Introduction to Plasma Physics (PY5012)
Lecture 5: Magnetic Mirroring

Dr. Peter T. Gallagher
Astrophysics Research Group
Trinity College Dublin

Adiabatic Invariance of magnetic moment

- Gyrating particle constitutes an electric current loop with a dipole moment:

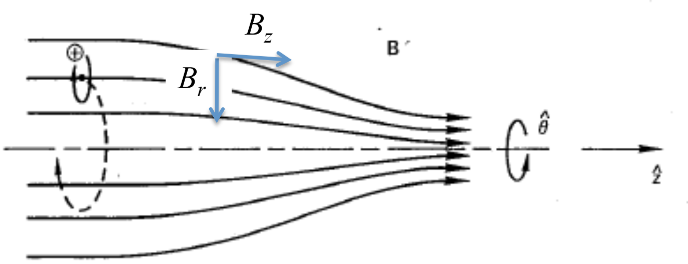
$$\mu = \frac{1/2 m v_\perp^2}{B}$$

- The dipole moment is conserved, i.e., is invariant. Called the *first adiabatic invariant.*

- μ = constant even if B varies spatially or temporally. If B varies, then v_{perp} varies to keep μ = constant => $v_{||}$ also changes.

- Gives rise to *magnetic mirroring*. Seen in planetary magnetospheres, magnetic bottles, coronal loops, etc.

- Right is geometry of mirror from Chen, Page 30.



Magnetic mirroring

- Consider B-field pointed primarily in z-direction and whose magnitude varies in z-direction. If field is axisymmetric, $B_\theta = 0$ and $d/d\theta = 0$.

- This has cylindrical symmetry, so write $\mathbf{B} = B_r\hat{\mathbf{r}} + B_z\hat{\mathbf{z}}$

- How does this configuration give rise to a force that can trap a charged particle?

- Can obtain B_r from $\nabla \cdot \mathbf{B} = 0$. In cylindrical polar coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_z}{\partial z} = 0$$

$$=> \frac{\partial}{\partial r}(rB_r) = -r\frac{\partial B_z}{\partial z}$$

- If $\partial B_z / \partial z$ is given at $r = 0$ and does not vary much with r, then

$$rB_r = -\int_0^r r\frac{\partial B_z}{\partial z}dr \approx -\frac{1}{2}r^2\left[\frac{\partial B_z}{\partial z}\right]_{r=0}$$

$$B_r = -\frac{1}{2}r\left[\frac{\partial B_z}{\partial z}\right]_{r=0} \qquad (5.1)$$

Magnetic mirroring

- Now have $\mathbf{B_r}$ in terms of $\mathbf{B_Z}$, which we can use to find Lorentz force on particle.

- The components of Lorentz force are:

$$\mathbf{F}_r = q(\underset{(1)}{v_\theta B_z} - v_z \cancel{B_\theta})$$

$$\mathbf{F}_\theta = q(\underset{(2)}{-v_r B_z} + \underset{(3)}{v_z B_r})$$

$$\mathbf{F}_z = q(v_r \cancel{B_\theta} - \underset{(4)}{v_\theta B_r})$$

- As $B_\theta = 0$, two terms vanish and terms (1) and (2) give rise to Larmor gyration. Term (3) vanishes on the axis and causes a drift in radial direction. Term (4) is therefore the one of interest.

- Substituting from Eqn. 5.1:

$$\mathbf{F}_z = -qv_\theta B_r = \frac{qv_\theta r}{2}\frac{\partial B_z}{\partial z}$$

Magnetic mirroring

- Averaging over one gyro-orbit, and putting $v_\theta = \mp v_\perp$ and $r = r_L$

$$\mathbf{F}_z = \mp \frac{q v_\perp r_L}{2} \frac{\partial B_z}{\partial z}$$

- This is called the *mirror force,* where -/+ arises because particles of opposite charge orbit the field in opposite directions.

- Above is normally written:
$$\mathbf{F}_z = \mp \frac{1}{2} q \frac{v_\perp^2}{\omega_c} \frac{\partial B_z}{\partial z} = -\frac{1}{2} \frac{m v_\perp^2}{B} \frac{\partial B_z}{\partial z}$$

or $\mathbf{F}_z = -\mu \frac{\partial B_z}{\partial z}$ where $\mu \equiv \frac{1/2 m v_\perp^2}{B}$ is the magnetic moment.

- In 3D, this can be generalised to: $\mathbf{F}_{||} = -\mu \frac{dB}{d\mathbf{s}} = -\mu \nabla_{||} B$

where $\mathbf{F}_{||}$ is the mirror force parallel to **B** and $d\mathbf{s}$ is a line element along **B**.

First adiabatic invariant

- As particle moves into regions of stronger or weaker **B**, Larmor radius changes, but μ remains invariant.

