Quantum Field Theory
and
Particle Physics

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• M.E.Peskin, D.V.Schroeder, An Introduction to Quantum Field Theory.
• J.Strathdee, Course on Quantum Electrodynamics, ICTP lecture notes.
• C.Itzykson, J-B.Zuber, Quantum Field Theory.
• J.Zinn-Justin, Quantum Field Theory and Critical Phenomena.
• A.M.Polyakov, Gauge Fields and Strings.
• C.Itzykson, J-M.Drouffe, From Brownian Motion to Renormalization and Lattice Gauge Theory.
• D.Griffiths, Introduction to Elementary Particles.
• W.Greiner, Relativistic Quantum Mechanics.Wave Equations.
• W.Greiner, J.Reinhardt, Field Quantization.
• W.Greiner, J.Reinhardt, Quantum Electrodynamics.
• S.Ranjbar-Daemi, Course on Quantum Field Theory, ICTP lecture notes.
• V.Radovanovic, Problem Book QFT.
Part I

Path Integrals, Gauge Fields and Renormalization Group
2.1 Feynman Path Integral

We consider a dynamical system consisting of a single free particle moving in one dimension. The coordinate is $x$ and the canonical momentum is $p = m \dot{x}$. The Hamiltonian is $H = p^2 / (2m)$. Quantization means that we replace $x$ and $p$ with operators $X$ and $P$ satisfying the canonical commutation relation $[X, P] = i\hbar$. The Hamiltonian becomes $H = P^2 / (2m)$. These operators act in a Hilbert space $\mathcal{H}$. The quantum states which describe the dynamical system are vectors on this Hilbert space whereas observables which describe physical quantities are hermitian operators acting in this Hilbert space. This is the canonical or operator quantization.

We recall that in the Schrödinger picture states depend on time while operators are independent of time. The states satisfy the Schrödinger equation, viz

$$H|\psi_s(t)\rangle = i\hbar \frac{\partial}{\partial t}|\psi_s(t)\rangle. \quad (2.1)$$

Equivalently

$$|\psi_s(t)\rangle = e^{-\frac{i}{\hbar}H(t-t_0)}|\psi_s(t_0)\rangle. \quad (2.2)$$

Let $|x\rangle$ be the eigenstates of $X$, i.e $X|x\rangle = x|x\rangle$. The completeness relation is $\int dx |x\rangle < x |x\rangle = 1$. The components of $|\psi_s(t)\rangle$ in this basis are $<x|\psi_s(t)\rangle$. Thus

$$|\psi_s(t)\rangle = \int dx <x|\psi_s(t)\rangle |x\rangle. \quad (2.3)$$

$$<x|\psi_s(t)\rangle = <x|e^{-\frac{i}{\hbar}H(t-t_0)}|\psi_s(t_0)\rangle = \int dx_0 G(x,t;x_0,t_0) <x_0|\psi_s(t_0)\rangle. \quad (2.4)$$
In above we have used the completeness relation in the form $\int dx_0 |x_0><x_0| = 1$. The Green function $G(x, t; x_0, t_0)$ is defined by

$$G(x, t; x_0, t_0) = <x|e^{-\frac{i}{\hbar}H(t-t_0)}|x_0>.$$  \hspace{1cm} (2.5)

In the Heisenberg picture states are independent of time while operators are dependent of time. The Heisenberg states are related to the Schrödinger states by the relation

$$|\psi_H(t) >= e^{\frac{i}{\hbar}Ht}|\psi_s(t) >.$$  \hspace{1cm} (2.6)

We can clearly make the identification $|\psi_H(t) >= |\psi_s(t_0) >$. Let $X(t)$ be the position operator in the Heisenberg picture. Let $|x, t >$ be the eigenstates of $X(t)$ at time $t$, i.e $X(t)|x, t >= x|x, t >$. We set

$$|x, t >= e^{\frac{i}{\hbar}Ht}|x >, |x_0, t_0 >= e^{\frac{i}{\hbar}Ht_0}|x_0 >.$$  \hspace{1cm} (2.7)

From the facts $X(t)|x, t >= x|x, t >$ and $X|x >= x|x >$ we conclude that the Heisenberg operators are related to the Schrödinger operators by the relation

$$X(t) = e^{\frac{i}{\hbar}Ht}Xe^{-\frac{i}{\hbar}Ht}.$$  \hspace{1cm} (2.8)

We immediately obtain the Heisenberg equation of motion

$$\frac{dX(t)}{dt} = e^{\frac{i}{\hbar}Ht}\frac{\partial X}{\partial t}e^{-\frac{i}{\hbar}Ht} + \frac{i}{\hbar}[H, X(t)].$$  \hspace{1cm} (2.9)

The Green function (2.5) can be put into the form

$$G(x, t; x_0, t_0) = <x, t|x_0, t_0>.$$  \hspace{1cm} (2.10)

This is the transition amplitude from the point $x_0$ at time $t_0$ to the point $x$ at time $t$ which is the most basic object in the quantum theory.

We discretize the time interval $[t_0, t]$ such that $t_j = t_0 + j\epsilon$, $\epsilon = (t - t_0)/N$, $j = 0, 1, ..., N$, $t_N = t_0 + N\epsilon = t$. The corresponding coordinates are $x_0, x_1, ..., x_N$ with $x_N = x$. The corresponding momenta are $p_0, p_1, ..., p_{N-1}$. The momentum $p_j$ corresponds to the interval $[x_j, x_{j+1}]$. We can show

$$G(x, t; x_0, t_0) = <x, t|x_0, t_0> = \int dx_1 <x, t|x_1, t_1><x_1, t_1|x_0, t_0> = \int dx_1 dx_2 ... dx_{N-1} \prod_{j=0}^{N-1} <x_{j+1}, t_{j+1}|x_j, t_j >.$$  \hspace{1cm} (2.11)
We compute (with $<p|x> = \exp(-ipx/\hbar)/\sqrt{2\pi\hbar}$)

$$<x_{j+1}, t_{j+1}|x_j, t_j> = <x_{j+1}|(1 - \frac{i}{\hbar} H\epsilon)|x_j>$$

$$= \int dp_j <x_{j+1}|p_j><p_j|(1 - \frac{i}{\hbar} H\epsilon)|x_j>$$

$$= \int dp_j (1 - \frac{i}{\hbar} H(p_j, x_j)\epsilon) <x_{j+1}|p_j><p_j|x_j>$$

$$= \int dp_j (1 - \frac{i}{\hbar} H(p_j, x_j)\epsilon) e^{ip_j x_{j+1}} e^{-ip_j x_j}$$

$$= \int dp_j \frac{2\pi}{2\pi\hbar} e^{\frac{i}{\hbar}(p_j x_{j+1} - H(x_j, p_j))\epsilon}. \quad (2.12)$$

In above $\dot{x}_j = (x_{j+1} - x_j)/\epsilon$. Therefore by taking the limit $N \to \infty$, $\epsilon \to 0$ keeping $t - t_0$ fixed we obtain

$$G(x, t; x_0, t_0) = \int \frac{dp_0 dp_1 dx_1}{2\pi\hbar} \cdots \frac{dp_{N-1} dx_{N-1}}{2\pi\hbar} e^{\frac{i}{\hbar} \sum_{j=0}^{N-1} (p_j x_j - H(p_j, x_j))\epsilon}$$

$$= \int DpDx e^{\frac{i}{\hbar} \int_0^t ds L(\dot{x}, x)}. \quad (2.13)$$

Now $\dot{x} = dx/ds$. In our case the Hamiltonian is given by $H = p^2/(2m)$. Thus by performing the Gaussian integral over $p$ we obtain

$$G(x, t; x_0, t_0) = N \int Dx e^{\frac{i}{\hbar} \int_0^t ds L(\dot{x}, x)}$$

$$= N \int Dx e^{\frac{i}{\hbar} S[x]}. \quad (2.14)$$

In the above equation $S[x] = \int dt L(x, \dot{x}) = m \int dt \dot{x}^2/2$ is the action of the particle. As it turns out this fundamental result holds for all Hamiltonians of the form $H = p^2/(2m) + V(x)$ in which case $S[x] = \int dt L(x, \dot{x}) = \int dt (m\dot{x}^2/2 - V(x))$.

This result is essentially the principle of linear superposition of quantum theory. The total probability amplitude for traveling from the point $x_0$ to the point $x$ is equal

\[ \begin{aligned}
\text{Exercise:} & \\
\text{• Show that} & \\
\int dp e^{-ap^2 + bp} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}. \\
\text{• Use the above result to show that} & \\
\int Dp e^{\frac{i}{\hbar} \int_0^t ds (p\dot{x} - \frac{p^2}{2m})} = N e^{\frac{i}{\hbar} \int_0^t ds \dot{x}^2(x)}. \\
\text{Determine the constant of normalization } N. \\
\text{Exercise: Repeat the analysis for a non-zero potential. See for example Peskin and Schroeder} & 
\end{aligned} \]
to the sum of probability amplitudes for traveling from \( x_0 \) to \( x \) through all possible paths connecting these two points. Clearly a given path between \( x_0 \) and \( x \) is defined by a configuration \( x(s) \) with \( x(t_0) = x_0 \) and \( x(t) = x \). The corresponding probability amplitude (wave function) is \( e^{i\frac{\hbar}{\pi}S[x(s)]} \). In the classical limit \( \hbar \to 0 \) only one path (the classical path) exists by the method of the stationary phase. The classical path is clearly the path of least action as it should be.

We note also that the generalization of the result (2.14) to matrix elements of operators is given by

\[
< x, t | T(X(t_1) \ldots X(t_n)) | x_0, t_0 > = \mathcal{N} \int \mathcal{D}x \ x(t_1) \ldots x(t_n) \ e^{i\frac{\hbar}{\pi}S[x]}.
\]  

(2.15)

The \( T \) is the time-ordering operator defined by

\[
T(X(t_1)X(t_2)) = X(t_1)X(t_2) \text{ if } t_1 > t_2.
\]

(2.16)

\[
T(X(t_1)X(t_2)) = X(t_2)X(t_1) \text{ if } t_1 < t_2.
\]

(2.17)

Let us now introduce the basis \( | n > \). This is the eigenbasis of the Hamiltonian, viz \( H|n >= E_n|n > \). We have the completeness relation \( \sum_n |n><n| = 1 \). The matrix elements (2.15) can be rewritten as

\[
\sum_{n,m} e^{-itE_n + it_0E_m} < x|n><m|x_0 > < n|T(X(t_1) \ldots X(t_n))|m > = \mathcal{N} \int \mathcal{D}x \ x(t_1) \ldots x(t_n) \ e^{i\frac{\hbar}{\pi}S[x]}.
\]  

(2.18)

In the limit \( t_0 \to -\infty \) and \( t \to \infty \) we observe that only the ground state with energy \( E_0 \) contributes, i.e. the rapid oscillation of the first exponential in this limit forces \( n = m = 0 \). Thus we obtain in this limit

\[
e^{itE_0(t_0-t)} < x|0><0|x_0 > < 0|T(X(t_1) \ldots X(t_n))|0 > = \mathcal{N} \int \mathcal{D}x \ x(t_1) \ldots x(t_n) \ e^{i\frac{\hbar}{\pi}S[x]}.
\]

(2.20)

We write this as

\[
< 0|T(X(t_1) \ldots X(t_n))|0 > = \mathcal{N}' \int \mathcal{D}x \ x(t_1) \ldots x(t_n) \ e^{i\frac{\hbar}{\pi}S[x]}.
\]

(2.21)

Exercise: Verify this explicitly. See for example Randjbar-Daemi lecture notes.

