Introduction to Plasma Physics (PY5012)
Lecture 9: Single-Fluid Theory of Plasmas - Magnetohydrodynamics

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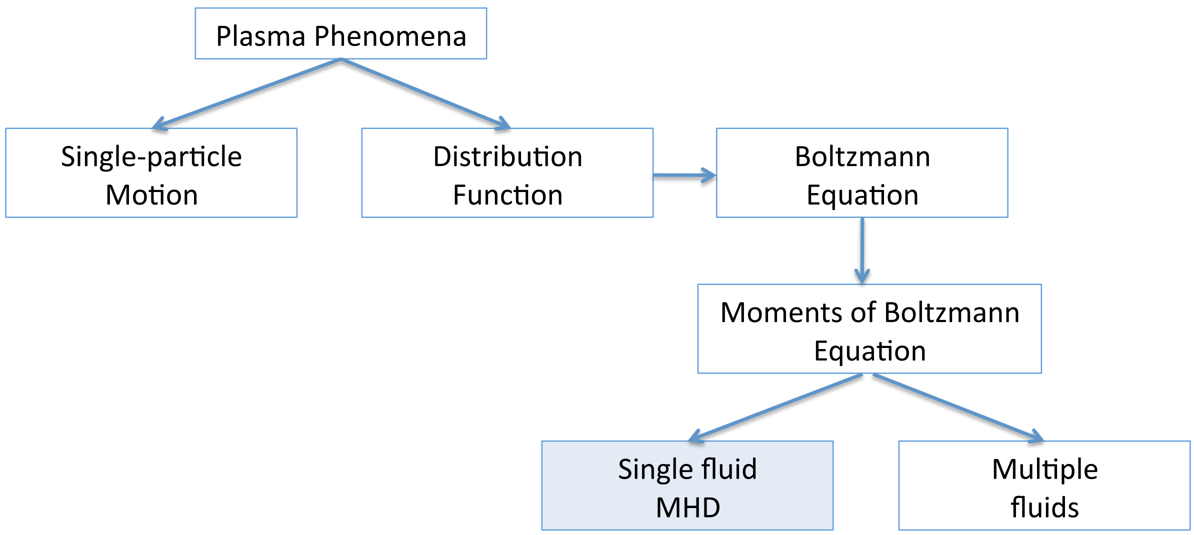
Hierarchy of plasma phenomena



Single-fluid Theory: MHD

- Under certain circumstances, appropriate to consider entire plasma as a single fluid.
- Do not differentiate between ions and electrons.
- Approach is called *magnetohydrodynamics (MHD).*
- General method for modelling highly conductive fluids, including salt water, coronal loops, ISM, tokamaks, etc.
- Single-fluid approach appropriate when dealing with slowly varying conditions.
- MHD is useful when plasma is highly ionised and electrons and ions are forced to act in unison, either because of frequent collisions or by the action of a strong external magnetic field.

Single-fluid equation for fully ionised plasma

- Can combine multiple-fluid equations into a set of equations for a single fluid.
- Assuming two-specials plasma of electrons and ions ($j = e$ or i):

$$\frac{\partial n_j}{\partial t} + \nabla \cdot (n_j \mathbf{v}_j) = 0 \qquad (9.1a)$$

$$m_j n_j \left[\frac{\partial \mathbf{v}_j}{\partial t} + (\mathbf{v}_j \cdot \nabla)\mathbf{v}_j \right] = -\nabla \cdot \mathbf{P}_j + q_j n_j (\mathbf{E} + \mathbf{v}_j \times \mathbf{B}) + P_{ij} \qquad (9.1b)$$

- For a fully ionised two-species plasma, total momentum must be conserved:

$$P_{ei} = -P_{ie}$$

- As $m_i >> m_e$ the time-scales in continuity and momentum equations for ions and electrons are very different. The characteristic frequencies of a plasma, such as plasma frequency or cyclotron frequency are much larger for electrons.

Single-fluid equation for fully ionised plasma

- When plasma phenomena are large-scale ($L >> \lambda_D$) and have relatively low frequencies ($\omega << \omega_{plasma}$ and $\omega << \omega_{cyclotron}$), plasma is on average electrically neutral ($n_i \approx n_e$). Independent motion of electrons and ions can then be neglected.

- Can therefore treat plasma as single conducting fluid, whose inertia is provided by mass of ions.

- Governing equations obtained by combining Eqns. 9.1.

