Introduction to Plasma Physics (PY5012)
Lecture 6: Kinetic Theory

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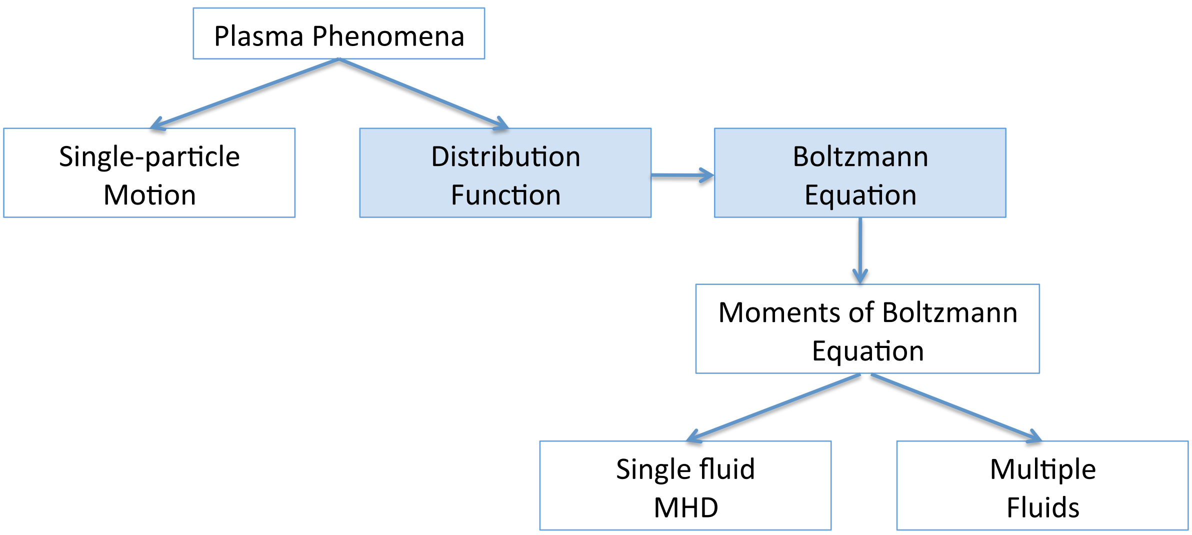
Hierarchy of plasma phenomena



Particle Phase Space

- A particle's dynamical state can be specified using its position and velocity:

$$\mathbf{r} = (\, x, y, z \,) \text{ and } \mathbf{v} = (\, v_x, v_y, v_z \,)$$

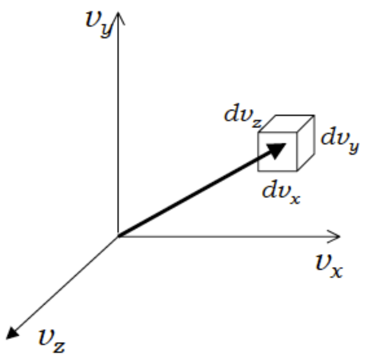
- Combining position and velocity information, gives particle's position in phase space:

$$(\, \mathbf{r}, \mathbf{v} \,) = (\, x, y, z, v_x, v_y, v_z \,)$$

- The state space for position and momentum or velocity is a 6D phase space

- Volume of a small element of velocity space is $dv_x dv_y dv_z = d^3v = d\mathbf{v}$

- Volume element in phase space is $d^3r d^3v$



Velocity Distribution Functions

- Single-particle approach has limited application where collective motion not important. Non-zero electric fields in a plasma generally arise self-consistently, so must consider collective motion of many plasma particles.

- State of plasma described by *velocity distribution function:*

$$f(\, x, y, z, v_x, v_y, v_z, t \,) \qquad \text{particles / m}^3$$

- Gives the number of particle per unit volume as a position **r** and a time t with velocities v_x, v_y, v_z. Has 7 independent variables, defining 6D phase space.