We consider the integral

\[
I = \int_{-\infty}^{\infty} dx \ F(x)e^{i\phi(x)}.
\]

(2.19)

The function \( \phi(x) \) is a rapidly-varying function over the range of integration while \( F(x) \) is slowly-varying by comparison. Evaluate this integral using the method of stationary phase.
In particular

\[
\langle 0|0 \rangle = N' \int Dx \ e^{i\pi S[x]}.
\] (2.22)

Hence

\[
\langle 0|T(X(t_1)\ldots X(t_n))|0 \rangle = \frac{\int Dx \ x(t_1)\ldots x(t_n) \ e^{i\pi S[x]}}{\int Dx \ e^{i\pi S[x]}}.
\] (2.23)

We introduce the path integral \( Z[J] \) in the presence of a source \( J(t) \) by

\[
Z[J] = \int Dx \ e^{i\pi S[x] + i\pi \int dt J(t)x(t)}.
\] (2.24)

This path integral is the generating functional of all the matrix elements \( \langle 0|T(X(t_1)\ldots X(t_n))|0 \rangle \). Indeed

\[
\langle 0|T(X(t_1)\ldots X(t_n))|0 \rangle = \frac{1}{Z[0]} \left( \frac{\hbar}{i} \right)^n \frac{\delta^n Z[J]}{\delta J(t_1)\ldots\delta J(t_n)} |_{J=0}.
\] (2.25)

From the above discussion \( Z[0] \) is the vacuum-to-vacuum amplitude. Therefore \( Z[J] \) is the vacuum-to-vacuum amplitude in the presence of the source \( J(t) \).

### 2.2 Scalar Field Theory

#### 2.2.1 Path Integral

A field theory is a dynamical system with \( N \) degrees of freedom where \( N \rightarrow \infty \). The classical description is given in terms of a Lagrangian and an action principle while the quantum description is given in terms of a path integral and correlation functions. In a scalar field theory the basic field has spin \( j = 0 \) with respect to Lorentz transformations.

It is well established that scalar field theories are relevant to critical phenomena and to the Higgs sector in the standard model of particle physics.

We start with the relativistic energy-momentum relation \( p^\mu p_\mu = M^2 c^2 \) where \( p^\mu = (p^0, \vec{p}) = (E/c, \vec{p}) \). We adopt the metric \((1, -1, -1, -1)\), i.e. \( p^\mu = (p_0, -\vec{p}) = (E/c, -\vec{p}) \). Next we employ the correspondence principle \( p_\mu \rightarrow i\hbar \partial_\mu \) where \( \partial_\mu = (\partial_0, \partial_i) \) and apply the resulting operator on a function \( \phi \). We obtain the Klein-Gordon equation

\[
\partial_\mu \partial^\mu \phi = -m^2 \phi, \quad m^2 = \frac{M^2 c^2}{\hbar^2}.
\] (2.26)

As a wave equation the Klein-Gordon equation is incompatible with the statistical interpretation of quantum mechanics. However the Klein-Gordon equation makes sense as an equation of motion of a classical scalar field theory with action and Lagrangian

\[
S = \int dt L, \quad L = \int d^3 x L \quad \text{where the lagrangian density} \ L \ \text{is given by}
\]

\[
L = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2.
\] (2.27)
So in summary $\phi$ is not really a wave function but it is a dynamical variable which plays the same role as the coordinate $x$ of the free particle discussed in the previous section.

The principle of least action applied to an action $S = \int dtL$ yields (with the assumption $\delta \phi|_{x_\mu=\pm\infty} = 0$) the result

$$\frac{\delta S}{\delta \phi} = \frac{\delta L}{\delta \phi} - \partial_\mu \frac{\delta L}{\delta (\partial_\mu \phi)} = 0.$$ (2.28)

It is not difficult to verify that this is the same equation as (2.26) if $L = \int d^3x L$ and $L$ is given by (2.27).

The free scalar field theory is a collection of infinite number of decoupled harmonic oscillators. To see this fact we introduce the fourier transform $\tilde{\phi}(t, \vec{k})$ of $\phi(t, \vec{x})$ as follows

$$\phi(t, \vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^3} \tilde{\phi}(t, \vec{k}) e^{i\vec{k}\cdot\vec{x}}, \quad \tilde{\phi}(t, \vec{k}) = \int d^3\vec{x} \phi(t, \vec{x}) e^{-i\vec{k}\cdot\vec{x}}.$$ (2.29)

Then the Lagrangian and the equation of motion can be rewritten as

$$L = \int \frac{d^3\vec{k}}{(2\pi)^3} \left( \frac{1}{2} \partial_0 \tilde{\phi} \partial_0 \tilde{\phi}^* - \frac{1}{2} \omega^2_k \tilde{\phi} \tilde{\phi}^* \right).$$ (2.30)

$$\partial_0 \tilde{\phi} + \omega_k^2 \tilde{\phi} = 0, \quad \omega_k^2 = \vec{k}^2 + m^2.$$ (2.31)

This is the equation of motion of a harmonic oscillator with frequency $\omega_k$. Using box normalization the momenta become discrete and the measure $\int d^3\vec{k}/(2\pi)^3$ becomes $\sum_k/V$. Reality of the scalar field $\phi$ implies that $\tilde{\phi}(t, \vec{k}) = \tilde{\phi}^*(t, \vec{k})$ and by writing $\tilde{\phi} = \sqrt{V}(X_k + iY_k)$ we end up with the Lagrangian

$$L = \frac{1}{V} \sum_{k_1>0} \sum_{k_2>0} \sum_{k_3>0} \left( \partial_0 X_k \partial_0 X_k^* - \omega_k^2 X_k^2 + \omega_k^2 \tilde{Y}_k \tilde{Y}_k \right).$$ (2.32)

The path integral of the two harmonic oscillators $X_k$ and $Y_k$ is immediately given by

$$Z[J_k, K_k] = \int \mathcal{D}X_k \mathcal{D}Y_k e^{\frac{i}{\hbar} S[X_k, Y_k]} e^{\frac{i}{\hbar} \int dt (J_k(t)X_k(t) + K_k(t)Y_k(t))}.$$ (2.33)

The action $S[X_k, Y_k]$ is obviously given by

$$S[X_k, Y_k] = \int_{t_0}^{t_\to +\infty} ds \left( \partial_0 X_k^2 - \omega_k^2 X_k^2 + \partial_0 Y_k^2 - \omega_k^2 Y_k^2 \right).$$ (2.34)
The definition of the measures $\mathcal{D}X_k$ and $\mathcal{D}Y_k$ must now be clear from our previous considerations. We introduce the notation $X_k(t_i) = x_i^{(k)}$, $Y_k(t_i) = y_i^{(k)}$, $i = 0, 1, ..., N - 1, N$ with the time step $\epsilon = t_i - t_{i-1} = (t - t_0)/N$. Then as before we have (with $N \rightarrow \infty, \epsilon \rightarrow 0$ keeping $t - t_0$ fixed) the measures

$$
\mathcal{D}X_k = \prod_{i=1}^{N-1} dx_i^{(k)}, \quad \mathcal{D}Y_k = \prod_{i=1}^{N-1} dy_i^{(k)}.
$$

(2.35)

The path integral of the scalar field $\phi$ is the product of the path integrals of the harmonic oscillators $X_k$ and $Y_k$ with different $k = (k_1, k_2, k_3)$, viz

$$
Z[J, K] = \prod_{k_1 > 0} \prod_{k_2 > 0} \prod_{k_3 > 0} Z[J_k, K_k]
$$

$$
= \int \prod_{k_1 > 0} \prod_{k_2 > 0} \prod_{k_3 > 0} \mathcal{D}X_k \mathcal{D}Y_k \exp \left( \frac{i}{\hbar} \sum_{k_1 > 0} \sum_{k_2 > 0} \sum_{k_3 > 0} S[X_k, Y_k] \right)
$$

$$
+ \frac{i}{\hbar} \int dt \sum_{k_1 > 0} \sum_{k_2 > 0} \sum_{k_3 > 0} (J_k(t)X_k(t) + K_k(t)Y_k(t)).
$$

(2.36)

The action of the scalar field is precisely the first term in the exponential, namely

$$
S[\phi] = \sum_{k_1 > 0} \sum_{k_2 > 0} \sum_{k_3 > 0} S[X_k, Y_k]
$$

$$
= \int_{t_0}^{t_{\rightarrow} + \infty} ds \sum_{k_1 > 0} \sum_{k_2 > 0} \sum_{k_3 > 0} \left( (\partial_0 X_k)^2 - \omega_k^2 X_k^2 + (\partial_0 Y_k)^2 - \omega_k^2 Y_k^2 \right)
$$

$$
= \int d^4x \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \right).
$$

(2.37)

We remark also that (with $J(t, \vec{x}) = \int d^3\vec{k}/(2\pi)^3 \ J(t, \vec{k}) \ e^{i\vec{k}\cdot\vec{x}}, \ \vec{J} = \sqrt{N} (J_k + iK_k)$) we have

$$
\sum_{k_1 > 0} \sum_{k_2 > 0} \sum_{k_3 > 0} (J_k(t)X_k(t) + K_k(t)Y_k(t)) = \int d^3x J(t, \vec{x}) \phi(t, \vec{x}).
$$

(2.38)

We write therefore the above path integral formally as

$$
Z[J] = \int \mathcal{D}\phi \ e^{\frac{i}{\hbar} S[\phi]} \frac{1}{i} \int d^4x J(x) \phi(x).
$$

(2.39)

This path integral is the generating functional of all the matrix elements $<0|T(\Phi(x_1)\ldots\Phi(x_n))|0>$ (also called $n-$point functions). Indeed

$$
<0|T(\Phi(x_1)\ldots\Phi(x_n))|0> = \frac{1}{Z[0]} \left( \frac{\hbar}{i} \right)^n \frac{\delta^n Z[J]}{\delta J(x_1)\ldots\delta J(x_n)}|_{J=0}
$$

$$
= \frac{\int \mathcal{D}\phi \ \phi(x_1)\ldots\phi(x_n) \ e^{\frac{i}{\hbar} S[\phi]} \left( \frac{\delta^n Z[J]}{\delta J(x_1)\ldots\delta J(x_n)} \right)}{\int \mathcal{D}\phi \ e^{\frac{i}{\hbar} S[\phi]}}.
$$

(2.40)
The interactions are added by modifying the action appropriately. The only renormalizable interacting scalar field theory in $d = 4$ dimensions is the quartic $\phi^4$ theory. Thus we will only consider this model given by the action

$$S[\phi] = \int d^4x \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 \right]. \quad (2.41)$$

### 2.2.2 The Free 2–Point Function

It is more rigorous to perform the different computations of interest on an Euclidean spacetime. Euclidean spacetime is obtained from Minkowski spacetime via the so-called Wick rotation. This is also called the imaginary time formulation which is obtained by the substitutions $t \rightarrow -i\tau$, $x^0 = ct \rightarrow -i\tau$, $\partial_0 \rightarrow i\partial_4$. Hence $\partial_\mu \phi \partial^\mu \phi \rightarrow -(\partial_\mu \phi)^2$ and $iS \rightarrow -S_E$ where

$$S_E[\phi] = \int d^4x \left[ \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \right]. \quad (2.42)$$

