Introduction to Plasma Physics (PY5012)
Lecture 8: Multiple-fluid theory of plasmas

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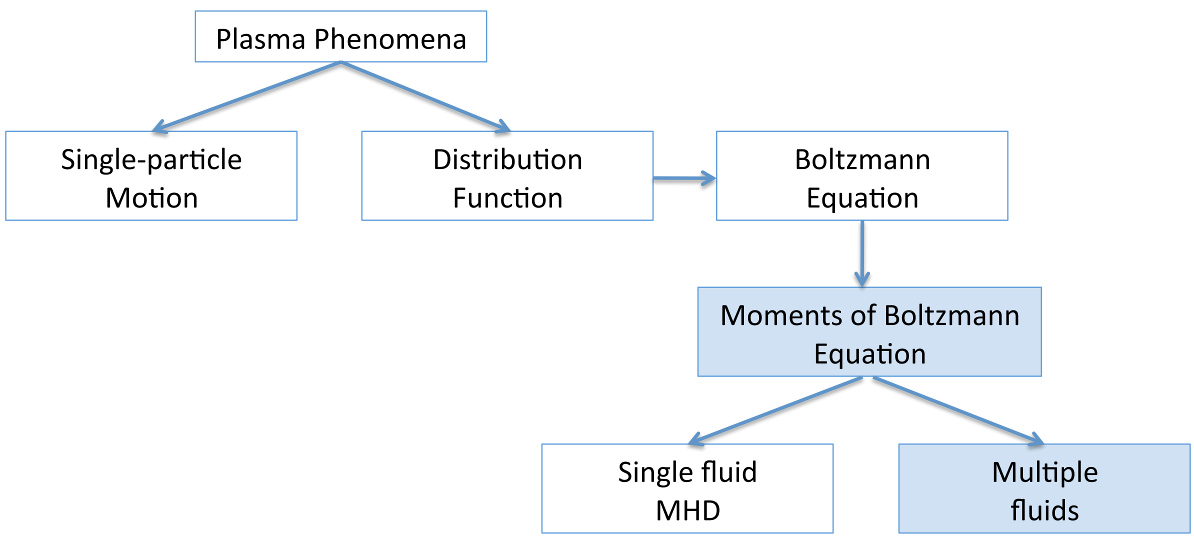
Hierarchy of plasma phenomena



Fluid Approach to Plasmas

- Fluid approach describes bulk properties of plasma. We do not attempt to solve unique trajectories of all particles in a plasma. This simplification works very well for majority of plasmas, despite gross simplifications made.

- Fluid theory follows directly from moments of the Boltzmann equation (Lecture 7).

- Each of the moments of the Boltzmann equation is a transport equation describing the dynamics of a quantity associated with a given power of **v**.

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0$$

Continuity of mass or charge transport

$$mn\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] = qn(\mathbf{E} + \mathbf{u}\times\mathbf{B}) - \nabla\cdot\mathbf{P} + \mathbf{P}_{ij}$$

Momentum Transport

$$\frac{\partial}{\partial t}\left[n\frac{1}{2}mu^2\right] + \nabla\cdot\left[n\frac{1}{2}m\langle u^2\mathbf{u}\rangle\right] - nq\langle\mathbf{E}\cdot\mathbf{u}\rangle = \frac{m}{2}\int u^2\left(\frac{\partial f}{\partial t}\right)_{coll} d\mathbf{u}$$

Energy Transport

Cold-Plasma Model

- Simplest set of macroscopic equations can be obtained by simplifying the momentum transfer equation and neglect thermal motions of particles.

- Here, set kinetic pressure tensor to zero, i.e., $\mathbf{P} = mn\langle\mathbf{ww}\rangle = 0$ as $\mathbf{w} = 0$.

- Remaining macroscopic variables are then n and $\mathbf{u}$, described by

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0$$

$$mn\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] = qn(\mathbf{E} + \mathbf{u}\times\mathbf{B}) + \mathbf{P}_{ij}$$

- Collision term $\boldsymbol{P}_{ij}$ can be approximated by by an “effective” collision frequency.

- Assumed that collisions cause a rate of decrease in momentum:

$$\mathbf{P}_{ij} = -mn\nu_{eff}\mathbf{u}$$

Warm-Plasma Model

- Alternative set of macroscopic equation is obtained by truncating energy conservation equation.
- Consider pressure tensor: $\mathbf{P} = mn\langle \mathbf{ww} \rangle = \begin{pmatrix} p_{xx} & p_{xy} & p_{xz} \\ p_{yx} & p_{yy} & p_{yz} \\ p_{zx} & p_{zy} & p_{zz} \end{pmatrix}$
- Components represent transport of momentum. Diagonal elements represent pressure, while off-diagonal represent shearing stresses.
- In warm-plasma model, only consider diagonal pressure elements, so

$$\nabla \cdot \mathbf{P} = \nabla p$$

- That is, viscous forces are neglected. We then have

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0$$

$$mn\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right] = qn(\mathbf{E} + \mathbf{u} \times \mathbf{B}) - \nabla p + \mathbf{P}_{ij}$$