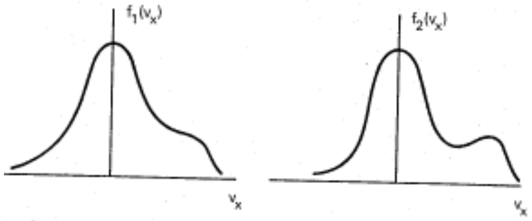
- Number of particles in a phase space volume $d^3r\, d^3v$ is

$$dn = f(\, r, v, t \,)\, dx\, dy\, dz\, dv_x\, dv_y\, dv_z = f(\, r, v, t \,)\, d^3r\, d^3v$$

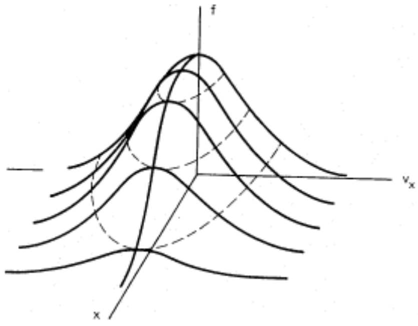
- The total number of particles is therefore $n = \int_{-\infty}^{\infty} f(\mathbf{r},\mathbf{v},t) d^3r d^3v$

Velocity Distribution Functions

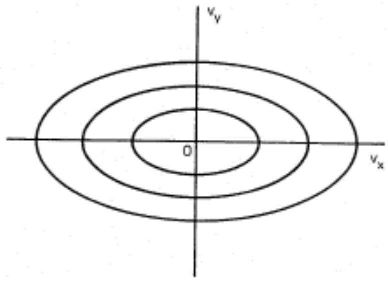
- Figures below from Chen.



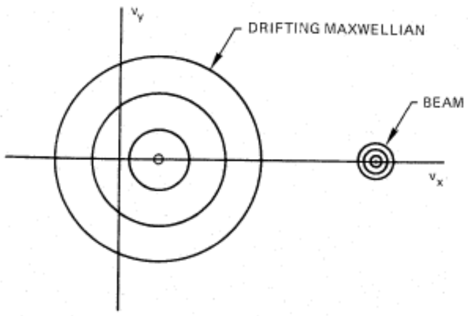
Examples of non-Maxwellian distribution functions.



A spatially varying one-dimensional distribution $f(x, v_x)$.



Contours of constant f for a two-dimensional, anisotropic distribution.



Contours of constant f for a drifting Maxwellian distribution and a "beam" FI in two dimensions.

Mathematical Aside: Moments

- Let *f(x)* be any function that is defined and positive on an interval *[a,b]*. The moments of this function are defined as

$$\begin{aligned}
&\text{Zeroth moment} && M_0 = \textstyle\int_a^b f(x)dx \\
&\text{First moment} && M_1 = \textstyle\int_a^b x f(x)dx \\
&\text{Second moment} && M_2 = \textstyle\int_a^b x^2 f(x)dx \\
& && \vdots \\
&\text{n}^{\text{th}}\text{ moment} && M_n = \textstyle\int_a^b x^n f(x)dx
\end{aligned}$$

- In particular case that distribution is a probability density, *p(x)*, then

$$\begin{aligned}
M_0 &= 1 \\
M_1 &= \textstyle\int_a^b x p(x)dx = \langle x \rangle = \text{mean}(x) \\
M_2 &= \textstyle\int_a^b x^2 p(x)dx = \text{variance}(x)
\end{aligned}$$

- Higher order moments correspond to skewness and kurtosis.

Moments of Distribution Function

- Velocity distribution function gives *microscopic* description of statistical information on particles. However, most important use is in determining *macroscopic* (i.e., ensemble averages) values such as as density, current, etc.

