Introduction to Plasma Physics (PY5012)
Lecture 4: Single-Particle Motion

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Single-Particle Motion

- Motion in a uniform **B** field
- **E** x **B** drift
- Motion in non-uniform **B** fields
  - Gradient drift
  - Curvature drift
  - Other gradients of **B**

  } Today's lecture
- Time-varying E field
- Time-varying B field
- Adiabatic Invariants

Drifts in uniform and non-uniform fields

- In last lecture, solved equations of motion for particles in uniform **B** and **E** fields.
- Showed that drift of guiding-centre is: $\mathbf{v}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2}$

  which can be extended to a form for a general force **F**:

$$\mathbf{v}_f = \frac{1}{q} \frac{\mathbf{F} \times \mathbf{B}}{B^2}$$

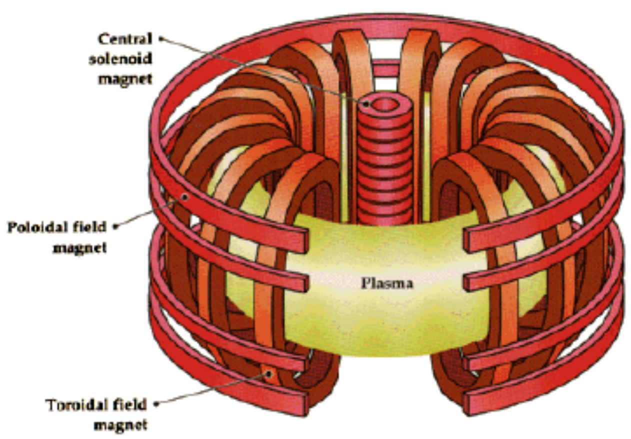
- In a gravitation field for example, $\mathbf{v}_g = \frac{m}{q} \frac{\mathbf{g} \times \mathbf{B}}{B^2}$
- Similar to the drift $\mathbf{v}_E$, in that drift is perpendicular to both forces, but in this case particles of opposite charge drift in opposite direction.
- Uniform fields provide poor descriptions for many phenomena, such as planetary fields, coronal loops, tokamaks, which have spatially and temporally varying fields.

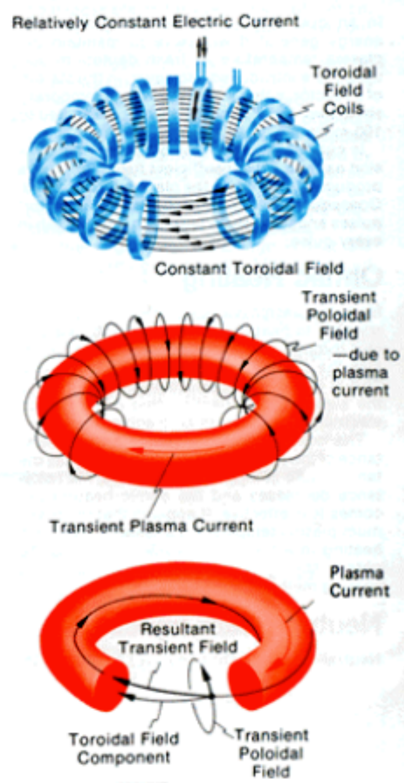
The configuration of a Tokamak.

TOroidalnaya KAmera ee MAgnitnaya Katushka,"

or "Toroidal Chamber in Magnetic Coil"

This is a device to magnetically confine a plasma, so that it can be heated to fusion temperatures.





Plasma confinement in Tokamak

Drifts in non-uniform magnetic fields

- Particle drifts in inhomogeneous fields are classified in several ways.

- In this lecture, consider two drifts associated with spatially non-uniform **B**: *gradient drift* and *curvature drift*. There are many others!