- First define macroscopic parameters of plasma fluid:

$\rho_m = n_e m_e + n_i m_i$	Mass density
$\mathbf{J} = n_e q_e \mathbf{v}_e + n_i q_i \mathbf{v}_i$	Electric current
$\mathbf{v} = \dfrac{n_e m_e \mathbf{v}_e + n_i m_i \mathbf{v}_i}{n_e m_e + n_i m_i}$	Mass velocity
$\mathsf{P} = \mathsf{P}_e + \mathsf{P}_i$	Total pressure tensor

MHD mass and charge conservation

- Using Eqn 9.1a: $$\frac{\partial n_j}{\partial t} + \nabla \cdot (n_j \mathbf{v}_j) = 0$$

- Multiply by q_i and q_e and add continuity equations to get:

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{J}) = 0}$$ Continuity of electric current

where **J** is the electric current density $\mathbf{J} = n_e q_e \mathbf{v}_e + n_i q_i \mathbf{v}_i$ and the electric charge density is $\rho = n_e q_e + n_i q_i$

- Now multiple Eqn. 9.1a by m_i and m_e,

$$\boxed{\frac{\partial \rho_m}{\partial t} + \rho_m \nabla \cdot (\mathbf{v}) = 0}$$ Mass conservation

where $\rho_m = n_e m_e + n_i m_i$ is the single-fluid mass density and **v** is the fluid mass velocity

$$\mathbf{v} = \frac{n_e m_e \mathbf{v}_e + n_i m_i \mathbf{v}_i}{n_e m_e + n_i m_i}$$

MHD Equation of motion

- Equation of motion for bulk plasma can be obtained by adding individual momentum transport equations for ions and electrons (Eqns. 9.1b).

$$(n_e m_e + n_i m_i)\frac{\partial \mathbf{v}}{\partial t} = -\nabla \cdot (\mathbf{P}_e + \mathbf{P}_i) + (n_e q_e + n_i q_i)\mathbf{E} + \mathbf{J} \times \mathbf{B} \qquad (9.2)$$

- Note that we have neglected $(\mathbf{v} \cdot \nabla)\mathbf{v}$ as we are dealing with small perturbations for which the gradients are negligible.

- Second term in Eqn. 9.2 is zero as plasma is neural. Therefore,

$$\rho_m \frac{\partial \mathbf{v}}{\partial t} = -\nabla \cdot \mathbf{P} + \mathbf{J} \times \mathbf{B}$$

Equation of motion

- For an isotropic plasma, $\nabla \cdot \mathbf{P} = \nabla p$ where the total pressure is $p = p_e + p_i$ and

$$\rho_m \frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \mathbf{J} \times \mathbf{B}$$

Equation of motion

Generalized Ohm's Law

- The final single-fluid equation describes the variation of current density **J**.

- Consider the momentum equations for electron and ions (Eqn. 9.1b):

$$m_j n_j \left[\frac{\partial \mathbf{v}_j}{\partial t} + (\mathbf{v}_j \cdot \nabla)\mathbf{v}_j \right] = -\nabla \cdot \mathbf{P}_j + q_j n_j (\mathbf{E} + \mathbf{v}_j \times \mathbf{B}) + P_{ij}$$

- Multiple electron equation by q_e / m_e and ion equation by q_i / m_i and add:

$$\begin{aligned}\frac{\partial \mathbf{J}}{\partial t} = &-\frac{q_e}{m_e}\nabla \cdot \mathbf{P}_e - \frac{q_i}{m_i}\nabla \cdot \mathbf{P}_i \\ &+ \left(\frac{n_e q_e^2}{m_e} + \frac{n_i q_i^2}{m_i} \right)\mathbf{E} \\ &+ \left(\frac{n_e q_e^2}{m_e}\mathbf{v}_e + \frac{n_i q_i^2}{m_i}\mathbf{v}_i \right) \times \mathbf{B} \\ &+ \frac{q_e}{m_e} P_{ei} + \frac{q_i}{m_i} P_{ie}\end{aligned}$$

Generalized Ohm's Law

- For an electrically neutral plasma $|q_e n_e| \approx |q_i n_i|$ and using $\mathbf{J} = n_e q_e \mathbf{v}_e + n_i q_i \mathbf{v}_i$

  and $\mathbf{v} = \dfrac{n_e m_e \mathbf{v}_e + n_i m_i \mathbf{v}_i}{n_e m_e + n_i m_i}$ we can write

$$\frac{\partial \mathbf{J}}{\partial t} = -\frac{q_e}{m_e}\nabla\cdot\mathbf{P}_e - \frac{q_i}{m_i}\nabla\cdot\mathbf{P}_i + \left(\frac{n_e q_e^2}{m_e} + \frac{n_i q_i^2}{m_i}\right)(\mathbf{E} + \mathbf{v}\times\mathbf{B}) + \left(\frac{q_e}{m_e} + \frac{q_i}{m_i}\right)(\mathbf{J}\times\mathbf{B}) + \left(\frac{q_e}{m_e} - \frac{q_i}{m_i}\right)P_{ei}$$

