY5012)

Lecture 3: Single-Particle Motion

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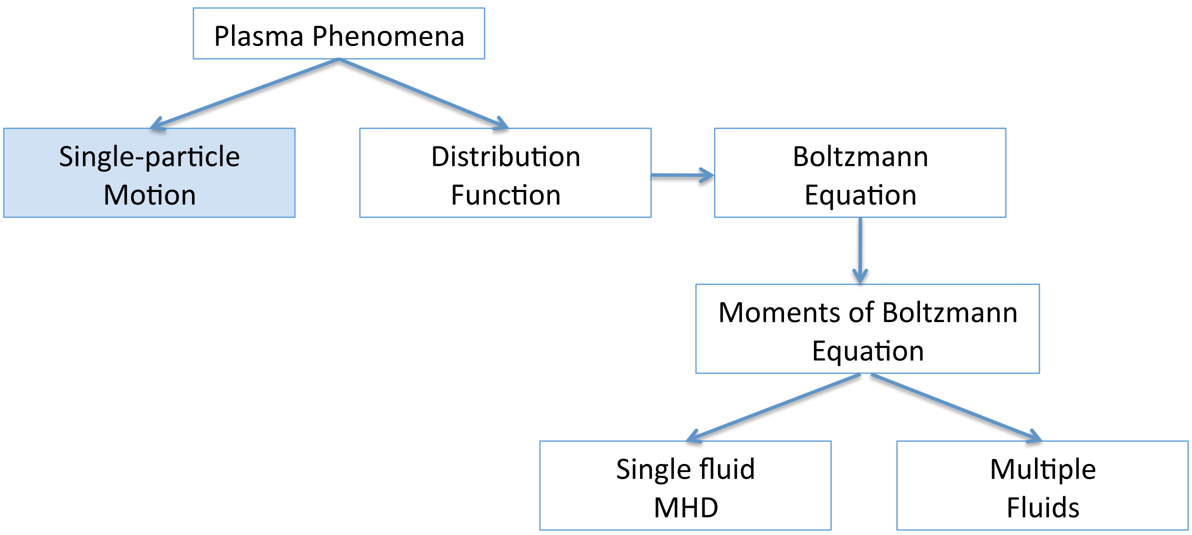
Hierarchy of plasma phenomena



Single-Particle Motion

- Motion in a uniform **B** field
- **E** x **B** drift

  } Today's lecture

- Motion in non-uniform **B** fields
  - Gradient drift
  - Curvature drift
  - Other gradient of **B**
- Time-varying E field
- Time-varying B field
- Adiabatic Invariants

Uniform E and B Fields: Gyration

- In dense plasmas, Coulomb forces couple particles, so bulk motion of plasma is significant.
- In rarefied plasmas charge particles do not interact with one another significantly, so so motion of each particle can be treated independently.
- In general, equation of motion of particle of mass *m* under influence of Lorentz force is:

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

  which is valid for non-relativistic motion ($\mathbf{v} << c$).

- If particles are relativistic, use $m = \frac{m_0}{(1 - v^2/c^2)^{1/2}}$

  where m_0 is is the rest mass.

Uniform B Fields: Gyration

- If motion only subject to static and uniform **B** field,

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v} \times \mathbf{B} \tag{3.1}$$

- Taking the dot product with **v**,

$$\mathbf{v} \cdot m\frac{d\mathbf{v}}{dt} = \mathbf{v} \cdot q(\mathbf{v} \times \mathbf{B})$$

$$m\frac{1}{2}\frac{d(\mathbf{v}\cdot\mathbf{v})}{dt} = q[\mathbf{v}\cdot(\mathbf{v}\times\mathbf{B})]$$

- RHS is zero, as $\mathbf{v} \perp \mathbf{B}$

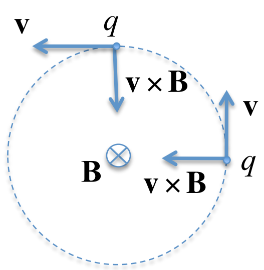
$$\Rightarrow \frac{d}{dt}\left(\frac{mv^2}{2}\right) = 0$$

- Therefore, a static magnetic field cannot change the kinetic energy of a particle since force is always perpendicular to direction of motion.

- Above is also true for spatially non-uniform **B** field.

Uniform B Fields: Gyration

- Now consider case of fields lines that are straight and parallel with constant magnetic field strength.