The path integral becomes

$$Z_E[J] = \int D\phi \ e^{-\frac{1}{\hbar}S_E[\phi] + \frac{1}{\pi} \int d^4x J(x)\phi(x)}. \quad (2.43)$$

The Euclidean $n$–point functions are given by

$$<0|T(\Phi(x_1)...\Phi(x_n))|0>_E = \frac{1}{Z[0]} \left( \frac{\hbar}{\pi} \right)^n \frac{\delta^n Z_E[J]}{\delta J(x_1)...\delta J(x_n)}|_{J=0} \quad (2.44)$$

The action of a free scalar field is given by

$$S_E[\phi] = \int d^4x \left[ \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 \right] = \frac{1}{2} \int d^4x \ [\phi(-\partial^2 + m^2)\phi]. \quad (2.45)$$

The corresponding path integral is (after completing the square)

$$Z_E[J] = \int D\phi \ e^{-\frac{1}{\hbar}S_E[\phi] + \frac{1}{\pi} \int d^4x J(x)\phi(x)} \quad (2.46)$$

In above $K$ is the operator defined by

$$K(-\partial^2 + m^2) = (-\partial^2 + m^2)K = 1. \quad (2.47)$$
After a formal change of variable given by $\phi \rightarrow \phi - KJ$ the path integral $Z[J]$ is reduced to (see next section for a rigorous treatment)

$$Z_E[J] = \mathcal{N} e^{\frac{1}{\hbar} \int d^4x (J^T K J)(x)} = \mathcal{N} e^{\frac{1}{\hbar} \int d^4x d^4y J(x)K(x,y)J(y)}.$$ (2.48)

The $\mathcal{N}$ is an unimportant normalization factor. The free 2−point function (the free propagator) is defined by

$$<0|T(\Phi(x_1)\Phi(x_2))|0>_E = \frac{\int D\phi \phi(x_1)\phi(x_2) e^{-\frac{1}{2} S_E[\phi]}}{\int D\phi e^{-\frac{1}{2} S_E[\phi]}} = \frac{1}{Z[0]} \hbar^2 \frac{\delta^2 Z_E[J]}{\delta J(x_1) \delta J(x_2)}|_{J=0}.$$ (2.49)

A direct calculation leads to

$$<0|T(\Phi(x_1)\Phi(x_2))|0>_E = \hbar K(x_1, x_2)$$ (2.50)

Clearly

$$(-\partial^2 + m^2) K(x, y) = \delta^4(x - y).$$ (2.51)

Using translational invariance we can write

$$K(x, y) = K(x - y) = \int \frac{d^4k}{(2\pi)^4} \tilde{K}(k) e^{ik(x-y)}.$$ (2.52)

By construction $\tilde{K}(k)$ is the fourier transform of $K(x, y)$. It is trivial to compute that

$$\tilde{K}(k) = \frac{1}{k^2 + m^2}.$$ (2.53)

The free euclidean 2−point function is therefore given by

$$<0|T(\Phi(x_1)\Phi(x_2))|0>_E = \int \frac{d^4k}{(2\pi)^4} \frac{\hbar}{k^2 + m^2} e^{ik(x-y)}.$$ (2.54)

### 2.2.3 Lattice Regularization

The above calculation of the 2−point function of a scalar field can be made more explicit and in fact more rigorous by working on an Euclidean lattice spacetime. The lattice provides a concrete non-perturbative definition of the theory.

We replace the Euclidean spacetime with a lattice of points $x_\mu = an_\mu$ where $a$ is the lattice spacing. In the natural units $\hbar = c = 1$ the action is dimensionless and hence the field is of dimension mass. We define a dimensionless field $\hat{\phi}_n$ by the relation $\hat{\phi}_n = a\phi_n$ where $\phi_n = \phi(x)$. The dimensionless mass parameter is $\hat{m}^2 = m^2a^2$. The integral over spacetime will be replaced with a sum over the points of the lattice, i.e

$$\int d^4x = a^4 \sum_n, \sum_n = \sum_{n_1} \ldots \sum_{n_4}.$$ (2.55)
The measure is therefore given by

\[ \int D\phi = \prod_n d\phi_n \cdot \prod_n = \prod_{n_1} \cdots \prod_{n_4}. \]  

(2.56)

The derivative can be replaced either with the forward difference or with the backward difference defined respectively by the equations

\[ \partial_\mu \phi = \frac{\phi_{n+\hat{\mu}} - \phi_n}{a}. \]  

(2.57)

\[ \partial_\mu \phi = \frac{\phi_n - \phi_{n-\hat{\mu}}}{a}. \]  

(2.58)

The \( \hat{\mu} \) is the unit vector in the direction \( x_\mu \). The Laplacian on the lattice is defined such that

\[ \partial^2 \phi = \frac{1}{a^2} \sum_\mu (\phi_{n+\hat{\mu}} + \phi_{n-\hat{\mu}} - 2\phi_n). \]  

(2.59)

The free Euclidean action on the lattice is therefore

\[ S_E[\hat{\phi}] = \frac{1}{2} \int d^4 x \phi [\partial^2 + m^2] \phi = \frac{1}{2} \sum_{n,m} \hat{\phi}_n K_{nm} \hat{\phi}_m. \]  

(2.60)

\[ K_{nm} = -\sum_\mu \left[ \delta_{n+\hat{\mu},m} + \delta_{n-\hat{\mu},m} - 2\delta_{n,m} \right] + m^2 \delta_{n,m}. \]  

(2.61)

The path integral on the lattice is

\[ Z_E[J] = \int \prod_n d\hat{\phi}_n e^{-S_E[\hat{\phi}]} + \sum_n J_n \hat{\phi}_n. \]  

(2.62)

The path integral of the free scalar field on the lattice can be computed in a closed form.

We find

\[ <0|T(\hat{\phi}_s...\hat{\phi}_t)|0>_E = \frac{1}{Z[0]} \delta^n Z_E[J]|_{J=0} \]  

(2.63)

\[ = \frac{\int \prod_n d\hat{\phi}_n \hat{\phi}_s...\hat{\phi}_t e^{-S_E[\hat{\phi}]}}{\int \prod_n d\hat{\phi}_n e^{-S_E[\hat{\phi}]}}. \]  

The path integral of the free scalar field on the lattice can be computed in a closed form.

We find

\[ \text{Exercise:} \]
\[ Z_E[J] = e^{\frac{1}{2} \sum_{n,m} J_n K_{nm}^{-1} J_m} \int \prod_n d\phi_n e^{-\frac{1}{2} \sum_{n,m} (\phi - JK^{-1})_n K_{nm} (\phi - K^{-1} J)_m} \]
\[ = \mathcal{N} e^{\frac{1}{2} \sum_{n,m} J_n K_{nm}^{-1} J_m}. \] (2.65)

The 2-point function is therefore given by
\[
<0|T(\hat{\Phi}_s \hat{\Phi}_t)|0> = \frac{1}{Z[0]} \frac{\delta^2 Z_E[J]}{\delta J_s \delta J_t} \bigg|_{J=0} = K_{st}^{-1}. \] (2.66)

We Fourier transform on the lattice as follows
\[
K_{st} = \int_{-\pi}^{\pi} \frac{d^4 \hat{k}}{(2\pi)^4} \hat{K}(k) e^{ik(s-t)}. \] (2.67)
\[
K_{st}^{-1} = \int_{-\pi}^{\pi} \frac{d^4 \hat{k}}{(2\pi)^4} G(k) e^{ik(s-t)}. \] (2.68)

For \( \hat{K}(k) = G(k) = 1 \) we obtain the identity, viz
\[
\delta_{st} = \int_{-\pi}^{\pi} \frac{d^4 \hat{k}}{(2\pi)^4} e^{ik(s-t)}. \] (2.69)

Furthermore we can show that \( K_{st} K_{tr}^{-1} = \delta_{sr} \) using the equations
\[
(2\pi)^4 \delta^4(\hat{k} - \hat{p}) = \sum_n e^{i(k-p)n}. \] (2.70)
\[
G(k) = \hat{K}^{-1}(k). \] (2.71)

Next we compute
\[
K_{nm} = -\sum_{\mu} \left[ \delta_{n+\hat{\mu},m} + \delta_{n-\hat{\mu},m} - 2\delta_{n,m} \right] + \hat{m}^2 \delta_{n,m}
\]
\[
= -\int_{-\pi}^{\pi} \frac{d^4 \hat{k}}{(2\pi)^4} e^{i(k-n-m)} \sum_{\mu} \left[ e^{ik\mu} + e^{-ik\nu} - 2 \right] + m^2 \int \frac{d^4 \hat{k}}{(2\pi)^4} e^{ik(n-m)}. \] (2.72)

- Perform explicitly the Gaussian integral
\[
I = \int \prod_{i=1}^{N} dx_i e^{-x_i^2} D_{ij} x_j. \] (2.64)

Try to diagonalize the symmetric and invertible matrix \( D \). It is also well advised to adopt an \( i\epsilon \) prescription (i.e. make the replacement \( D \rightarrow D + i\epsilon \)) in order to regularize the integral.
- Use the above result to determine the constant of normalization \( \mathcal{N} \) in equation (2.63).
Thus
\[ \hat{K}(k) = 4 \sum_{\mu} \sin^2 \left( \hat{k}_{\mu} \frac{\pi}{2} \right) + \hat{m}^2. \] (2.73)

Hence
\[ G(k) = \frac{1}{4 \sum_{\mu} \sin^2 \left( \hat{k}_{\mu} \frac{\pi}{2} \right) + \hat{m}^2}. \] (2.74)

The 2-point function is then given by
\[ <0 | T(\hat{\Phi}_s \hat{\Phi}_t) | 0>_E = \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \frac{1}{4 \sum_{\mu} \sin^2 \left( \hat{k}_{\mu} \frac{\pi}{2} \right) + \hat{m}^2} e^{i \hat{k}(s-t)}. \] (2.75)

In the continuum limit \( a \rightarrow 0 \) we scale the fields as follows \( \hat{\Phi}_s = a \phi(x), \hat{\Phi}_t = a \phi(y) \) where \( x = as \) and \( y = at \). The momentum is scaled as \( \hat{k} = ak \) and the mass is scaled as \( \hat{m}^2 = a^2 m^2 \). In this limit the lattice mass \( \hat{m}^2 \) goes to zero and hence the correlation length \( \hat{\xi} = 1/\hat{m} \) diverges. In other words the continuum limit is realized at a critical point of a second order phase transition. The physical 2-point function is given by
\[ <0 | T(\hat{\Phi}(x) \hat{\Phi}(y)) | 0>_E = \lim_{a \rightarrow 0} \frac{1}{a^2} \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 + m^2} e^{ik(x-y)}. \] (2.76)

This is the same result obtained from continuum considerations in the previous section.

