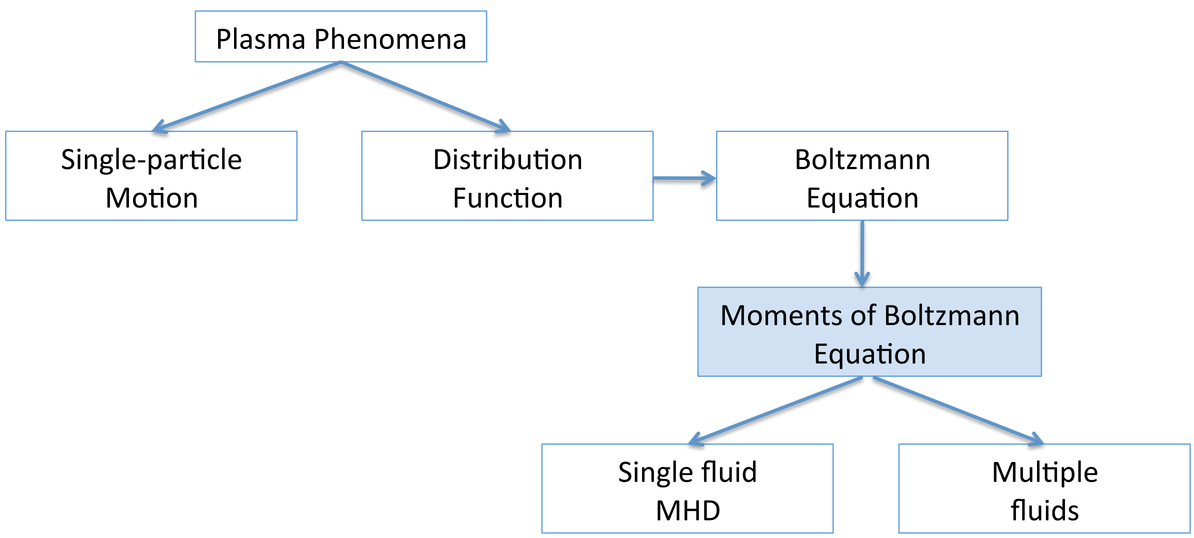
Introduction to Plasma Physics (PY5012)
Lecture 7: Moments of Boltzmann-Vlasov Equation

Dr. Peter T. Gallagher
Astrophysics Research Group
Trinity College Dublin

Hierarchy of plasma phenomena



Moments of Boltzmann-Vlasov Equation

- Under certain assumptions not necessary to obtain actual distribution function if only interested in the macroscopic values.

- Instead of solving Boltzmann or Vlasov equation for distribution function and integrating, can take integrals over *collisional Boltzmann-Vlasov* equation and solve for the quantities of interest.

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{q}{m}[\mathbf{E} + (\mathbf{v} \times \mathbf{B})] \cdot \frac{\partial f}{\partial \mathbf{v}} = \left(\frac{\partial f}{\partial t}\right)_{coll} \qquad (7.1)$$

- Called "taking the moments of the Boltzmann-Vlasov equation" .

- Resulting equations known as the macroscopic transport equations, and form the foundation of plasma fluid theory.

- Results in derivation of the *equations of magnetohydrodynamics (MHD)*.

Zeroth-order Moment: Continuity Equation

- Lowest order moment obtained by integrating Eqn. 7.1:

$$\int \frac{\partial f}{\partial t} d\mathbf{v} + \int \mathbf{v} \cdot \nabla f d\mathbf{v} + \int \frac{q}{m}[\mathbf{E} + (\mathbf{v} \times \mathbf{B})] \cdot \frac{\partial f}{\partial \mathbf{v}} d\mathbf{v} = \int \left(\frac{\partial f}{\partial t}\right)_{coll} d\mathbf{v}$$

- The first term gives $\int \frac{\partial f}{\partial t} d\mathbf{v} = \frac{\partial}{\partial t} \int f d\mathbf{v} = \frac{\partial n}{\partial t} \qquad (7.2)$

- Since **v** and **r** are independent, **v** is not effected by gradient operator:

$$\int \mathbf{v} \cdot \nabla f d\mathbf{v} = \nabla \cdot \int \mathbf{v} f d\mathbf{v}$$

- From before, the first order moment of distribution function is

$$\mathbf{u} = \frac{1}{n} \int \mathbf{v} f(\mathbf{r}, \mathbf{v}, t) d\mathbf{v}$$

therefore,

$$\int \mathbf{v} \cdot \nabla f d\mathbf{v} = \nabla \cdot (n\mathbf{u}) \qquad (7.3)$$