- Can decompose velocity into components parallel to and perpendicular to **B**:

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp}$$

- Re-writing Eqn. 3.1 as $\frac{d\mathbf{v}_{\parallel}}{dt} + \frac{d\mathbf{v}_{\perp}}{dt} = \frac{q}{m}(\mathbf{v}_{\perp} \times \mathbf{B})$

- This equation can be split in two independent equations:

$$\frac{d\mathbf{v}_{\parallel}}{dt} = 0 \Rightarrow \mathbf{v}_{\parallel} = const$$

$$\frac{d\mathbf{v}_{\perp}}{dt} = \frac{q}{m}(\mathbf{v}_{\perp} \times \mathbf{B})$$

- These imply that **B** field has no effect on the motion along it (||). Only effects particle velocity perpendicular to it.

Uniform B Fields: Cyclotron Frequency

- To examine perpendicular motion, we consider $\mathbf{B} = (0,0,\mathbf{B}_z) = \hat{\mathbf{z}}B$

- Now re-write Eqn 3.1 in component form:

$$m\frac{dv_x}{dt} = qBv_y \qquad (3.2)$$

$$m\frac{dv_y}{dt} = -qBv_x \qquad (3.3)$$

$$m\frac{dv_z}{dt} = 0$$

- To determine time variation of v_x and v_y we take derivatives of Eqns. 3.2 and 3.3, we can write

$$\frac{d^2v_x}{dt^2} + \omega_c^2 v_x = 0 \qquad (3.4)$$

$$\frac{d^2v_y}{dt^2} + \omega_c^2 v_y = 0 \qquad (3.5)$$

where $\boxed{\omega_c = -\frac{qB}{m}}$ is the *gyrofrequency* or *cyclotron frequency*.

Uniform B Fields: Cyclotron Frequency

- *Gyrofrequency* or *cyclotron frequency*: $\boxed{\omega_c = -\frac{qB}{m}}$

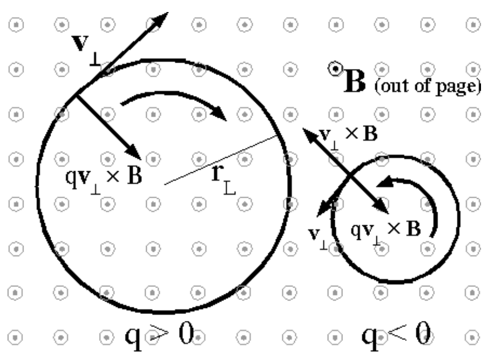
- Indicative of field strength, and charge and mass of particles.

- Does not depend on kinetic energy.

- For electrons, ω_c is positive and electrons rotate in the right-hand sense.

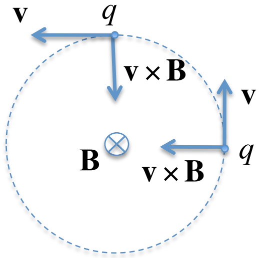
- Plasma can have several cyclotron frequencies.

- Figure from Schwartz et al., Page 23.



Uniform B Fields: Cyclotron Frequency

- The **v** x **B** force is centripetal, so



$$-\frac{mv_\perp^2}{r} = q\mathbf{v}\times\mathbf{B}$$
$$= qv_\perp B$$

Larmor radius or *gyroradius*

$$=> r_L = \frac{mv_\perp}{|q|B} = \frac{v_\perp}{\omega_c}$$

- Particles with higher velocities orbit in circles of larger radii.

- For electrons, the *gyrofrequency* can be written:

$$f_{ce} = \frac{\omega_c}{2\pi} \cong 2.8\times 10^6 B \quad \text{Hz}$$

where B is in units of Gauss.

Uniform B Fields: Helical Motion

- What is path of electrons? Solutions to Eqns. 3.4 and 3.5 are harmonic:

$$v_x = v_\perp e^{i\omega_c t} = \dot{x} \tag{3.6a}$$
$$v_y = \frac{m}{qB} v_x^2 = \pm\frac{1}{\omega_c}\dot{v}_x = \pm i v_\perp e^{i\omega_c t} = \dot{y} \tag{3.6b}$$

where $v_\perp = \sqrt{v_x^2 + v_y^2}$ is a constant speed in plan perpendicular to **B**.

- Integrating, we have $x - x_0 = -i\frac{v_\perp}{\omega_c}e^{i\omega_c t}$

$$y - y_0 = \pm\frac{v_\perp}{\omega_c}e^{i\omega_c t}$$

- Using Larmor radius (r_L), and taking the real parts of above:

$$x - x_0 = r_L \sin(\omega_c t)$$
$$y - y_0 = \pm r_L \cos(\omega_c t) \tag{3.7}$$

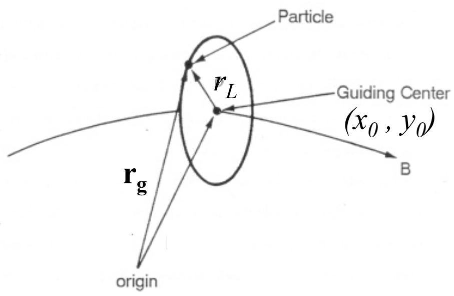
- These describe a circular orbit about a *guiding centre* (x_0, y_0).