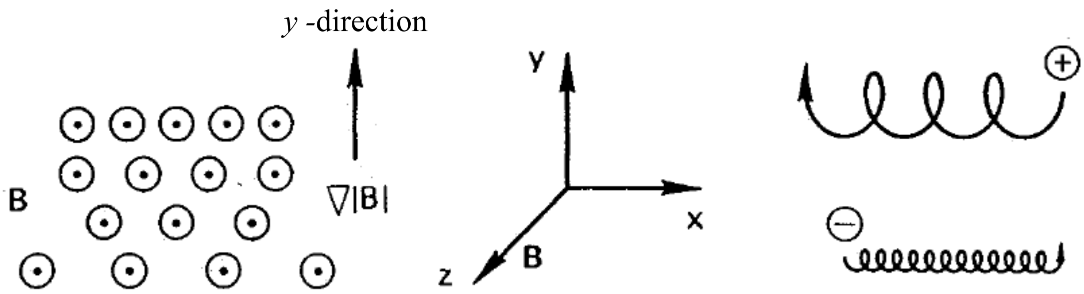
- As soon as introduce inhomogeneity, too complicated to obtain exact solutions for guiding centre drifts.

- Therefore use *orbit theory* approximation:

  - Within one Larmor orbit, **B** is approximately uniform, i.e., the typical length-scale, L, over which **B** varies is such that $L \gg r_{L.}$ => gyro-orbit is nearly a circle.

Grad-B Drift

- Assume lines of forces are straight, but their density increases in y direction (see figure below from Chen, Page 27).



- Gradient in |**B**| causes the Larmor radius ($r_L = mv/qB$) to be larger at the bottom of the orbit than the top, which leads to a drift.

- Drift should be perpendicular to grad B and **B** and ions and electrons drift in opposite directions.

Grad-B Drift

- Consider spatially-varying magnetic field, **B** = (0, 0, B_z(y)). i.e., **B** only has *z*-component, the strength of which varies with *y*.

- Assume that **E** = 0, so equation of motion is $\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$

- Separating into components,

$$F_x = q(v_y B_z) \qquad (4.1a)$$
$$F_y = -q(v_x B_z) \qquad (4.1b)$$
$$F_z = 0$$

- Now the gradient of B_z is

$$\frac{dB_z}{dy} \approx \frac{B_z}{L} << \frac{B_z}{r_L}$$
$$=> r_L \frac{dB_z}{dy} << B_z$$

- This means that the magnetic field strength can be expanded in a Taylor expansion for distances $y < r_L$:

$$B_z(y) = B_0 + y\frac{dB_z}{dy} + \ldots$$

Grad-B Drift

- Expanding B_z to first order in Eqns. 4.1a and 4.1b:

$$F_x = qv_y\left(B_0 + y\frac{dB_z}{dy}\right) \qquad (4.2)$$

$$F_y = -qv_x\left(B_0 + y\frac{dB_z}{dy}\right)$$

- Particles in **B**-field travelling about a guide centre (0,0) have a helical trajectory:

$$x = r_L \sin(\omega_c t)$$
$$y = \pm r_L \cos(\omega_c t)$$

- The velocities can be written in a similar form: $v_x = v_\perp \cos(\omega_c t)$

$$v_y = \pm v_\perp \sin(\omega_c t)$$

- Substituting these into Eqn. 4.2 gives: $F_x = -qv_\perp \sin(\omega_c t)\left(B_0 \pm r_L \cos(\omega_c t)\frac{dB_z}{dy}\right)$

$$F_y = -qv_\perp \cos(\omega_c t)\left(B_0 \pm r_L \cos(\omega_c t)\frac{dB_z}{dy}\right)$$

Grad-B Drift

- Since we are only interested in the guiding centre motion, we average force over a gyroperiod. Therefore, in the x-direction:

$$\langle F_x \rangle = -qv_\perp\left[B_0\langle \sin(\omega_c t)\rangle \pm r_L\langle \sin(\omega_c t)\cos(\omega_c t)\rangle\frac{dB_z}{dy}\right]$$

- But $< \sin(\omega_c t) > = 0$ and $< \sin(\omega_c t) \cos(\omega_c t) > = 0$ => $<F_x> = 0$.