Warm-Plasma Model

- The previous system of equations does not form a closed set, since scalar pressure is now a third variable. Usually determined by energy equation
- If plasma is *isothermal,* assume *equation of state* of form:

$$p = nk_B T \quad \text{and} \quad \nabla p = k_B T \nabla n$$

- Holds for slow time variations, allowing temperatures to reach equilibrium.
- If plasma does not exchange energy with its surrounds, assume it is *adiabatic:*

$$p\, n^{-\gamma} = constant \quad \text{and} \quad \frac{\nabla p}{p} = \gamma \frac{\nabla n}{n}$$

where γ is the ratio of the specific heats at constant pressure.

Simplified Energy Equation

- Note, the energy equation can be written

$$\frac{\partial[1/2nm < w^2 >]}{\partial t} + \nabla\cdot\left(1/2nm < w^2 > \mathbf{u}\right) + (\mathbf{P}\cdot\nabla)\mathbf{u} + \nabla\cdot\mathbf{q} = P_{ij}$$

  where **q** is the heat flow vector. For electrons, commonly used approximation for **q** is

$$\mathbf{q} = K\nabla T$$

  where K is the thermal *Spitzer conductivity.*

- As average energy of plasma is *1/2m* <**ww**> = *3/2* k_B T and using $p = n\, k_B\, T$ => *3/2 p* = *1/2nm*<**ww**>. Energy equation can then be written

$$\frac{\partial(3/2p)}{\partial t} + \nabla\cdot(3/2p\mathbf{u}) - p\nabla\cdot\mathbf{u} + \nabla\cdot\mathbf{q} = P_{ij}$$

- The quantity 3/2p**u** represents the flow of energy density at the fluid velocity.

Complete set of two-fluid equations

- Consider plasma of two species; ions and electrons, in which fluid is fully ionised, isotropic and collisionless. The charge and current densities are

$$\sigma = n_i q_i + n_e q_e$$
$$\mathbf{j} = n_i q_i \mathbf{v}_i + n_e q_e \mathbf{v}_e$$

- Using **v** = **u**, complete set of electrodynamics equations are then (j = i or e)

$$m_j n_j\left[\frac{\partial\mathbf{v}_j}{\partial t} + (\mathbf{v}_j\cdot\nabla)\mathbf{v}_j\right] = -\nabla p_j - q_j n_j(\mathbf{E} + \mathbf{v}_j\times\mathbf{B})$$
$$\frac{\partial n_j}{\partial t} + \nabla\cdot(n_j\mathbf{v}_j) = 0$$
$$\varepsilon_0\nabla\cdot\mathbf{E} = \sigma$$
$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$
$$\nabla\cdot\mathbf{B} = 0$$
$$\frac{1}{\mu_0}\nabla\times\mathbf{B} = \mathbf{j} + \varepsilon_0\frac{\partial\mathbf{E}}{\partial t}$$
$$p_j = C_j n_j^{\gamma j}$$

Fluids drifts perpendicular to B

- Since a fluid element is composed of mane individual particles, expect drifts perpendicular to **B**. But, the grad (p) term results in a fluid drift called *diamagnetic drift.*

- Consider momentum equation for each species:

$$mn\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] = -\nabla p - qn(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

(1) (2) (3)

- Consider ratio of terms (1) to (3):

$$\frac{(1)}{(3)} \approx \left|\frac{mni\omega v_\perp}{qnv_\perp B}\right| \approx \frac{\omega}{\omega_c}$$

- Here we have used $\partial/\partial t = i\omega$. If only consider slow drifts compared to time-scale of the gyrofrequency, can set (1) to zero.

Fluids drifts perpendicular to B

- Therefore can write $0 \approx -qn(\mathbf{E} + \mathbf{v}_\perp\times\mathbf{B}) - \nabla p$

  where $\mathbf{v}\times\mathbf{B} = (\mathbf{v}_\perp + \mathbf{v}_\parallel)\times\mathbf{B} = \mathbf{v}_\perp\times\mathbf{B}$

- Taking the cross-product $0 = qn[\mathbf{E}\times\mathbf{B} + (\mathbf{v}_\perp\times\mathbf{B})\times\mathbf{B}] - \nabla p\times\mathbf{B}$

- Using the identity $(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = (\mathbf{C}\cdot\mathbf{A})\mathbf{B} - (\mathbf{C}\cdot\mathbf{B})\mathbf{A}$ we can write

$$0 = qn[\mathbf{E}\times\mathbf{B} + (\mathbf{v}_\perp\cdot\mathbf{B})\mathbf{B} - (\mathbf{B}\cdot\mathbf{B})\mathbf{v}_\perp] - \nabla p\times\mathbf{B}$$