Uniform B Fields: Helical Motion

- In addition to this motion, there is a velocity v_z along **B** which is not effected by **B**.

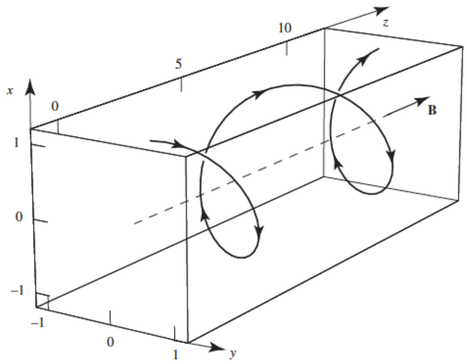
  Combined with Eqn. 3.7, this gives rise to *helical motion* about a guiding center

$$\mathbf{r_g} = \hat{\mathbf{x}}x_0 + \hat{\mathbf{y}}y_0 + \hat{\mathbf{z}}(z_0 + v_{\|}t)$$



- Guiding center moves linearly along z with constant velocity ($v_{||}$).

- Right: Electron guiding center motion in a uniform **B**-field (Inan & Golkowski).



Uniform B Fields: Helical Motion

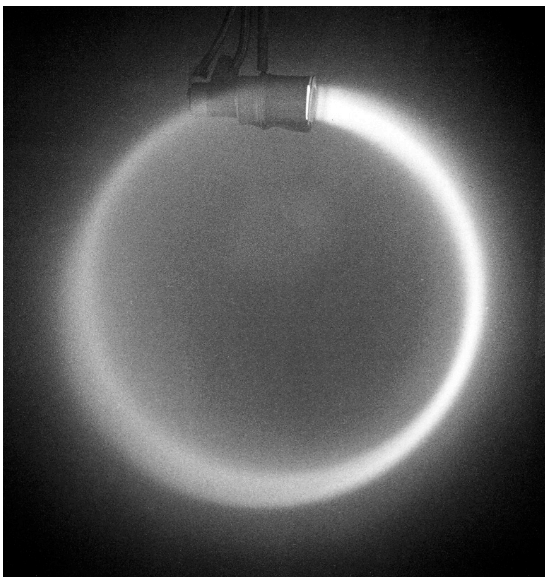
- Right: A 100 eV electron beam injected perpendicular to a DC **B**-field.

- Sense of the cyclotron orbit implies that **B**-field points into the plane.

$$\omega_c = -\frac{qB}{m}$$

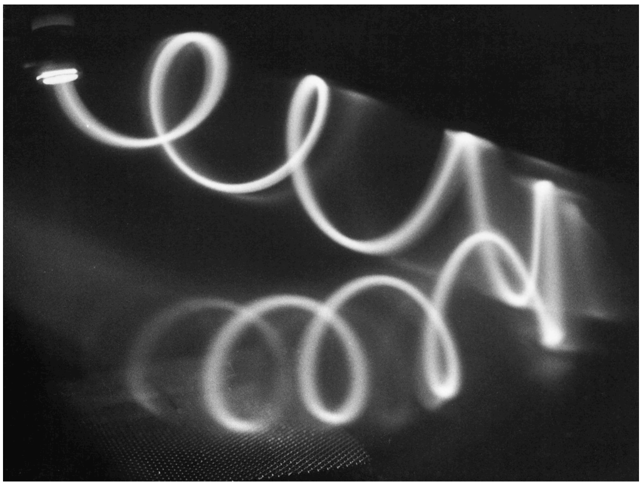
- From beam energy and the cyclotron radius the field strength can be calculated (Reiner Stenzel; UCLA).

$$r_L = \frac{mv_\perp}{|q|B}$$



Uniform B Fields: Helical Motion

- Mirror reflection of a stronger electron beam in a magnetic field which converges to the right. Note that the guiding center (axis of spiral) of the reflected beam does not coincide with that of the incident. This is due to the gradient and curvature drift in a nonuniform field – more on this in next lecture.

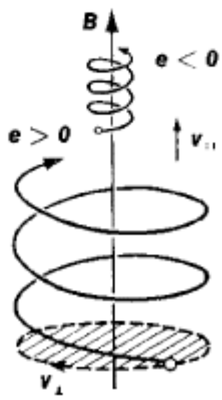


- http://www.physics.ucla.edu/plasma-exp/beam/

Helical Motion: Particle Pitch Angle

- The *pitch angle* of helix is defined as

$$\alpha = \tan^{-1}\left(\frac{v_\perp}{v_\parallel}\right)$$



- Revolution of ions and electrons in Larmor spirals weakens the external magnetic field.