- In the y-direction: $\langle F_y \rangle = -qv_\perp\left[B_0\langle \cos(\omega_c t)\rangle \pm r_L\langle \cos^2(\omega_c t)\rangle\frac{dB_z}{dy}\right]$

$$= \mp\frac{qv_\perp r_L}{2}\frac{dB_z}{dy} \qquad (4.3)$$

where $< \cos(\omega_c t) > = 0$ but $< \cos^2(\omega_c t) > = 1/2$.

Grad-B Drift

- In general, drift of guiding-centre is $\mathbf{v}_f = \dfrac{1}{q}\dfrac{\mathbf{F}\times\mathbf{B}}{B^2}$
- Subing in using Eqn 4.3:

$$\mathbf{v}_{\nabla B} = \frac{1}{q}\frac{\langle F_y\rangle \hat{\mathbf{y}} \times \mathbf{B}_z \hat{\mathbf{z}}}{B_z^2}$$

$$= \mp \frac{v_\perp r_L}{2B_z}\frac{dB_z}{dy}\hat{\mathbf{x}}$$

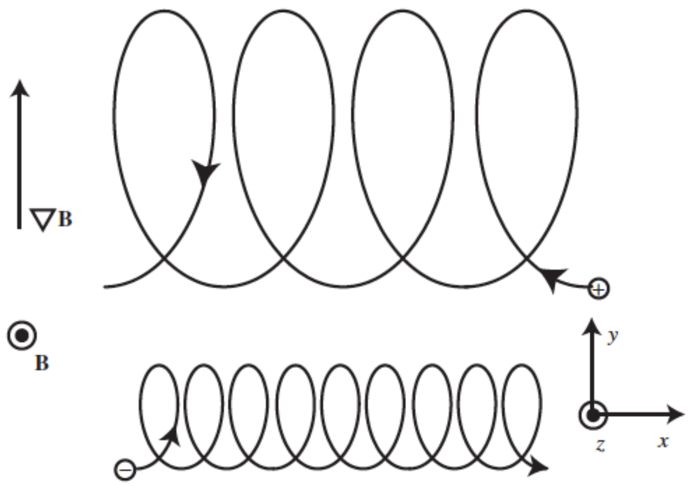
- So +ve particles drift in $-x$ direction and –ve particles drift in $+x$.
- In 3D, the result can be generalised to:

$$\mathbf{v}_{\nabla B} = \pm\frac{1}{2} v_\perp r_L \frac{\mathbf{B}\times\nabla B}{B^2} \qquad \textit{Grad-B Drift}$$

- The ± stands for sign of charge. The grad-B drift in in opposite directions for electrons and ions and causes a current transverse to **B**.

Grad-B Drift

- Below are shown particle drifts due to a magnetic field gradient, where **B**(y) = **z** B_z (y). Page 31 of Inan & Golkowski.



- Consider $r_L = mv/qB$. Local gyroradius is large where **B** is small and visa versa which gives rise to a drift. Note also that charge determines direction of drift.

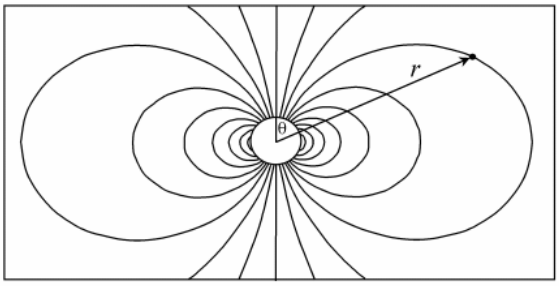
Grad-B Drift: Planetary Ring Current

- Gradient drift is responsible for current in inner parts of planetary magnetospheres.