Zeroth-order Moment: Continuity Equation

- For the third term, consider **E** and **B** separately. **E** term vanishes as

$$\int \mathbf{E}\cdot\frac{\partial f}{\partial \mathbf{v}}d\mathbf{v} = \int \frac{\partial}{\partial \mathbf{v}}\cdot(f\mathbf{E})d\mathbf{v} = \int f\mathbf{E}\cdot d\mathbf{S} = 0 \qquad (7.4a)$$

using Gauss' Divergence Theorem in velocity space. The surface area in velocity space goes as v^2. As $v \to \infty$, $f \to 0$ more quickly than $\mathbf{S} \to \infty$ (e.g., f typically goes as $1/v^2$. A Maxwellian goes as e^{-v^2}). Integral to **v** = infinity therefore goes to zero.

- Using vector identity $\nabla\cdot(a\mathbf{A}) = \mathbf{A}\cdot\nabla a + a\nabla\cdot\mathbf{A}$ the $\mathbf{v}\times\mathbf{B}$ term is

$$\int(\mathbf{v}\times\mathbf{B})\cdot\frac{\partial f}{\partial \mathbf{v}}d\mathbf{v} = \int \frac{\partial}{\partial \mathbf{v}}\cdot(f\mathbf{v}\times\mathbf{B})d\mathbf{v} - \int f\frac{\partial}{\partial \mathbf{v}}\cdot(\mathbf{v}\times\mathbf{B})d\mathbf{v}$$

$$= \int f(\mathbf{v}\times\mathbf{B})\cdot d\mathbf{S} - \int f\frac{\partial}{\partial \mathbf{v}}\cdot(\mathbf{v}\times\mathbf{B})d\mathbf{v} = 0 \qquad (7.4b)$$

- The first term on right again vanishes as $f \to 0$ more quickly than $\mathbf{S} \to \infty$. The second vanishes as $\mathbf{v}\times\mathbf{B}$ is perpendicular to $\partial/\partial\mathbf{v}$.

Zeroth-order Moment: Continuity Equation

- Last term is on right hand side of Eqn. 7.1:

$$\int\left(\frac{\partial f}{\partial t}\right)_{coll}d\mathbf{v} = \frac{\partial}{\partial t}\left[\int f d\mathbf{v}\right] = 0 \qquad (7.5)$$

- This is assuming that the total number of particles remains constant as collisions proceed.

- Combining Eqns. 7.2 – 7.5 yields the *equation of continuity*:

$$\boxed{\frac{\partial n}{\partial t} + \nabla\cdot(n\mathbf{u}) = 0} \qquad (7.6)$$

- First term represents rate of change of particle concentration within a volume, while the second term represents the divergence of particles of the flow of particles out of the volume.

- Eqn. 7.6 is the *first of the equations of magnetohydrodynamics (MHD)*. Eqn. 7.6 is a continuity equation for mass or charge transport if we multiply *m* or *q*.

First-order Moment: Momentum Transport

- Re-write Eqn. 7.1:

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f + \frac{q}{m}[\mathbf{E}+(\mathbf{v}\times\mathbf{B})]\cdot\frac{\partial f}{\partial \mathbf{v}} = \left(\frac{\partial f}{\partial t}\right)_{coll}$$

- Next moment of the Boltzmann equation is obtained by multiplying Eqn. 7.1 by $m\mathbf{v}$ and integrating over d**v**.

$$m\int \mathbf{v}\frac{\partial f}{\partial t}d\mathbf{v} + m\int \mathbf{v}(\mathbf{v}\cdot\nabla)fd\mathbf{v} + q\int \mathbf{v}[\mathbf{E}+(\mathbf{v}\times\mathbf{B})]\cdot\frac{\partial f}{\partial \mathbf{v}}d\mathbf{v} = \int m\mathbf{v}\left(\frac{\partial f}{\partial t}\right)_{coll}d\mathbf{v} \qquad (7.7)$$

- The right-hand side is the change of the momentum due to collisions and will be given the term $\mathbf{P}_{ij}$.

- The first term gives

$$m\int \mathbf{v}\frac{\partial f}{\partial t}d\mathbf{v} = m\frac{\partial}{\partial t}\int \mathbf{v}fd\mathbf{v}$$
$$= m\frac{\partial(n\mathbf{u})}{\partial t} \qquad (7.8)$$

First-order Moment: Momentum Transport

- Next consider third term:

$$\int \mathbf{v}[\mathbf{E}+(\mathbf{v}\times\mathbf{B})]\cdot\frac{\partial f}{\partial \mathbf{v}}d\mathbf{v} = \int \frac{\partial}{\partial \mathbf{v}}\cdot[f\mathbf{v}(\mathbf{E}+\mathbf{v}\times\mathbf{B})]d\mathbf{v}$$
$$-\int f\mathbf{v}\frac{\partial}{\partial \mathbf{v}}\cdot(\mathbf{E}+\mathbf{v}\times\mathbf{B})d\mathbf{v}$$
$$-\int f(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial}{\partial \mathbf{v}}\mathbf{v}d\mathbf{v}$$

- The first to integrals on the right vanish for same reasons as before.