2.3 The Effective Action

2.3.1 Formalism

We are interested in the \( \phi^4 \) theory on a Minkowski spacetime given by the classical action
\[ S[\phi] = \int d^4x \left[ \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 \right]. \] (2.77)

The quantum theory is given by the path integral
\[ Z[J] = \int D\phi \ e^{\hat{\pi} S[\phi] + \hat{\pi} J \int d^4x J(x) \phi(x)}. \] (2.78)

The functional \( Z[J] \) generates all Green functions, viz
\[ <0 | T(\Phi(x_1) ... \Phi(x_n)) | 0> = \left. \frac{1}{Z[0]} \left( \frac{\hbar}{i} \right)^n \frac{\delta^n Z[J]}{\delta J(x_1) ... \delta J(x_n)} \right|_{J=0} = \frac{\int D\phi \ \phi(x_1) ... \phi(x_n) \ e^{\hat{\pi} S[\phi]}}{\int D\phi \ e^{\hat{\pi} S[\phi]}}. \] (2.79)
The path integral $Z[J]$ generates disconnected as well as connected graphs and it generates reducible as well as irreducible graphs. Clearly the disconnected graphs can be obtained by putting together connected graphs whereas reducible graphs can be decomposed into irreducible components. All connected Green functions can be generated from the functional $W[J]$ (vacuum energy) whereas all connected and irreducible Green functions (known also as the 1–particle irreducible) can be generated from the functional $\Gamma[\phi_c]$ (effective action). The vacuum energy $W[J]$ is defined through the equation

$$Z[J] = e^{i\frac{\pi}{\hbar}W[J]}.$$  \hspace{1cm} (2.80)

In order to define the effective action we introduce the notion of the classical field. This is defined by the equation

$$\phi_c(x) = \frac{\delta W[J]}{\delta J(x)}.$$  \hspace{1cm} (2.81)

This is a functional of $J$. It becomes the vacuum expectation value of the field operator $\Phi$ at $J = 0$. Indeed we compute

$$\phi_c(x)|_{J=0} = \hbar \int \frac{1}{Z[0]} \frac{\delta Z[J]}{\delta J(x)}|_{J=0} = <0|\Phi(x)|0>.$$  \hspace{1cm} (2.82)

The effective action $\Gamma[\phi_c]$ is the Legendre transform of $W[J]$ defined by

$$\Gamma[\phi_c] = W[J] - \int d^4xJ(x)\phi_c(x).$$  \hspace{1cm} (2.83)

This is the quantum analogue of the classical action $S[\phi]$. The effective action generates all the 1–particle irreducible graphs from which the external legs have been removed. These are the connected, irreducible and amputated graphs.

The classical equations of motion are obtained from the principle of least action applied to the classical action $S[\phi] + \int d^4xJ(x)\phi(x)$. We obtain

$$\frac{\delta S[\phi]}{\delta \phi(x)} = -J(x).$$  \hspace{1cm} (2.84)

Similarly the quantum equations of motion are obtained from the principle of least action applied to the quantum action $\Gamma[\phi_c]$. We obtain

$$\frac{\delta \Gamma[\phi_c]}{\delta \phi_c(x)} = 0.$$  \hspace{1cm} (2.85)

In the presence of source this generalizes to

$$\frac{\delta \Gamma[\phi_c]}{\delta \phi_c(x)} = -J(x).$$  \hspace{1cm} (2.86)
The proof goes as follows:

\[
\frac{\delta \Gamma}{\delta \phi_c(x)} = \frac{\delta W}{\delta \phi_c(x)} - \int d^4y \frac{\delta J(y)}{\delta \phi_c(x)} \phi_c(y) - J
\]
\[
= \frac{\delta W}{\delta \phi_c(x)} - \int d^4y \frac{\delta J(y)}{\delta \phi_c(x)} \frac{\delta W}{\delta \phi_c(x)} - J
\]
\[
= -J(x) \quad (2.87)
\]

A more explicit form of the quantum equation of motion can be obtained as follows. We start from the identity

\[
0 = \int D\phi \frac{\hbar}{i} \delta \left( \frac{\delta S}{\delta \phi(x)} + \frac{\hbar}{i} J(x) \right) e^{\frac{\hbar}{i} S[\phi]} e^{\frac{\hbar}{i} \int d^4x J(x) \phi(x)}
\]
\[
= \int D\phi \left( \frac{\delta S}{\delta \phi(x)} + J \right) e^{\frac{\hbar}{i} S[\phi]} e^{\frac{\hbar}{i} \int d^4x J(x) \phi(x)}
\]
\[
= \left( \frac{\delta S}{\delta \phi(x)} e^{\frac{\hbar}{i} S[\phi]} \right) \left( \frac{\delta S}{\delta \phi(x)} e^{\frac{\hbar}{i} \int d^4x J(x) \phi(x)} \right)
\]
\[
= e^{\frac{\hbar}{i} W[J]} \left( \frac{\delta S}{\delta \phi(x)} e^{\frac{\hbar}{i} \int d^4x J(x) \phi(x)} \right)
\].
\[
(2.88)
\]

In the last line above we have used the identity

\[
F(\partial_x) e^{g(x)} = e^{g(x)} F(\partial_x g + \partial_x).
\]
\[
(2.89)
\]

We obtain the equation of motion

\[
\frac{\delta S}{\delta \phi(x)} e^{\frac{\hbar}{i} S[\phi]} = -J = \frac{\delta \Gamma[\phi_c]}{\delta \phi_c}.
\]
\[
(2.90)
\]

By the chaine rule we have

\[
\frac{\delta}{\delta J(x)} = \int d^4y \frac{\delta \phi_c(y)}{\delta J(x)} \frac{\delta}{\delta \phi_c(y)}
\]
\[
= \int d^4y G^{(2)}(x, y) \frac{\delta}{\delta \phi_c(y)}.
\]
\[
(2.91)
\]

The \(G^{(2)}(x, y)\) is the connected 2−point function in the presence of the source \(J(x)\), viz

\[
G^{(2)}(x, y) = \frac{\delta \phi_c(y)}{\delta J(x)} = \frac{\delta^2 W[J]}{\delta J(x) \delta J(y)}.
\]
\[
(2.92)
\]

The quantum equation of motion becomes

\[
\frac{\delta S}{\delta \phi(x)} e^{\frac{\hbar}{i} \int d^4y G^{(2)}(x, y) \phi_c(y) + \phi_c(x)} = -J = \frac{\delta \Gamma[\phi_c]}{\delta \phi_c}.
\]
\[
(2.93)
\]
The connected \( n \)-point functions and the proper \( n \)-point vertices are defined as follows. The connected \( n \)-point functions are defined by

\[
G^{(n)}(x_1, \ldots, x_n) = G^{i_1 \ldots i_n} = \frac{\delta^n W[J]}{\delta J(x_1) \ldots \delta J(x_n)}. \tag{2.94}
\]

The proper \( n \)-point vertices are defined by

\[
\Gamma^{(n)}(x_1, \ldots, x_n) = \Gamma_{,i_1 \ldots i_n} = \frac{\delta^n \Gamma[\phi_c]}{\delta \phi_c(x_1) \ldots \delta \phi_c(x_n)}. \tag{2.95}
\]

These are connected 1–particle irreducible \( n \)-point functions from which the external legs are removed (amputated).

The proper 2–point vertex \( \Gamma^{(2)}(x, y) \) is the inverse of the connected 2–point function \( G^{(2)}(x, y) \). Indeed we compute

\[
\int d^4 z \ G^{(2)}(x, z) \Gamma^{(2)}(z, y) = \int d^4 z \ \frac{\delta \phi_c(z)}{\delta J(x)} \frac{\delta^2 \Gamma[\phi_c]}{\delta \phi_c(z) \delta \phi_c(y)} = -\int d^4 z \ \frac{\delta \phi_c(z)}{\delta J(x)} \frac{\delta J(y)}{\delta \phi_c(z)} = -\delta^4(x - y). \tag{2.96}
\]

We write this as

\[
G^{ik} \Gamma_{,kj} = -\delta^i_j. \tag{2.97}
\]

We remark the identities

\[
\frac{\delta G^{i_1 \ldots i_n}}{\delta J_{i_{n+1}}} = G^{i_1 \ldots i_n i_{n+1}}. \tag{2.98}
\]

\[
\frac{\delta \Gamma_{,i_1 \ldots i_n}}{\delta J_{i_{n+1}}} = \frac{\delta \Gamma_{,i_1 \ldots i_n}}{\delta \phi_c^k} \frac{\delta \phi_c^k}{\delta J_{i_{n+1}}} = \Gamma_{,i_1 \ldots i_n k} G^{ki_{n+1}}. \tag{2.99}
\]

By differentiating (2.97) with respect to \( J_l \) we obatin

\[
G^{ikl} \Gamma_{,kj} + G^{ik} \Gamma_{,kjr} G^{rl} = 0. \tag{2.100}
\]

Next by multiplying with \( G^{js} \) we get the 3–point connected function as

\[
G^{isl} = G^{ik} G^{rl} G^{js} \Gamma_{,kjr}. \tag{2.101}
\]

Now by differentiating (2.101) with respect to \( J_m \) we obatin the 4–point connected function as

\[
G^{isml} = \left( G^{ikm} G^{rl} G^{js} + G^{ik} G^{rlm} G^{js} + G^{ik} G^{rl} G^{jsm} \right) \Gamma_{,kjr} + G^{ik} G^{rl} G^{js} \Gamma_{,kjrn} G^{nm}. \tag{2.102}
\]
By using again (2.101) we get

\[ G^{islm} = \Gamma_{k'j'r} G^{ik'} G^{kj'} G^{mr} G^{jr} \Gamma_{kjr} + \text{two permutations} + G^{ik} G^{jr} \Gamma_{kjr} G^{im}. \]  

(2.103)

The diagrammatic representation of (2.94), (2.95), (2.97), (2.101) and (2.103) is shown on figure 1.

2.3.2 Perturbation Theory

In this section we will consider a general scalar field theory given by the action

\[ S[\phi] = S_i \phi_i^2 + \frac{1}{2!} S_{ij} \phi_j \phi^i \phi^j + \frac{1}{3!} S_{ijk} \phi_j \phi^i \phi^k \phi^l + \frac{1}{4!} S_{ijkl} \phi_j \phi^i \phi^k \phi^l \phi^m + ... \]  

(2.104)

We need the first derivative of \( S[\phi] \) with respect to \( \phi^i \), viz

\[ S[\phi],_i = S_i + S_{ij} \phi_j^i + \frac{1}{2!} S_{ijk} \phi_j^i \phi^k + \frac{1}{3!} S_{ijkl} \phi_j^i \phi^k \phi^m + \frac{1}{4!} S_{ijklm} \phi_j^i \phi^k \phi^m \phi^n + ... \]  

(2.105)

Thus

\[ \Gamma[\phi^c],_i = S[\phi],_i |_{\phi = \phi^c} + \frac{1}{2!} G^{im} \frac{\delta}{\delta \phi^c_0} \frac{\delta}{\delta \phi^c_0} S_i + \frac{1}{3!} S_{ijkl} \phi_j^i \phi^k + \frac{1}{4!} S_{ijklm} \phi_j^i \phi^k \phi^m + ... \]  

(2.106)

We find up to the first order in \( \hbar \) the result \(^8\)

\[ \Gamma[\phi^c],_i = S[\phi],_i |_{\phi = \phi^c} + \frac{1}{2!} G^{jk} \left( S_{ij} + S_{ijkl} \phi_j^i \phi^k \phi^m + ... \right) + O\left( \frac{\hbar^2}{2} \right). \]  

(2.107)

In other words

\[ \Gamma[\phi^c],_i = S[\phi],_i |_{\phi = \phi^c} + \frac{1}{2!} G^{jk} S[\phi^c],_j + O\left( \frac{\hbar^2}{2} \right). \]  