- *Zeroth order moment* of *f*(**r**, **v**, *t*) is $n(\mathbf{r},t) = \int_{-\infty}^{\infty} f(\mathbf{r},\mathbf{v},t)d^3v$

- *First order moment* is the bulk velocity: $\mathbf{u} = \frac{1}{n}\int \mathbf{v}f(\mathbf{r},\mathbf{v},t)d^3v$

- Charge and current density of species (s) can be expressed in using moments:

$$\rho = \sum_s q_s n_s$$

$$\mathbf{j} = \sum_s q_s n_s \mathbf{u}$$

- *Second order moment* relates to kinetic energy:

$$\left\langle \frac{1}{2}m\mathbf{v}^2 \right\rangle = \frac{1}{n}\int \frac{1}{2}m\mathbf{v}^2 f(\mathbf{r},\mathbf{v},t)d^3v$$

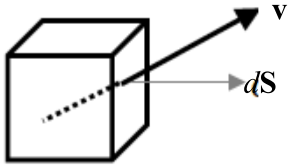
Derivation of Boltzmann Equation

- Evolution of *f*(**v**, **r**, *t*) is described by the *Boltzmann Equation.*

- Consider particles entering and leaving a small volume of phase space. As **r** and **v** are independent, can treat separately.

- Position: Number of particles leaving d^3r per second through its surface $d\mathbf{S}$ is

$$\int f(\mathbf{r},\mathbf{v},t)\dot{\mathbf{r}}\cdot d\mathbf{S} = \int f(\mathbf{r},\mathbf{v},t)\mathbf{v}\cdot d\mathbf{S}$$



- Velocity: Number of particles leaving d^3v per second through its surface $d\mathbf{S}_v$ is

$$\int f(\mathbf{r},\mathbf{v},t)\dot{\mathbf{v}}\cdot d\mathbf{S}_v = \int f(\mathbf{r},\mathbf{v},t)\mathbf{a}\cdot d\mathbf{S}_v$$

- So, the net number of particles leaving the phase space volume $d^3r d^3v$ is

$$\int f(\mathbf{r},\mathbf{v},t)\mathbf{v}\cdot d\mathbf{S}d^3v + \int f(\mathbf{r},\mathbf{v},t)\mathbf{a}\cdot d\mathbf{S}_v d^3r$$

Derivation of Boltzmann Equation

- The rate of change of particle number in d^3rd^3v is:

$$\frac{\partial}{\partial t}\left[\int f d^3r d^3v\right] = -\left[\int f\mathbf{v}\cdot d\mathbf{S}d^3v + \int f\mathbf{a}\cdot d\mathbf{S}_v d^3r\right]$$

- As total number of particles in d^3rd^3v is conserved:

$$\frac{\partial}{\partial t}\left[\int f d^3r d^3v\right] + \left[\int f\mathbf{v}\cdot d\mathbf{S}d^3v + \int f\mathbf{a}\cdot d\mathbf{S}_v d^3r\right] = 0$$

- Recall Gauss' Divergence Theorem: $\int_V (\nabla\cdot\mathbf{F})dV = \int_S (\mathbf{F}\cdot\mathbf{n})dS$

- Can now change integral over $d\mathbf{S}$ to d^3r:

$$\frac{\partial}{\partial t}\left[\int f d^3r d^3v\right] + \left[\int \nabla_r\cdot(f\mathbf{v})d^3rd^3v + \int \nabla_v\cdot(f\mathbf{a})d^3vd^3r\right] = 0$$

or

$$\frac{\partial}{\partial t}\left[\int f d^3r d^3v\right] + \left[\int \frac{\partial}{\partial\mathbf{r}}\cdot(f\mathbf{v})d^3rd^3v + \int \frac{\partial}{\partial\mathbf{v}}\cdot(f\mathbf{a})d^3vd^3r\right] = 0$$

Derivation of Boltzmann Equation

- Now, the phase space volume can be made arbitrarily small, such that the integrands are constant within the volume. Therefore, we have

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial\mathbf{r}}\cdot(f\mathbf{v}) + \frac{\partial}{\partial\mathbf{v}}\cdot(f\mathbf{a}) = 0$$

- But since **r** and **v** are independent variables, we can take **v** outside $d/d\mathbf{r}$ and similarly for **a**, we can write

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial\mathbf{r}} + \mathbf{a}\cdot\frac{\partial f}{\partial\mathbf{v}} = 0$$

- Replacing **a** with **F** / m, we have

$$\boxed{\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial\mathbf{r}} + \frac{\mathbf{F}}{m}\cdot\frac{\partial f}{\partial\mathbf{v}} = 0} \qquad (6.1)$$

- This is the *collisionless Boltzman equation.* Can be used in hot plasmas where collisions can be neglected.