- As $m_e << m_i$ => $q_e / m_e >> q_i / m_i$ and $n_e\, q_e^2 / m_e >> n_i\, q_i^2 / m_i$. In thermal equilibrium, kinetic pressures of electrons is similar to ion pressure ($\mathbf{P}_e \approx \mathbf{P}_i$)

$$\frac{\partial \mathbf{J}}{\partial t} = -\frac{q_e}{m_e}\nabla\cdot\mathbf{P}_e + \frac{n_e q_e^2}{m_e}(\mathbf{E} + \mathbf{v}\times\mathbf{B}) + \frac{q_e}{m_e}(\mathbf{J}\times\mathbf{B}) + \frac{q_e}{m_e}P_{ei} \qquad (9.3)$$

Generalized Ohm's Law

- The collisional term can be written: $P_{ei} = \eta q^2 n_e^2 (\mathbf{v}_i - \mathbf{v}_e)$

  where η is the specific resistivity, q^2 relates to fact that collisions result from Coulomb force between ions (q_i) and electrons (q_e) and total momentum transferred to electrons in an elastic collision with an ion is $\mathbf{v}_i - \mathbf{v}_e$.

- Now $q_i = -q_e$ and $n_e = n_i$ and $\mathbf{J} = n_e\, q_e\, (\mathbf{v}_e - \mathbf{v}_i)$ =>

$$P_{ei} = -n_e q_e \eta \mathbf{J}$$

- Can therefore write Eqn. 9.4 as

$$\boxed{\frac{\partial \mathbf{J}}{\partial t} = -\frac{q_e}{m_e}\nabla\cdot\mathbf{P}_e + \frac{n_e q_e^2}{m_e}(\mathbf{E} + \mathbf{v}\times\mathbf{B}) + \frac{q_e}{m_e}(\mathbf{J}\times\mathbf{B}) - \frac{n_e q_e^2}{m_e}\hat{\eta}\cdot\mathbf{J}} \qquad (9.5)$$

  where η is a tensor. This is *generalised Ohm's law.*

Generalized Ohm's Law

- For a steady current in a uniform **B**, $\partial \mathbf{J}/\partial t = 0, \nabla \cdot \mathbf{P} = 0$ and **B** = 0 so that

$$\mathbf{E} = \eta\, \mathbf{J} \quad => \quad \mathbf{J} = 1/\eta\, E$$

- The electric field **E** can be found from Eqn. 9.5:

$$E = -\mathbf{v} \times \mathbf{B} - \frac{\mathbf{J} \times \mathbf{B}}{n_e q_e} + \frac{\nabla \cdot \mathbf{P}}{n_e q_e} + \hat{\eta} \cdot \mathbf{J} + \frac{m_e}{n_e q_e} \frac{\partial \mathbf{J}}{\partial t}$$

- Consider right hand side of this equation:

  - Term 1: **E** associated with plasma motion.
  - Term 2: Hall effect.
  - Term 3: Ambipolar diffusion from E-field generated by density gradients.
  - Term 4: Ohmic losses/Joule heating
  - Term 5: Electron inertia

Simplified MHD Equations

- We have now derived the following:

$$\frac{\partial \rho_m}{\partial t} + \rho_m \nabla \cdot (\mathbf{v}) = 0$$

$$\rho_m \frac{\partial \mathbf{v}}{\partial t} = -\nabla \cdot \mathbf{P} + \mathbf{J} \times \mathbf{B}$$

$$\frac{\partial \mathbf{J}}{\partial t} = -\frac{q_e}{m_e} \nabla \cdot \mathbf{P}_e + \frac{n_e q_e^2}{m_e} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \frac{q_e}{m_e} (\mathbf{J} \times \mathbf{B}) - \frac{n_e q_e^2}{m_e} \hat{\eta} \cdot \mathbf{J}$$

- Now assume plasma is isotropic, so that $\nabla \cdot \mathbf{P} = \nabla p$. Also neglect Hall effect and ambipolar diffusion in generalised Ohm's law since not important in most cases. For slow variations, **J** = constant, so can write generalised Ohm's law as:

$$0 = \frac{n_e q_e^2}{m_e} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) - \frac{n_e q_e^2}{m_e} \eta \mathbf{J}$$

- Rearranging gives, $\boxed{\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B})}$

Simplified MHD Equations

- The simplified MHD equations can therefore be written:

$$\frac{\partial \rho_m}{\partial t} + \rho_m \nabla \cdot \mathbf{v} = 0$$

$$\rho_m \frac{\partial \mathbf{v}_m}{\partial t} = -\nabla p + \mathbf{J} \times \mathbf{B}$$

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

- Fluid equations must be solved with reduced Maxwell's equations for fields:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

- Here we have assumed that there is no accumulation of space charge (i.e., $\rho = 0$).

- Complete set of equations only when *equation of state* for relationship between p and n is specified.