(2.108)

We expand

\[ \Gamma = \Gamma_0 + \frac{\hbar}{2} \Gamma_1 + \left( \frac{\hbar}{2} \right)^2 \Gamma_2 + ... \]  

(2.109)

\(^8\)Exercise: Verify this equation.
\[ G^{ij} = G_0^{ij} + \frac{\hbar}{i} G_1^{ij} + \left( \frac{\hbar}{i} \right)^2 G_2^{ij} + \ldots \]  
(2.110)

Immediately we find

\[ \Gamma_0[\phi_c],i = S[\phi_c],i. \]  
(2.111)

\[ \Gamma_1[\phi_c],i = \frac{1}{2} G_0^{jk} S[\phi_c],ijk. \]  
(2.112)

Equation (2.111) can be trivially integrated. We obtain

\[ \Gamma_0[\phi_c] = S[\phi_c]. \]  
(2.113)

Let us recall the constraint \( G^{ik}\Gamma_{kj} = -\delta^i_j. \) This is equivalent to the constraints

\[ \begin{align*}
G_0^{ik}\Gamma_{0,kj} &= -\delta^i_j \\
G_0^{ik}\Gamma_{1,kj} + G_1^{ik}\Gamma_{0,kj} &= 0 \\
G_0^{ik}\Gamma_{2,kj} + G_1^{ik}\Gamma_{1,kj} + G_2^{ik}\Gamma_{0,kj} &= 0 \\
\end{align*} \]  
(2.114)

The first constraint gives \( G_0^{ik} \) in terms of \( \Gamma_0 = S \) as

\[ G_0^{ik} = -S^{-1}_{ik}. \]  
(2.115)

The second constraint gives \( G_1^{ik} \) in terms of \( \Gamma_0 \) and \( \Gamma_1 \) as

\[ G_1^{ij} = G_0^{ik} G_0^{jl} \Gamma_{kl}. \]  
(2.116)

The third constraint gives \( G_2^{ik} \) in terms of \( \Gamma_0, \Gamma_1 \) and \( \Gamma_2 \). Hence the calculation of the 2–point function \( G^{ik} \) to all orders in perturbation theory requires the calculation the effective action to all orders in perturbation theory, viz the calculation of the \( \Gamma_n \). In fact the knowledge of the effective action will allow us to calculate all proper \( n \)–point vertices to any order in perturbation theory.

We are now in a position to integrate equation (2.112). We have

\[ \begin{align*}
\Gamma_1[\phi_c],i &= \frac{1}{2} G_0^{jk} \frac{\delta S[\phi_c],jk}{\delta \phi_{ci}} \\
&= -\frac{1}{2} G_0^{jk} \frac{\delta (G_0^{-1})_{jk}}{\delta \phi_{ci}} \\
&= -\frac{1}{2} \frac{\delta}{\delta \phi_{ci}} \ln \det G_0^{-1}. \end{align*} \]  
(2.117)
Thus
\[ \Gamma_1[\phi_c] = -\frac{1}{2} \ln \det G_0^{-1}. \]  

The effective action up to the 1-loop order is
\[ \Gamma = \Gamma_0 + \frac{i}{2} \frac{\hbar}{i} \ln \det G_0 + \ldots \]  

This is represented graphically by the first two diagrams on figure 2.

### 2.3.3 Analogy with Statistical Mechanics

We start by making a Wick rotation. The Euclidean vacuum energy, classical field, classical equation of motion, effective action and quantum equation of motion are defined by
\[
Z_E[J] = e^{-\frac{i}{\hbar} W_E[J]},
\]
\[
\phi_c(x)|_{J=0} = -\frac{\delta W_E[J]}{\delta J(x)}|_{J=0} = <0|\Phi(x)|0>_{E}.
\]
\[
\frac{\delta S_E[\phi]}{\delta \phi(x)} = J(x).
\]
\[
\Gamma_E[\phi_c] = W_E[J] + \int d^4x J(x)\phi_c(x).
\]
\[
\frac{\delta \Gamma_E[\phi_c]}{\delta \phi_c(x)} = J(x).
\]

Let us now consider the following statistical mechanics problem. We consider a magnetic system consisting of spins \( s(x) \). The spin energy density is \( \mathcal{H}(s) \). The system is placed in a magnetic field \( H \). The partition function of the system is defined by
\[
Z[H] = \int Ds \ e^{-\beta \int dx \mathcal{H}(s) + \beta \int dx H(x)s(x)}.
\]

The spin \( s(x) \), the spin energy density \( \mathcal{H}(s) \) and the magnetic field \( H(x) \) play in statistical mechanics the role played by the scalar field \( \phi(x) \), the Lagrangian density \( \mathcal{L}(\phi) \) and the source \( J(x) \) respectively in field theory. The free energy of the magnetic system is defined through the equation
\[
Z[H] = e^{-\beta F[H]}.
\]
This means that $F$ in statistical mechanics is the analogue of $W$ in field theory. The magnetization of the system is defined by

$$\left. -\frac{\delta F}{\delta H} \right|_{\beta=\text{fixed}} = \frac{1}{Z} \int dx \int Ds(x) e^{-\beta \int dx(H(x) - H_s(x))} = \int dx <s(x)> = M.$$  \hfill (2.127)

Thus the magnetization $M$ in statistical mechanics plays the role of the effective field $-\phi_c$ in field theory. In other words $\phi_c$ is the order parameter in the field theory. Finally the Gibbs free energy in statistical mechanics plays the role of the effective action $\Gamma[\phi_c]$ in field theory. Indeed $G$ is the Legendre transform of $F$ given by

$$G = F + MH.$$  \hfill (2.128)

Furthermore we compute

$$\frac{\delta G}{\delta M} = H.$$  \hfill (2.129)

The thermodynamically most stable state (the ground state) is the minimum of $G$. Similarly the quantum mechanically most stable state (the vacuum) is the minimum of $\Gamma$. The thermal fluctuations from one side correspond to quantum fluctuations on the other side.

### 2.4 The $O(N)$ Model

In this section we will consider a generalization of the $\phi^4$ model known as the linear sigma model. We are interested in the $(\phi^2)^2$ theory with $O(N)$ symmetry given by the classical action

$$S[\phi] = \int d^4x \left[ \frac{1}{2} \partial_\mu \phi_i \partial^\mu \phi_i - \frac{1}{2} m^2 \phi_i^2 - \frac{\lambda}{4!}(\phi_i^2)^2 \right].$$  \hfill (2.130)

This classical action is of the general form studied in the previous section, viz

$$S[\phi] = \frac{1}{2!} S_{IJ} \phi^I \phi^J + \frac{1}{4!} S_{IJKL} \phi^I \phi^J \phi^K \phi^L.$$  \hfill (2.131)

The index $I$ stands for $i$ and the spacetime index $x$, i.e $I = (i, x)$, $J = (j, y)$, $K = (k, z)$ and $L = (l, w)$. We have

$$S_{IJ} = -\delta_{ij} (\Delta + m^2) \delta^4(x - y)$$
$$S_{IJKL} = -\frac{\lambda}{3} \delta_{ijkl} \delta^4(y - x) \delta^4(z - x) \delta^4(w - x) , \; \delta_{ijkl} = \delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}.$$  \hfill (2.132)
The effective action up to the 1-loop order is
\[ \Gamma[\phi] = S[\phi] + \frac{1}{2} \hbar \ln \det G_0. \] (2.133)

The proper "n"-point vertex is defined now by setting \( \phi = 0 \) after taking the
\( n \) derivatives, viz.
\[ \Gamma^{(n)}_{i_1 \ldots i_n}(x_1, \ldots, x_n) = \frac{\delta^n \Gamma[\phi]}{\delta \phi_{i_1}(x_1) \ldots \delta \phi_{i_n}(x_n)} |_{\phi = 0}. \] (2.134)

### 2.4.1 The 2-Point and 4-Point Proper Vertices

The proper 2-point vertex is defined by
\[ \Gamma^{(2)}_{ij}(x, y) = \frac{\delta^2 \Gamma[\phi]}{\delta \phi_i(x) \delta \phi_j(y)} |_{\phi = 0} = \frac{\delta^2 S[\phi]}{\delta \phi_i(x) \delta \phi_j(y)} |_{\phi = 0} + \frac{\hbar}{i} \frac{\delta^2 \Gamma[\phi]}{\delta \phi_i(x) \delta \phi_j(y)} |_{\phi = 0} \]
\[ = -\delta_{ij} (\Delta + m^2) \delta^4(x - y) + \frac{\hbar}{i} \frac{\delta^2 \Gamma[\phi]}{\delta \phi_i(x) \delta \phi_j(y)} |_{\phi = 0}. \] (2.135)

The one-loop correction can be computed using the result
\[ \Gamma_1[\phi]_{IJ} = \frac{1}{2} G_0^{mn} S[\phi]_{ij} \delta_{mn} + \frac{1}{2} G_0^{mn} G_0^{mn} S[\phi]_{ij} \delta_{mn} S[\phi]_{k0} \delta_{mn}. \] (2.136)

We get by setting \( \phi = 0 \) the result
\[ \Gamma_1[\phi]_{IJ} = \frac{1}{2} G_0^{mn} S[\phi]_{ij} \delta_{mn}. \]
\[ = \frac{\lambda}{6} \int d^4 z d^4 w G_0^{mn}(z, w) \left( \delta_{ij} \delta_{mn} + \delta_{im} \delta_{jn} + \delta_{in} \delta_{jm} \right) \delta^4(y - x) \delta^4(z - x) \delta^4(w - x). \]
\[ = \frac{\lambda}{6} \left( \delta_{ij} G_0^{mm}(x, y) + 2 G_0^{ij}(x, y) \right) \delta^4(x - y). \] (2.137)

We have
\[ G_0^{IJ} = -S_{IJ}^{-1}. \] (2.138)

Since \( S_{IJ} = S_{JI} \) and \( S_{IJ} = -\delta_{ij} S(x, y) \) where \( S(x, y) = (\Delta + m^2) \delta^4(x - y) \) we can write
\[ G_0^{IJ} = \delta_{ij} G_0(x, y). \] (2.139)

Clearly \( \int d^4 x G_0(x, y) S(y, z) = \delta^4(x - y) \). We obtain
\[ \Gamma_1[\phi]_{IJ} = -\frac{\lambda}{6} (N + 2) \delta_{ij} G_0(x, y) \delta^4(x - y). \] (2.140)
Now we compute the 4-point proper vertex. Clearly the first contribution will be given precisely by the second equation of (2.132). Indeed we have

\[
\begin{align*}
\Gamma^{(4)}_{i_1 \ldots i_4}(x_1, \ldots, x_4) &= \frac{\delta^4 \Gamma[\phi]}{\delta \phi_{i_1}(x_1) \ldots \delta \phi_{i_4}(x_4)}|_{\phi=0} \\
&= \frac{\delta^4 S[\phi]}{\delta \phi_{i_1}(x_1) \ldots \delta \phi_{i_4}(x_4)}|_{\phi=0} + \frac{\hbar}{i} \frac{\delta^4 \Gamma[\phi]}{\delta \phi_{i_1}(x_1) \ldots \delta \phi_{i_4}(x_4)}|_{\phi=0} \\
&= S_{i_1 \ldots i_4} + \frac{\hbar}{i} \delta^2 \Gamma[\phi] |_{\phi=0}.
\end{align*}
\]