The Vlasov Equation

- Previous equation written in terms of generalised force. For plasmas, Lorentz force is of interest, so

$$\boxed{\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} + \frac{q}{m}[\mathbf{E} + (\mathbf{v}\times\mathbf{B})]\cdot\frac{\partial f}{\partial \mathbf{v}} = 0}$$

- This is called the *Vlasov equation.* Can also be written

$$\boxed{\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f + \frac{q}{m}[\mathbf{E} + (\mathbf{v}\times\mathbf{B})]\cdot\frac{\partial f}{\partial \mathbf{v}} = 0} \qquad (6.2)$$

- This is one of the most important and widely used equations in kinetic theory of plasmas.

- Maxwell's equations for **E** and **B** and the Vlasov equation represent a complete set of self-consistent equations.

Convective Derivative in Phase Space

- Distribution function *f(* **r**, **v**, t) depends on 7 independent variables. Total time derivative of *f* is:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$
$$+ \frac{\partial f}{\partial v_z}\frac{\partial v_z}{\partial t} + \frac{\partial f}{\partial v_z}\frac{\partial v_z}{\partial t} + \frac{\partial f}{\partial v_z}\frac{\partial v_z}{\partial t}$$

- This can be written as $\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} + \mathbf{a}\cdot\frac{\partial f}{\partial \mathbf{v}}$

- To appreciate meaning of this equation, consider *f* = *f(***r**, t):

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} \equiv \frac{Df}{Dt}$$

- Called the *convective derivative* or *Lagrangian derivative*. Second term gives change in *f* measured by an observed moving in the fluid frame.

Phase Space Evolution

- A plasma particle's state (**r**, **v**) evolves in phase space. In absence of collisions, points move along continuous corves and f obeys the continuity equation:

$$\frac{\partial f}{\partial t} + \nabla_{\mathbf{r},\mathbf{v}} \cdot \left[(\dot{\mathbf{r}}, \dot{\mathbf{v}}) f \right] = 0$$

- Called *Liouville Equation.*

- The Liouville equation describes the time evolution of the phase space distribution function. Liouvilles' theorem states that flows in phase space are incompressible.

- In Cartesian coordinates, equation above reduces to

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{r}} \cdot (f \dot{\mathbf{r}}) + \frac{\partial}{\partial \mathbf{v}} \cdot (f \dot{\mathbf{v}}) = 0$$

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0$$

- Which is in the form of the collisionless Boltzmann equation. The Boltzmann and Vlasov equations follow from Liouville's equation.

Collisional Boltzmann and Vlasov Equations

- In the presence of collisions, the Boltzmann equation (Eqn 6.1) can be written

$$\boxed{\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{\mathbf{F}}{m} \cdot \frac{\partial f}{\partial \mathbf{v}} = \left(\frac{\partial f}{\partial t} \right)_{coll}}$$

  where the term on the right is the time rate of change of f due to collisions. This is the *collisional Boltzmann equation.*

- Similarly, the Vlasov equation (Eqn. 6.2) can be written

$$\boxed{\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q}{m} [\mathbf{E} + (\mathbf{v} \times \mathbf{B})] \cdot \frac{\partial f}{\partial \mathbf{v}} = \left(\frac{\partial f}{\partial t} \right)_{coll}}$$

- This is the *collisional Vlasov equation.* Describes change in particle distribution due to short-range interactions.

- When there are collisions with neutral atoms: $\left(\frac{\partial f}{\partial t} \right)_{coll} \approx \frac{f_n - f}{\tau}$

  where f_n is the neutral atom distribution function and τ is the collision time. Called Krook collision model.

Hierarchy of plasma phenomena