- To prove this, consider component of equation of motion along **B**:

$$m \frac{dv_{||}}{dt} = -\mu \frac{dB}{ds}$$

- Multiplying by $v_{||}$:

$$m v_{||} \frac{dv_{||}}{dt} = -\mu v_{||} \frac{dB}{ds}$$

- Then,

$$\frac{d}{dt}\left(\frac{1}{2} m v_{||}^2\right) = -\mu \frac{ds}{dt} \frac{dB}{ds} = -\mu \frac{dB}{dt} \qquad (5.2)$$

- The particle's energy must be conserved, so $\frac{d}{dt}\left(\frac{1}{2} m v_{||}^2 + \frac{1}{2} m v_\perp^2\right) = 0$

- Using $\mu \equiv \frac{1/2 m v_\perp^2}{B}$

$$\frac{d}{dt}\left(\frac{1}{2} m v_{||}^2 + \mu B\right) = 0$$

First adiabatic invariant

- Using Eqn 5.2,

$$-\mu\frac{dB}{dt}+\frac{d}{dt}(\mu B)=0$$

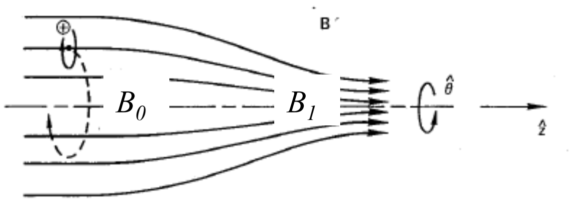
$$-\mu\frac{dB}{dt}+\mu\frac{dB}{dt}+B\frac{d\mu}{dt}=0$$

$$=> B\frac{d\mu}{dt}=0$$

- As B is not equal to 0, this implies that $\frac{d\mu}{dt}=0$
- That is, μ = constant in time (invariant).
- μ *is* known as the *first adiabatic invariant* of the particle orbit.
- As a particle moves from a weak-field to a strong-field region, it sees B increasing and therefore v_{perp} must increase in order to keep μ constant. Since total energy must remain constant, $v_{||}$ must decrease.
- If B is high enough, $v_{||}$ eventually -> 0 and particle is reflected back to weak-field.

Consequence of invariance of μ: Magnetic mirroring

- Consider B_0 and B_1 in the weak- and strong-field regions. Associated speeds are v_0 and v_1.



- The conservation of μ implies that $\mu=\frac{mv_0^2}{2B_0}=\frac{mv_1^2}{2B_1}$
- So, as B increases, the perpendicular component of the particle velocity increases: particle moves more and more perpendicular to B.
- However, since we have $E=0$, the total particle energy cannot increase. Thus as $v_\perp$ increases, $v_{||}$ must decrease. The particle slows down in its motion along the field.
- If field convergence is strong enough, at some point the particle may have $v_{||}$ = 0.

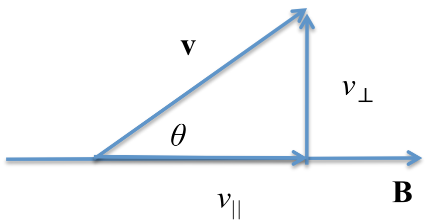
Consequence of invariance of μ: Magnetic mirroring

- Say that at B_1 we have $v_{1,||} = 0$. From conservation of energy:

$$v_1^2 = v_{1,\perp}^2 = v_0^2$$

- Using $\mu = \frac{mv_0^2}{2B_0} = \frac{mv_1^2}{2B_1}$ we can write

$$\frac{B_0}{B_1} = \frac{v_{0,\perp}^2}{v_{1,\perp}^2} = \frac{v_{0,\perp}^2}{v_0^2}$$



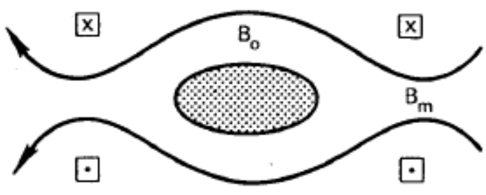
- But $\sin(\theta) = v_\perp / v_0$ where θ is the *pitch angle.*

- Therefore $\frac{B_0}{B_1} = \sin^2(\theta)$

- Particles with smaller θ will mirror in regions of higher B. If θ is too small, $B_1 >> B_0$ and particle does not mirror.

Consequence of invariance of μ: Magnetic mirroring

- *Mirror ratio* is defined $R_m = \frac{B_m}{B_0}$



A plasma trapped between magnetic mirrors.