(2.141)

In order to compute the first correction we use the identity

\[
\frac{\delta G^{mn}_{0}}{\delta \phi_{el}} = G^{mno}_{0} G^{nmo}_{0} S[\phi],_{lmn0}.
\]

(2.142)

We compute

\[
\Gamma[\phi]_{j_0k_0l_0} = \left[ \frac{1}{2} G^{mno}_{0} G^{nmo}_{0} S[\phi],_{j_0k_0m_0} + \frac{1}{2} G^{mno}_{0} G^{nmo}_{0} S[\phi],_{j_0l_0m_0} \right] |_{\phi=0}.
\]

(2.143)

Thus

\[
\frac{\delta^4 \Gamma[\phi]}{\delta \phi_{i_1}(x_1) \ldots \delta \phi_{i_4}(x_4)}|_{\phi=0} = \frac{1}{2} \left( \frac{\lambda}{3} \right)^2 \delta^4(x_1 - x_2) \delta^4(x_3 - x_4) \left( (N + 2) \delta_{i_1,i_2} \delta_{i_3,i_4} + 2 \delta_{i_1,i_2,i_3,i_4} \right) G_0(x_1, x_3)^2 \\
+ \frac{1}{2} \left( \frac{\lambda}{3} \right)^2 \delta^4(x_1 - x_3) \delta^4(x_2 - x_4) \left( (N + 2) \delta_{i_1,i_2} \delta_{i_3,i_4} + 2 \delta_{i_1,i_2,i_3,i_4} \right) G_0(x_1, x_2)^2 \\
+ \frac{1}{2} \left( \frac{\lambda}{3} \right)^2 \delta^4(x_1 - x_4) \delta^4(x_2 - x_3) \left( (N + 2) \delta_{i_1,i_2} \delta_{i_3,i_4} + 2 \delta_{i_1,i_2,i_3,i_4} \right) G_0(x_1, x_2)^2.
\]

(2.144)

### 2.4.2 Momentum Space Feynman Graphs

The proper 2-point vertex up to the 1-loop order is

\[
\Gamma^{(2)}_{ij}(x, y) = -\delta_{ij}(\Delta + m^2) \delta^4(x - y) - \frac{\hbar \lambda}{6} (N + 2) \delta_{ij} G_0(x, y) \delta^4(x - y).
\]

(2.145)

The proper 2-point vertex in momentum space \(\Gamma^{(2)}_{ij}(p)\) is defined through the equations

\[
\int d^4 x \Gamma^{(2)}_{ij}(x, y) e^{ipx + iky} = (2\pi)^4 \delta^4(p + k) \Gamma^{(2)}_{ij}(p, k) \\
= (2\pi)^4 \delta^4(p + k) \Gamma^{(2)}_{ij}(p, -p) \\
= (2\pi)^4 \delta^4(p + k) \Gamma^{(2)}_{ij}(p).
\]

(2.146)
The delta function is due to translational invariance.

From the definition $S(x, y) = (\Delta + m^2)\delta^4(x - y)$ we have

$$S(x, y) = \int \frac{d^4p}{(2\pi)^4}(-p^2 + m^2) e^{ip(x-y)}. \quad (2.147)$$

Then by using the equation $\int d^4y \ G_0(x, y)S(y, z) = \delta^4(x - y)$ we obtain

$$G_0(x, y) = \int \frac{d^4p}{(2\pi)^4} \frac{1}{-p^2 + m^2} e^{ip(x-y)}. \quad (2.148)$$

We get

$$\Gamma_{ij}^{(2)}(p) = -\delta_{ij}(-p^2 + m^2) - \frac{\hbar \lambda}{i} \frac{1}{6} (N + 2) \delta_{ij} \int \frac{d^4p_1}{(2\pi)^4} \frac{1}{-p_1^2 + m^2}. \quad (2.149)$$

The corresponding Feynman diagrams are shown on figure 4.

The proper 4-point vertex up to the 1-loop order is

$$\Gamma_{i_1...i_4}^{(4)}(x_1, ..., x_4) = -\frac{\lambda}{3} \left( \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \right) \delta^4(y - x) \delta^4(z - x) \delta^4(w - x) + \frac{1}{2} \left( \frac{\hbar}{i} \right) \left( \frac{\lambda}{3} \right)^2 \delta^4(x_1 - x_2) \delta^4(x_3 - x_4) \left( (N + 2) \delta_{i_1i_2} \delta_{i_3i_4} + 2\delta_{i_1i_2i_3i_4} \right) G_0(x_1, x_3)^2 + \delta^4(x_1 - x_3) \delta^4(x_2 - x_4) \left( (N + 2) \delta_{i_1i_3} \delta_{i_2i_4} + 2\delta_{i_1i_2i_3i_4} \right) G_0(x_1, x_2)^2 + \delta^4(x_1 - x_4) \delta^4(x_2 - x_3) \left( (N + 2) \delta_{i_1i_4} \delta_{i_2i_3} + 2\delta_{i_1i_2i_3i_4} \right) G_0(x_1, x_4)^2 \right]. \quad (2.150)$$

The proper 4-point vertex in momentum space $\Gamma_{i_1...i_4}^{(4)}(p_1...p_4)$ is defined through the equation

$$\int d^4x_1...d^4x_4 \Gamma_{i_1...i_4}^{(4)}(x_1, ..., x_4) e^{ip_1x_1 + ... + ip_4x_4} = (2\pi)^4 \delta^4(p_1 + ... + p_4) \Gamma_{i_1...i_4}^{(2)}(p_1, ..., p_4). \quad (2.151)$$

We find (with $p_{12} = p_1 + p_2$ and $p_{14} = p_1 + p_4$, etc)

$$\Gamma_{i_1...i_4}^{(4)}(p_1, ..., p_4) = -\frac{\lambda}{3} \delta_{i_1i_2i_3i_4} + \frac{\hbar}{i} \left( \frac{\lambda}{3} \right)^2 \frac{1}{2} \left[ \left( (N + 2) \delta_{i_1i_2} \delta_{i_3i_4} + 2\delta_{i_1i_2i_3i_4} \right) \int \frac{1}{(k^2 + m^2)(-(p_{12} - k)^2 + m^2)} \right] + 2 \text{ permutations}. \quad (2.152)$$

The corresponding Feynman diagrams are shown on figure 5.
2.4.3 Cut-off Regularization

At the one-loop order we have then

\[ \Gamma_{ij}^{(2)}(p) = -\delta_{ij}(-p^2 + m^2) - \frac{h \lambda}{6}(N + 2)\delta_{ij}I(m^2). \quad (2.153) \]

\[ \Gamma_{i_1...i_4}^{(4)}(p_1,...,p_4) = -\frac{\lambda}{3}\delta_{i_1i_2i_3i_4} + \frac{h}{i}(\frac{\lambda}{3})^2 \frac{1}{2} \left[ (N + 2)\delta_{i_1i_2}\delta_{i_3i_4} + 2\delta_{i_1i_2i_3i_4} \right] J(p_{i_1i_2}^2, m^2) \]

\[ + \text{ 2 permutations } \right] , \quad (2.154) \]

where

\[ \Delta(k) = \frac{1}{-k^2 + m^2} , \quad I(m^2) = \int \frac{d^4k}{(2\pi)^4} \Delta(k) , \quad J(p_{i_1i_2}^2, m^2) = \int \frac{d^4k}{(2\pi)^4} \Delta(k)\Delta(p_{i_1i_2} - k). \quad (2.155) \]

It is not difficult to convince ourselves that the first integral \( I(m^2) \) diverges quadratically whereas the second integral \( J(p_{i_1i_2}^2, m^2) \) diverges logarithmically. To see this more carefully it is better we Wick rotate to Euclidean signature. Formally this is done by writing \( k_0 = ik_4 \) which is consistent with \( x^0 = -ix^4 \). As a consequence we replace \( k^2 = k_0^2 - k^2 \) with \( -k_2^2 - \vec{k}^2 = -k^2 \). The Euclidean expressions are

\[ \Gamma_{ij}^{(2)}(p) = \delta_{ij}(p^2 + m^2) + \frac{h\lambda}{6}(N + 2)\delta_{ij}I(m^2). \quad (2.156) \]

\[ \Gamma_{i_1...i_4}^{(4)}(p_1,...,p_4) = \frac{\lambda}{3}\delta_{i_1i_2i_3i_4} - h(\frac{\lambda}{3})^2 \frac{1}{2} \left[ (N + 2)\delta_{i_1i_2}\delta_{i_3i_4} + 2\delta_{i_1i_2i_3i_4} \right] J(p_{i_1i_2}^2, m^2) \]

\[ + \text{ 2 permutations } \right] , \quad (2.157) \]

where now

\[ \Delta(k) = \frac{1}{k^2 + m^2} , \quad I(m^2) = \int \frac{d^4k}{(2\pi)^4} \Delta(k) , \quad J(p_{i_1i_2}^2, m^2) = \int \frac{d^4k}{(2\pi)^4} \Delta(k)\Delta(p_{i_1i_2} - k). \quad (2.158) \]

Explicitly we have

\[ I(m^2) = \int_0^\infty d\alpha e^{-\alpha m^2} \int \frac{d^4k}{(2\pi)^4} e^{-\alpha k^2} \]

\[ = \int_0^\infty d\alpha e^{-\alpha m^2} \frac{1}{8\pi^2} \int k^2 dk e^{-\alpha k^2} \]

\[ = \frac{1}{16\pi^2} \int_0^\infty d\alpha e^{-\alpha m^2}. \quad (2.159) \]
To calculate the divergences we need to introduce a cut-off \( \Lambda \). In principle we should use the regularized propagator

\[
\Delta(k, \Lambda) = \frac{e^{-\frac{k^2}{\Lambda^2}}}{k^2 + m^2}.
\]  

(2.160)

Alternatively we can introduce the cut-off \( \Lambda \) as follows

\[
I(m^2, \Lambda) = \frac{1}{16\pi^2} \int_0^\infty \frac{d\alpha}{\alpha^2} e^{-\alpha m^2} \left( \Lambda^2 - m^2 \right) = \frac{1}{16\pi^2} \left( \Lambda^2 + m^2 Ei\left( -\frac{m^2}{\Lambda^2} \right) \right). 
\]

(2.161)

This diverges quadratically. The exponential-integral function is defined by

\[
Ei(x) = \int_{-\infty}^x \frac{e^t}{t} dt.
\]

(2.162)

Also by using the same method we compute

\[
J(p^2_{T2}, m^2) = \int d\alpha_1 d\alpha_2 e^{-m^2(\alpha_1 + \alpha_2) - \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2} p^2_{T2}} \int \frac{d^4k}{(2\pi)^4} e^{-(\alpha_1 + \alpha_2)k^2} = \frac{1}{(4\pi)^2} \int d\alpha_1 d\alpha_2 \frac{e^{-m^2(\alpha_1 + \alpha_2) - \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2} p^2_{T2}}}{(\alpha_1 + \alpha_2)^2}.
\]

(2.163)

We introduce the cut-off \( \Lambda \) as follows

\[
J(p^2_{T2}, m^2, \Lambda) = \frac{1}{(4\pi)^2} \int d\alpha_1 d\alpha_2 \frac{e^{-m^2(\alpha_1 + \alpha_2) - \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2} p^2_{T2}}}{(\alpha_1 + \alpha_2)^2} = \frac{1}{(4\pi)^2} \int dx dx_2 \frac{e^{-m^2(x + x_2) - \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2} p^2_{T2}}}{(x + x_2)^2}.
\]