- Approximate field is a dipole:

$$B_r = \frac{\mu_0}{4\pi}\frac{2m}{r^3}\cos(\theta)$$

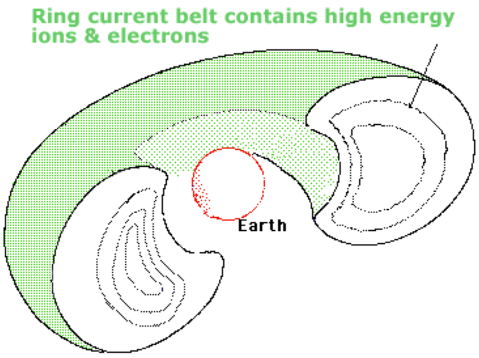
$$B_\theta = \frac{\mu_0}{4\pi}\frac{2m}{r^3}\sin(\theta)$$

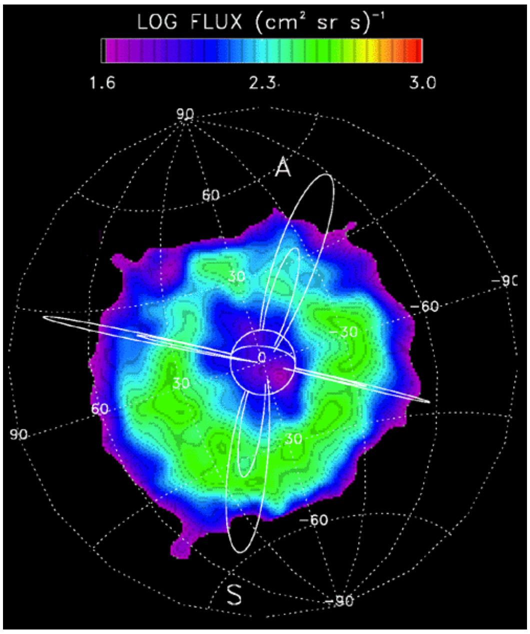


- In equatorial plane, $B_r = 0$ and $B_\theta = B_0 / r^3$. There is therefore a positive gradient in B_θ directed radialy outwards.

- There is therefore a grad-B drift perpendicular to **B** and grad-B, which produces a ring current circulating about a planet.

Grad-B Drift: Planetary Ring Current





- Right: Ring current viewed from north pole with NASA's Image satellite.

Curvature Drift

- When charged particles move along curved magnetic field lines, experience centrifugal force perpendicular to magnetic field (Fig Inan & Golkowski, Page 34).

- Assume radius of curvature (R_c) is >> r_L

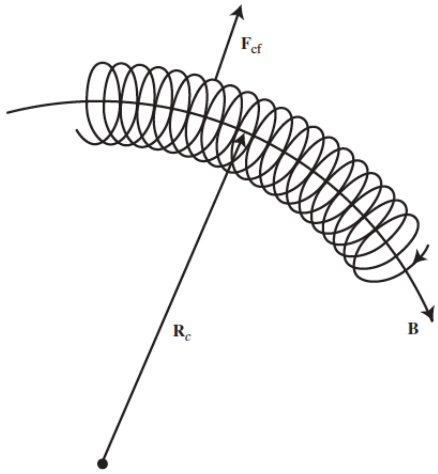
- The outward centrifugal force is

$$\mathbf{F}_{cf} = \frac{mv_{\parallel}^2}{R_c}\hat{\mathbf{r}}$$

- This can be directly inserted into general form for guiding-centre drift

$$\mathbf{v}_f = \frac{1}{q}\frac{\mathbf{F}\times\mathbf{B}}{B^2}$$

$$\Rightarrow \mathbf{v}_R = \frac{mv_{\parallel}^2}{qR_c^2}\frac{\mathbf{R}_c\times\mathbf{B}}{B^2}$$



Curvature Drift

- Drift is therefore into or out of page depending on sign of q.

Curvature and grad-B drift

- Type of field configuration studied above – having curved but parallel field lines – will never occur in reality, since $\nabla\cdot\mathbf{B}\neq 0$ for this field.

- In practice, curved field lines will always be converging/diverging. Thus a curvature drift will always be accompanied by a grad-B drift.

- Where ∇B is in the opposite direction to R_c, the grad-B and curvature drifts act in the same direction.

- Right is a cross-section of a tokamak magnetic field.

- Dashed lines are field lines. In the highlighed section, the field is lower at edge than in the centre. So curvature and gradient drift act in same direction - outwards for ions – so ions hit the vessel walls and are lost.