- As $\mathbf{v}_\perp$ is perpendicular to **B**, $\mathbf{v}_\perp$. **B** = 0. Therefore

$$\mathbf{v}_\perp = \frac{\mathbf{E}\times\mathbf{B}}{B^2} - \frac{\nabla p\times\mathbf{B}}{qnB^2}$$
$$= \mathbf{v}_E + \mathbf{v}_D$$

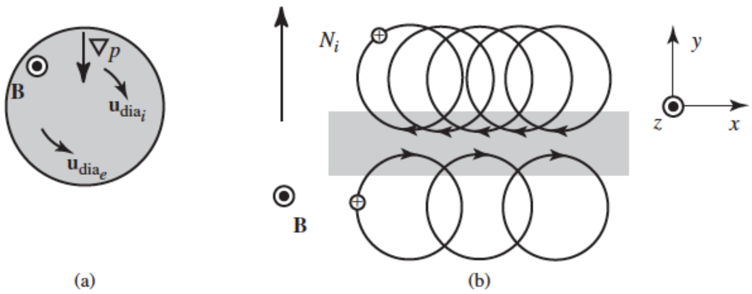
Fluids drifts perpendicular to B

- In previous equation is $\mathbf{v}_E = \dfrac{\mathbf{E}\times\mathbf{B}}{B^2}$ **E** × **B** *drift*

  and $\mathbf{v}_D = -\dfrac{\nabla p\times\mathbf{B}}{qnB^2}$ is *diamagnetic drift.*

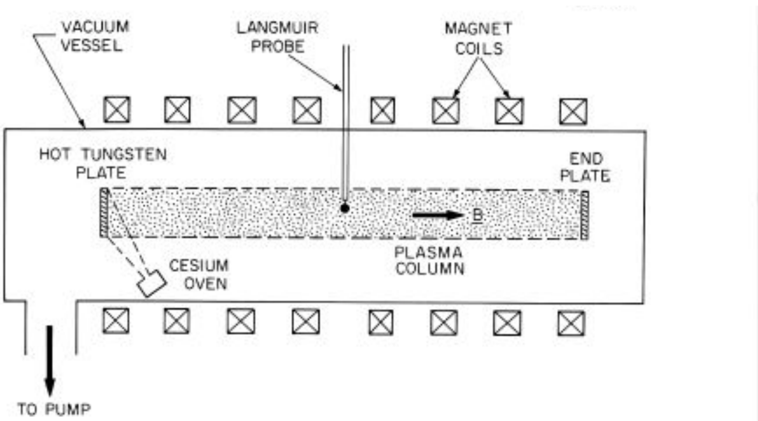
- The $\mathbf{v}_E$ drift is same as for guiding centres, but there is now a new drift, called the diamagnetic drift. Is in opposite directions for ions and electrons.

- Gives currents in plasma that reduce magnetic field in plasma. More ions moving to left in shaded area that to the right (Inan & Golkowski, Page 111).



Diamagnetic Drift in Q-machines

- Diamagnetic drift first measured in *Q*-machines



- See http://www.physics.uiowa.edu/xplasma/Qmachine.html

Fluid drifts parallel to B

- Consider z component of fluid equation of motion:

$$mn\left[\frac{\partial v_z}{\partial t}+(\mathbf{v}\cdot\nabla)v_z\right]=-qnE_z-\frac{\partial p}{\partial z}$$

- The convective term can be neglected as is is much smaller than $\partial v_z/\partial t$

- Using $p = n\, k_B\, T$ or $\frac{\partial p}{\partial z}=k_B T\frac{\partial n}{\partial z}$ we can write

$$\frac{\partial v_z}{\partial t}=\frac{q}{m}E_z-\frac{\gamma k_B T}{mn}\frac{\partial n}{\partial z}$$

- This shows that the fluid is accelerated along **B** under the combined electrostatic and pressure gradient forces.

Fluid drifts parallel to B

- Taking the limit as m -> 0 and $q = -e$ and $E_z=-\partial\phi/dz$ we have

$$eE_z=e\frac{\partial\phi}{dz}=\frac{\gamma k_B T}{n}\frac{\partial n}{dz}$$

- Electrons are so mobile that their heat conductivity is almost infinite.

- Assuming isothermal electrons and taking γ = 1, we can integrate to get

$$e\phi=k_B T\ln(n)+C$$

- We can therefore write

$$\boxed{n=n_0\exp(e\phi/k_B T)}$$

- This is called the *Boltzmann relation* or *Boltzmann factor* for electrons.

- Implies that electrons have a tendency to move rapidly in response to and external force (i.e., electrostatic potential gradient).