(2.164)

The integral can be rewritten as 2 times the integral over the symmetric region \( x_2 > x \). We can also perform the change of variables \( x_2 = xy \) to obtain

\[
J(p^2_{T2}, m^2, \Lambda) = \frac{2}{(4\pi)^2} \int_1^\infty \frac{dx}{x} \int_0^1 \frac{dy}{1 + y} e^{-\frac{m^2}{\Lambda^2} x(1 + y)} - \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2} \frac{p^2_{T2}}{\Lambda^2}.
\]

(2.165)
In above $a = \frac{m^2}{\Lambda^2}$ and $b = \frac{p^2_q}{\Lambda^2}$. We have

$$J(p^2_{12}, m^2, \Lambda) = \frac{2}{(4\pi)^2} \int_0^1 \frac{d\rho}{x} \int_0^\infty e^{-x} \left( \frac{a}{x} + b(1-\rho) \right)$$

$$= -\frac{1}{8\pi^2} \int_0^1 d\rho \, \text{Ei}\left(-\frac{a}{\rho} - (1-\rho)b\right). \quad (2.166)$$

The exponential-integral function is such that

$$\text{Ei}\left(-\frac{a}{\rho} - (1-\rho)b\right) = C + \ln \left( \sqrt{\frac{a}{4} + \frac{b}{4} - \sqrt{b\rho}} \right) + \ln \left( \sqrt{\frac{a}{4} - \frac{b}{4} + \sqrt{b\rho}} \right) - \ln \rho. \quad (2.167)$$

By using the integral $\int_0^1 d\rho \ln(A + B\rho) = \frac{1}{B}(A + B) \ln(A + B) - A \ln A - 1$ we find

$$\int_0^1 d\rho \, \text{Ei}\left(-(1-\rho)b - \frac{a}{\rho}\right) = C + \ln a + \sqrt{1 + \frac{4a}{b} \ln \left( 1 + \frac{b}{2a} + \frac{1}{2a} \sqrt{b(b+4a)} \right)} + 1. \quad (2.169)$$

Hence we have

$$-\int_0^1 d\rho \, \text{Ei}\left(-(1-\rho)b - \frac{a}{\rho}\right) = -\ln a + ... = \ln \frac{\Lambda^2}{m^2} + ... \quad (2.170)$$

Equivalently

$$J(p^2_{12}, m^2, \Lambda) = \frac{1}{16\pi^2} \ln \frac{\Lambda^2}{2m^2} + ... \quad (2.171)$$

This is the logarithmic divergence.

In summary we have found two divergences at one-loop order. A quadratic divergence in the proper 2–point vertex and a logarithmic divergence in the proper 4–point vertex. All higher $n$–point vertices are finite in the limit $\Lambda \to \infty$.

### 2.4.4 Renormalization at 1–Loop

To renormalize the theory, i.e. to remove the above two divergences we will assume that:

- 1) The theory comes with a cut-off $\Lambda$ so that the propagator of the theory is actually given by (2.160).
• 2) The parameters of the model \( m^2 \) and \( \lambda \) which are called from now on bare parameters will be assumed to depend implicitly on the cut-off \( \Lambda \).

• 3) The renormalized (physical) parameters of the theory \( m^2_R \) and \( \lambda^R \) will be determined from specific conditions imposed on the 2– and 4–proper vertices.

In the limit \( \Lambda \to \infty \) the renormalized parameters remain finite while the bare parameters diverge in such a way that the divergences coming from loop integrals are canceled. In this way the 2– and 4–proper vertices become finite in the large cut-off limit \( \Lambda \to \infty \).

Since only two vertices are divergent we will only need two conditions to be imposed. We choose the physical mass \( m^2_R \) to correspond to the zero momentum value of the proper 2–point vertex, viz

\[
\Gamma^{(2)}_{ij}(0) = \delta_{ij} m^2_R = \delta_{ij} m^2 + \hbar \frac{\lambda^R}{6} (N + 2) \delta_{ij} I(m^2, \Lambda). \tag{2.172}
\]

We solve for the bare parameters in terms of the renormalized parameters we find

\[
m^2 = m^2_R - \hbar \frac{\lambda^R}{6} (N + 2) I(m^2_R, \Lambda). \tag{2.174}
\]

\[
\frac{\lambda}{3} = \frac{\lambda^R}{3} + \hbar \left( \frac{\lambda^R}{3} \right) \frac{N + 8}{2} J(0, m^2_R, \Lambda). \tag{2.175}
\]

The 2– and 4–vertices in terms of the renormalized parameters are

\[
\Gamma^{(2)}_{ij}(p) = \delta_{ij} (p^2 + m^2_R). \tag{2.176}
\]

\[
\Gamma^{(4)}_{i_1 \ldots i_4}(p_1, \ldots, p_4) = \frac{\lambda^R}{3} \delta_{i_1 i_2 i_3 i_4} - \hbar \left( \frac{\lambda^R}{3} \right)^2 \frac{1}{2} \left[ (N + 2) \delta_{i_1 i_2} \delta_{i_3 i_4} + 2 \delta_{i_1 i_2 i_3 i_4} \right] \left( J(p_2^2, m^2_R, \Lambda) - J(0, m^2_R, \Lambda) \right) + \text{2 permutations}. \tag{2.177}
\]
2.5 Two-Loop Calculations

2.5.1 The Effective Action at 2–Loop

By extending equation (2.107) to the second order in $\hbar$ we get

$$\Gamma[\phi_c],i = O(1) + O\left(\frac{\hbar}{i}\right)^2 \left[ G^{ij} \frac{\delta G^{kl}}{\delta \phi_{cj}} \left( S_{ijkl} + S_{ijklm} \phi^m_c + \frac{1}{2} S_{ijklmn} \phi^m_c \phi^n_c + \ldots \right) ight. + \left. \frac{3}{4} \left( S_{ijklm} + S_{ijklmn} \phi^k_c + \ldots \right) G^{jk} G^{lm} \right] + O\left(\left(\frac{\hbar}{i}\right)^3\right). \quad (2.178)$$

Equivalently

$$\Gamma[\phi_c],i = O(1) + O\left(\frac{\hbar}{i}\right)^2 \left[ -G^{jj} G^{kk} G^{ll} G^{mm} \Gamma,mj_0 \right. + \left. \frac{3}{4} S[\phi_c],ijklm G^{jk} G^{lm} \right] + O\left(\left(\frac{\hbar}{i}\right)^3\right). \quad (2.179)$$

We use the identity

$$\frac{\delta G^{kl}}{\delta \phi_{cj}} = \frac{\delta G^{kl}}{\delta J_m} \phi_{cj} = -G^{klm} \Gamma,ml_0 \Gamma,kl_0 \phi_{mj_0}.$$

Thus

$$\Gamma[\phi_c],i = O(1) + O\left(\frac{\hbar}{i}\right)^2 \left[ -G^{jj} G^{kk} G^{ll} G^{mm} \Gamma,ml_0 \Gamma,kl_0 \phi_{mj_0} S[\phi_c],ijkl ight. + \left. \frac{3}{4} S[\phi_c],ijklm G^{jk} G^{lm} \right] + O\left(\left(\frac{\hbar}{i}\right)^3\right). \quad (2.181)$$

By substituting the expansions (2.109) and (2.110) we get at the second order in $\hbar$ the equation

$$\Gamma_2[\phi_c],i = \frac{1}{2} G^{ij} S[\phi_c],ijkl + \frac{1}{6} G^{ij} G^{kl} G^{lm} \Gamma,kl_0 \Gamma,ml_0 \phi_{mj_0} S[\phi_c],ijkl + \frac{3}{4} S[\phi_c],ijklm G^{jk} G^{lm}. \quad (2.182)$$

Next we compute $G^{ij}_1$. Therefore we must determine $\Gamma_{ikl}$. By differentiating equation (2.112) with respect to $\phi_{cl}$ we get

$$\Gamma_1[\phi_c],kl = \frac{1}{2} G^{mn}_0 S[\phi_c],klmn + \frac{1}{2} \frac{\delta G^{mn}_0}{\delta \phi_{cl}} S[\phi_c],kmn. \quad (2.183)$$

Exercise: Verify this equation.
By using the identity
\[
\frac{\delta G_{mn}}{\delta \phi_{,kl}} = G_{0}^{mn} G_{0}^{0\alpha} S[\phi_{c}, \lambda_{\alpha}] .
\] (2.184)

We get
\[
\Gamma_{1}[\phi_{c}, j_{,k}] = \frac{1}{2} G_{0}^{mn} S[\phi_{c}, j_{,kmn}] + \frac{1}{2} G_{0}^{mno} G_{0}^{nno} S[\phi_{c}, j_{,mn}] S[\phi_{c}, k_{,mon}].
\] (2.185)

Hence
\[
G_{1}^{jk} = G_{0}^{jko} G_{0}^{kko} \left( \frac{1}{2} G_{0}^{mn} S[\phi_{c}, j_{,kmn}] + \frac{1}{2} G_{0}^{mno} G_{0}^{nno} S[\phi_{c}, j_{,mn}] S[\phi_{c}, k_{,mon}] \right). \] (2.186)

Equation (2.182) becomes
\[
\Gamma_{2}[\phi_{c}, i] = \frac{1}{2} S[\phi_{c}, i_{,jkl}] G_{0}^{ij} G_{0}^{kl} + \frac{1}{12} S[\phi_{c}, ikm] G_{0}^{ij} G_{0}^{kl} G_{0}^{mn} S[\phi_{c}, jln].
\] (2.188)

Integration of this equation yields \(^{10}\)
\[
\Gamma_{2}[\phi_{c}] = \frac{1}{8} S[\phi_{c}, i_{,jkl}] G_{0}^{ij} G_{0}^{kl} + \frac{1}{12} S[\phi_{c}, ikm] G_{0}^{ij} G_{0}^{kl} G_{0}^{mn} S[\phi_{c}, jln].
\] (2.189)

The effective action upto the 2–loop order is
\[
\Gamma = \Gamma_{0} + \frac{\hbar}{2i} \ln \det G_{0} + \left( \frac{\hbar}{i} \right)^{2} S[\phi_{c}, i_{,jkl}] G_{0}^{ij} G_{0}^{kl} + \frac{1}{12} S[\phi_{c}, ikm] G_{0}^{ij} G_{0}^{kl} G_{0}^{mn} S[\phi_{c}, jln] + ...
\] (2.189)

This is represented graphically on figure 2.

### 2.5.2 The Linear Sigma Model at 2–Loop

The proper 2–point vertex upto 2–loop is given by \(^{11}\)
\[
\Gamma_{ij}^{(2)}(x, y) = O(1) + O \left( \frac{\hbar}{i} \right)^{2} + \left( \frac{\hbar}{i} \right)^{2} \frac{\delta^{2} \Gamma_{2}[\phi]}{\delta \phi_{i}(x) \delta \phi_{j}(y)} \bigg|_{\phi=0}.
\] (2.190)

\(^{10}\) Exercise: verify this result.