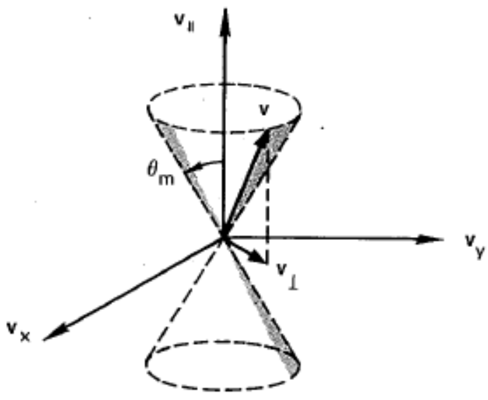
- So, the smallest θ of a confined particle is

$$\sin^2(\theta) = \frac{B_0}{B_m} = \frac{1}{R_m}$$

- This defines region in velocity space in the shape of a cone, called the *loss cone.*

The loss cone

- Particles are confined in a mirror with ratio R_m if they have $\theta > \theta_m$.
- Otherwise they escape the mirror. That portion of velocity space occupied by escaping particles is called the *loss-cone.*
- The opening angle of the loss cone is not dependent on mass or charge. Electrons and protons are lost equally, if the plasma is collisionless.



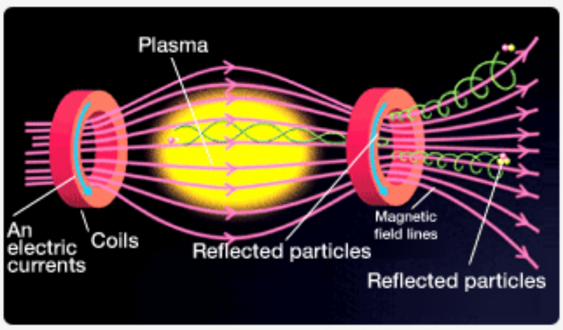
The loss cone.

- Coulomb collisions scatter charged particles, changing their pitch angle. Thus a particle which originally lay outside the loss cone can be scattered into it.
- Electrons are more readily Coulomb-scattered than ions; thus electrons will be scattered out of the mirror trap more quickly.

Bounce motion between two mirrors

- Early magnetic confinement machines involved designs using magnetic mirror. The original idea was based on the fact that an electric current generates a magnetic field, and that the currents flowing in the plasma will "pinch" the plasma, containing it within its own magnetic field.

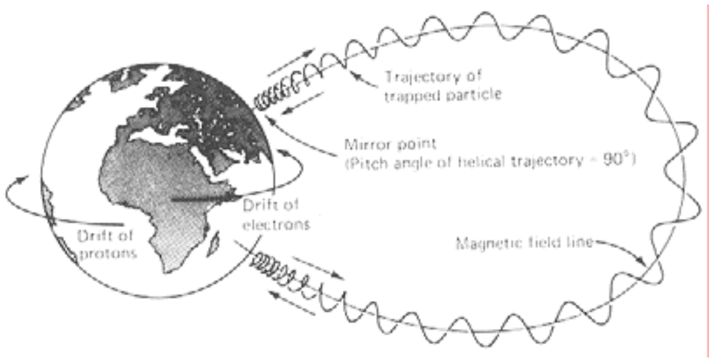
Magnetic Mirror



When two mirrors are placed at either end of a solenoid to increase the internal volume of the reaction area, magnetic field lines appear as in the figure. The mirror effect results in a tendency for charged particles to bounce back from the high field region. Therefore, the plasma can be confined between coils. However, some particles of plasma escape from the ends.

Bounce motion between two mirrors

- A particle outside the loss-cone in a field structure which is convergent at both ends (such as the Earth's magnetic field) will be reflected by both mirrors and bounce between them.

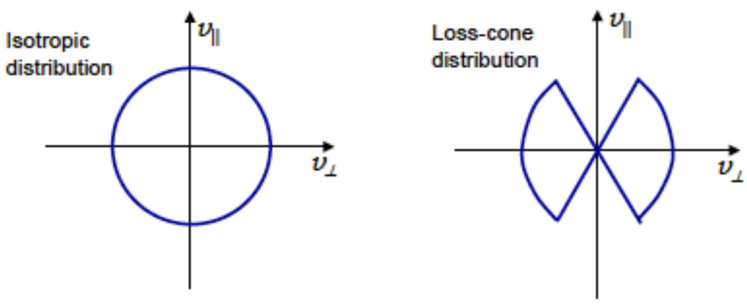


- Superposed on this bounce will be drift motions. For example, particle orbits in the Earth's magnetic field are a combination of gyromotion, bounce motion, and grad-B drift.

Loss cone distributions and instabilities

- Consider an initially Maxwellian distribution of particles inside a magnetic
- mirror. In velocity space, the distribution is spherical.

- As particles exit via the loss cone, the distribution becomes anisotropic



- The distribution will try to relax back to an isotropic one (higher entropy); one of the ways it does this is by radiating energy in the form of plasma waves. These waves are somewhat more complex than cold plasma wave - they involve the distribution functions of particles. They are generally called *kinetic plasma waves, driven by a kinetic instability.*