- Therefore have,

$$q\int \mathbf{v}[\mathbf{E}+(\mathbf{v}\times\mathbf{B})]\cdot\frac{\partial f}{\partial \mathbf{v}}d\mathbf{v} = -q\int f(\mathbf{E}+\mathbf{v}\times\mathbf{B})d\mathbf{v}$$
$$= -qn(\mathbf{E}+\mathbf{u}\times\mathbf{B}) \qquad (7.9)$$

- To evaluate the second integral of Eqn. 7.7, use fact that **v** does not depend on gradient operator:

$$\int \mathbf{v}(\mathbf{v}\cdot\nabla)fd\mathbf{v} = \int \nabla\cdot(f\mathbf{v}\mathbf{v})d\mathbf{v}$$
$$= \nabla\cdot\int f\mathbf{v}\mathbf{v}d\mathbf{v}$$

First-order Moment: Momentum Transport

- Since the average of a quantity is $1/n$ times its weighted integral over **v**, we have

$$\nabla \cdot \int f\mathbf{vv}d\mathbf{v} = \nabla \cdot (n < \mathbf{vv} >)$$

- Now separate **v** into average fluid velocity **u** and a thermal velocity **w**:

$$\mathbf{v} = \mathbf{u} + \mathbf{w}$$

- Since **u** is already and average, we have

$$\nabla \cdot (n < \mathbf{vv} >) = \nabla \cdot (n\mathbf{uu}) + \nabla \cdot (n < \mathbf{ww} >) + 2\nabla \cdot (n\mathbf{u} < \mathbf{w} >) \qquad (7.10)$$

- The average thermal velocity is zero => $< \mathbf{w} > = 0$ and

$$\mathsf{P} = mn < \mathbf{ww} > \qquad (7.11)$$

  is the *stress tensor*. Also called the *pressure tensor* or *dyad*.

- P is a measure of the thermal motion in a fluid. If all particles moved with same steady velocity **v**, then $\mathbf{w} = 0$ and thus $\mathsf{P} = 0$ (i.e., a cold plasma).

First-order Moment: Momentum Transport

- Remaining term in Eqn. 7.7 can be written

$$\nabla \cdot (n\mathbf{uu}) = \mathbf{u}\nabla \cdot (n\mathbf{u}) + n(\mathbf{u} \cdot \nabla)\mathbf{u} \qquad (7.12)$$

- Collecting Eqns. 7.8, 7.9, 7.11 and 7.12, we have

$$m\frac{\partial}{\partial t}(n\mathbf{u}) + m\mathbf{u}\nabla \cdot (n\mathbf{u}) + mn(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla \cdot \mathsf{P} - qn(\mathbf{E} + \mathbf{u} \times \mathbf{B}) = \mathbf{P}_{ij}$$

- Combining the first two terms, we obtain the *fluid equation of motion*:

$$\boxed{mn\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right] = qn(\mathbf{E} + \mathbf{u} \times \mathbf{B}) - \nabla \cdot \mathsf{P} + \mathbf{P}_{ij}} \qquad (7.13)$$

- This describes flow of momentum – also called *momentum transport equation.*

- Eqn. 7.13 is a statement of conservation of momentum and represents force balance on components of plasma. On right are the Lorentz force, pressure, and collisions.

Summary of Moments of Vlasov Equation

- Equations of MHD and multi-fluid theory are obtained by taking the moments of the Vlasov equation, corresponding to mass, momentum and energy.

$\int$ (Vlasov equation) $d\mathbf{v}$ => conservation of mass

$\int$ (Vlasov equation) $\mathbf{v}\, d\mathbf{v}$ => conservation of momentum

$\int$ (Vlasov equation) $\mathbf{v}^2/2\, d\mathbf{v}$ => conservation of energy

- Zeroth moment of the Vlasov equation results in the MHD mass continuity equation (Eqn. 7.6).
- First moment of Vlasov equation gives MHD momentum equation (Eqn. 7.13).
- Second moment of Vlasov equation give MHD energy transport equation.

Hierarchy of plasma phenomena