- The radius of revolution of ion with a charge $q > 0$ is greater than that of an electron ($q < 0$).

- Charged particle in motion about **B** has a magnetic moment:

$$\mu = iA$$
$$= \frac{q}{\tau_c}\pi r_L^2 = \frac{q\omega_c}{2\pi}\pi r_L^2$$

- Therefore, $$\mu = \frac{1/2 m v_\perp^2}{B}$$ *Magnetic Moment* or *First Adiabatic Invariant*

Uniform E and B Fields: E x B Drift

- When **E** is finite, motion will be sum of two motions: circular Larmor gyration plus drift of the guiding centre.

- Choose **E** to lie in *x*-*z* plane, so that $E_y = 0$. Equation of motion is then:

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

- *z*-component of velocity is then $\frac{dv_z}{dt} = \frac{q}{m}E_z$

$$=> v_z = \frac{qE_z}{m}t + v_{z0}$$

- This is a straight acceleration along **B**. The transverse components are:

$$\frac{dv_x}{dt} = \frac{q}{m}E_x \pm \omega_c v_y$$
$$\frac{dv_y}{dt} = 0 \pm \omega_c v_x$$

Uniform E and B Fields: E x B Drift

- Differentiating, with constant **E**, $\ddot{v}_x = -\omega_c^2 v_x$

$$\ddot{v}_y = \pm\omega_c\left(\frac{q}{m}E_x \pm \omega_c v_y\right)$$
$$= -\omega_c^2\left(\frac{E_x}{B} + v_y\right)$$

- We can write this as $\frac{d^2}{dt^2}\left(v_y + \frac{E_x}{B}\right) = -\omega_c^2\left(v_y + \frac{E_x}{B}\right)$

  using $v_y = v_y + E_x/B$. In a form similar to Eqns. 3.6a and 3.6b:

$$v_x = v_\perp e^{i\omega_c t}$$
$$v_y = \pm i v_\perp e^{i\omega_c t} - \frac{E_x}{B}$$

- Larmor motion is similar to case when $E = 0$, but now there is a superimposed drift $\mathbf{v_g}$ of the guiding center in the $-y$ direction.

Uniform E and B Fields: E x B Drift

- To obtain a general formula for $\mathbf{v}_g$, we can solve equation of motion

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

- As $m\, d\mathbf{v}/dt$ gives rise to circular motion, already understand its effect, so set to zero.

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$$

- Taking the cross product with **B**,

$$\mathbf{E} \times \mathbf{B} = \mathbf{B} \times (\mathbf{v} \times \mathbf{B}) = \mathbf{v}B^2 - \mathbf{B}(\mathbf{v} \cdot \mathbf{B})$$

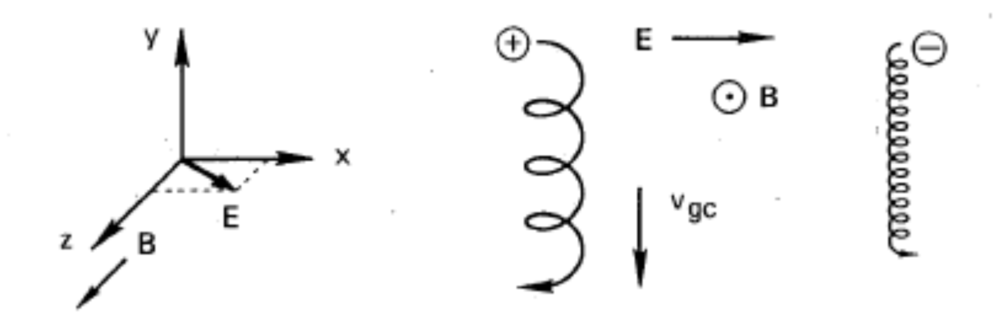
- The transverse components of this equation are

$$\mathbf{v}_{\perp gc} = \mathbf{E} \times \mathbf{B} / B^2 \equiv \mathbf{v}_E$$

  where $\mathbf{v}_E$ is the electric field drift of the guiding center, which has magnitude

$$v_E = \frac{E(V/m)}{B(T)} ms^{-1}$$

Particle drifts in crossed E and B fields



$$r_L = \frac{mv_\perp}{qB}$$

- Figure above from Chen, Introduction to Plasma Physics.

- As +ve charged particle moves to right in first part of orbit, is velocity is parallel to **E**, is therefore gains energy, and r_L increases. Opposite in second part of orbit. Acceleration and deceleration causes instantaneous gyroradius to change. Difference between radius of curvature at left and right of orbit gives drift.

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