\(^{11}\) Exercise: Verify all equations of this section.
The 2–loop correction can be computed using the result

\[
\Gamma_2[\phi]_{ij} = \frac{1}{2} G^{j0}_{G_0} G^{k0}_{G_0} \left[ \frac{1}{2} G^{mn}_{G_0} S[\phi]_{jaka} + \frac{1}{2} G^{mn}_{G_0} G^{mno}_{G_0} S[\phi]_{jaka} S[\phi]_{kama} \right] S[\phi]_{ijkl} + \frac{1}{6} \left[ -G^{j0}_{G_0} G^{k0}_{G_0} G^{d0}_{G_0} G^{m0}_{G_0} S[\phi]_{kada} S[\phi]_{mada} S[\phi]_{ijkl} + \frac{3}{4} S[\phi]_{ijklm} G^{jk}_{G_0} G^{lm}_{G_0} \right].
\]

(2.191)

By setting \( \phi = 0 \) we obtain

\[
\Gamma_2[\phi]_{ij} = \frac{1}{4} G^{i0}_{G_0} G^{j0}_{G_0} G^{m0}_{G_0} S[\phi]_{jaka} S[\phi]_{iakj} - \frac{1}{6} G^{i0}_{G_0} G^{j0}_{G_0} G^{d0}_{G_0} G^{m0}_{G_0} S[\phi]_{kada} S[\phi]_{mada} S[\phi]_{ijkl} = \frac{1}{4} \left( \frac{\lambda}{3} \right)^2 \left( N + 2 \right)^2 \delta_{ij} \delta^4(x - y) G_0(w, w) \int d^4 z G_0(x, z) G_0(y, z) + \frac{N + 2}{2} \left( \frac{\lambda}{3} \right)^2 \delta_{ij} G_0(x, y)^3.
\]

(2.192)

We have then

\[
\Gamma^{(2)}_{ij}(x, y) = O(1) + O\left( \frac{\hbar}{i} \right) + \left( \frac{\hbar}{i} \right)^2 \left( \frac{\lambda}{3} \right)^2 \left( N + 2 \right)^2 \delta_{ij} \left( \frac{N + 2}{2} \right) \delta^4(x - y) G_0(w, w) \int d^4 z G_0(x, z) G_0(y, z) + G_0(x, y)^3.
\]

(2.193)

Next we write this result in momentum space. The proper 2–point vertex in momentum space \( \Gamma^{(2)}_{ij}(p) \) is defined through the equations

\[
\int d^4 x d^4 y \Gamma^{(2)}_{ij}(x, y) e^{ipx + iky} = (2\pi)^4 \delta^4(p + k) \Gamma^{(2)}_{ij}(p).
\]

(2.194)

We compute immediately

\[
\Gamma^{(2)}_{ij}(p) = O(1) + O\left( \frac{\hbar}{i} \right) + \left( \frac{\hbar}{i} \right)^2 \left( \frac{\lambda}{3} \right)^2 \left( N + 2 \right)^2 \delta_{ij} \left[ \frac{N + 2}{2} \right] \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \frac{1}{(-p_1^2 + m^2)(-p_2^2 + m^2)}.
\]

(2.195)

The corresponding Feynman diagrams are shown on figure 4.

The 4–point proper vertex upto 2–loop is given by

\[
\Gamma^{(4)}_{i_1 \cdots i_4}(x_1, \ldots, x_4) = O(1) + O\left( \frac{\hbar}{i} \right) + \left( \frac{\hbar}{i} \right)^2 \delta^2 \frac{\Gamma_2[\phi]}{\delta \phi_{i_1}(x_1) \cdots \delta \phi_{i_4}(x_4)}|_{\phi = 0}.
\]

(2.196)
We compute

\[
\Gamma_2[\phi]_{ijkl}\big|_{\phi=0} = \frac{1}{2} G_0^{ij} G_0^{jk} G_0^{km} S_{jikm} \left[ S_{dljk} S_{jlmn} + S_{ikjl} S_{jlmn} + S_{ijlk} S_{kmn} \right] + \frac{1}{4} G_0^{ij} G_0^{jk} G_0^{km} S_{jikm} \left[ S_{dljk} S_{jlmn} + S_{ikjl} S_{jlmn} + S_{ijlk} S_{kmn} \right] + \frac{1}{2} G_0^{ij} G_0^{jk} G_0^{km} S_{jikm} \left[ S_{dljk} S_{jlmn} + S_{ikjl} S_{jlmn} + S_{ijlk} S_{kmn} \right] + S_{ijlk} S_{jkmn} + S_{ijlk} S_{jkmn} + S_{ijlk} S_{jkmn} \right]. 
\]

(2.197)

Thus

\[
\frac{\delta^4 \Gamma_2[\phi]}{\delta \phi_{i_1}(x_1) \cdots \delta \phi_{i_4}(x_4)}\big|_{\phi=0} = -\frac{1}{2} \left( \frac{\lambda}{3} \right)^3 \left[ (N + 2)^2 (N + 2) \delta_{i_1 i_4} \delta_{i_2 i_3} + 2 \delta_{i_1 i_2 i_3 i_4} \right] \delta^4(x_1 - x_4) \delta^4(x_2 - x_3) 
\times G_0(x_1, x_2) \int d^4 z G_0(z, z) G_0(x_1, z) G_0(x_2, z) + 2 \text{ permutations} 
- \frac{1}{4} \left( \frac{\lambda}{3} \right)^3 \left[ (N + 2)(N + 4) \delta_{i_1 i_4} \delta_{i_2 i_3} + 4 \delta_{i_1 i_2 i_3 i_4} \right] \delta^4(x_1 - x_4) \delta^4(x_2 - x_3) 
\times \int d^4 z G_0(x_1, z)^2 G_0(x_2, z)^2 + 2 \text{ permutations} 
- \frac{1}{2} \left( \frac{\lambda}{3} \right)^3 \left[ 2(N + 2) \delta_{i_1 i_4} \delta_{i_2 i_3} + (N + 6) \delta_{i_1 i_2 i_3 i_4} \right] \delta^4(x_1 - x_4) 
\times G_0(x_1, x_2) G_0(x_1, x_3) G_0(x_2, x_3)^2 + 5 \text{ permutations} \right]. 
\]

(2.198)

The proper 4-point vertex in momentum space \(\Gamma^{(4)}_{i_1 \ldots i_4}(p_1 \ldots p_4)\) is defined through the equation

\[
\int d^4 x_1 \ldots d^4 x_4 \Gamma^{(4)}_{i_1 \ldots i_4}(x_1, \ldots, x_4) e^{ip_1x_1 + \ldots + ip_4x_4} = (2\pi)^4 \delta^4(p_1 + \ldots + p_4) \Gamma^{(2)}_{i_1 \ldots i_4}(p_1, \ldots, p_4).
\]

(2.199)
Thus we obtain in momentum space (with $p_{12} = p_1 + p_2$ and $p_{14} = p_1 + p_4$, etc)

$$\Gamma_{i_1...i_4}^{(4)}(p_1,...,p_4) = O(1) + O\left(\frac{\hbar}{\ell}\right)$$

$$- \left(\frac{\hbar}{\ell}\right)^2 \frac{N + 2}{2} \left(\frac{\lambda}{3}\right)^3 \left[(N + 2)\delta_{i_1i_4}\delta_{i_2i_3} + 2\delta_{i_1i_2i_3i_4}\right) \int \frac{1}{-\ell^2 + m^2}$$

$$\times \left[ \int \frac{1}{(-k^2 + m^2)^2} \left(\frac{N + 2}{2}\delta_{i_1i_4}\delta_{i_2i_3} + 2\delta_{i_1i_2i_3i_4}\right) \right] + 2 \text{ permutations}$$

$$\cdot \left[ \int \frac{1}{(-k^2 + m^2)^2} \left(\frac{N + 2}{2}\delta_{i_1i_4}\delta_{i_2i_3} + 2\delta_{i_1i_2i_3i_4}\right) \right] + 2 \text{ permutations}$$

$$\cdot \left[ \int \frac{1}{(-k^2 + m^2)^2} \left(\frac{N + 2}{2}\delta_{i_1i_4}\delta_{i_2i_3} + 2\delta_{i_1i_2i_3i_4}\right) \right] + 5 \text{ permutations}$$

The corresponding Feynman diagrams are shown on figure 5.

2.5.3 The 2–Loop Renormalization of the 2–Point Proper Vertex

The Euclidean expression of the proper 2–point vertex at 2–loop is given by

$$\Gamma^{(2)}_{ij}(p) = \delta_{ij}(p^2 + m^2) + \hbar \frac{\lambda}{6} (N + 2) \delta_{ij} I(m^2) - \hbar^2 \left(\frac{\lambda}{3}\right)^2 \frac{N + 2}{2} \delta_{ij} \left[\frac{N + 2}{2} I(m^2) J(0, m^2) + K(p^2, m^2)\right].$$

(2.201)

$$K(p^2, m^2) = \int \frac{d^4k}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} \Delta(k) \Delta(l) \Delta(k + l - p).$$

(2.202)

We compute

$$K(p^2, m^2) = \int d\alpha_1 d\alpha_2 d\alpha_3 e^{-m^2(\alpha_1 + \alpha_2 + \alpha_3) - \alpha_1 \alpha_2 \alpha_3 \alpha_4 / \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4} \int \frac{d^4k}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} e^{-(\alpha_1 + \alpha_3)k^2} e^{\alpha_1 \alpha_2 \alpha_3 \alpha_4 / \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4} \left(\frac{\alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3}{\alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3}\right)^2.$$

(2.203)

We have used the result

$$\int \frac{d^4k}{(2\pi)^4} e^{-ak^2} = \frac{1}{16\pi^2 a^2}.$$

(2.204)
We introduce the cut-off $\Lambda$ as follows

$$K(p^2, m^2) = \frac{1}{(4\pi)^4} \int \frac{d\alpha_1 d\alpha_2 d\alpha_3}{x^2} \frac{e^{-m^2(\alpha_1 + \alpha_2 + \alpha_3) - \frac{\alpha_1 \alpha_2 \alpha_3}{\alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3}}}{(\alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3)^2} p^2$$

$$= \frac{m^2}{(4\pi)^4} \int \frac{dx_1 dx_2 dx_3}{x^2} \frac{e^{x_1 - x_2 - x_3 - \frac{x_1 x_2 + x_1 x_3 + x_2 x_3}{m^2}}}{(x_1 x_2 + x_1 x_3 + x_2 x_3)^2}$$

$$= \frac{m^2}{(4\pi)^4} \left( A + B \frac{p^2}{m^2} + C \frac{p^2}{m^2} + \ldots \right). \quad (2.205)$$

We have

$$A = \frac{\Lambda^2}{m^2} \int \frac{dx_1 dx_2 dx_3}{x^2} \frac{e^{-x_1 - x_2 - x_3}}{(x_1 x_2 + x_1 x_3 + x_2 x_3)^2}$$

$$= \frac{\Lambda^2}{m^2} \int \frac{dx dx dx}{x^2} \frac{e^{-\frac{m^2}{\Lambda^2}(x + x_2 + x_3)}}{(x x_2 + x x_3 + x_2 x_3)^2}. \quad (2.206)$$

The integrand is symmetric in the three variables $x$, $x_2$ and $x_3$. The integral can be rewritten as $6$ times the integral over the symmetric region $x_3 > x_2 > x$. We can also perform the change of variables $x_2 = xy$ and $x_3 = x y z$, i.e. $dx_2 dx_3 = x^2 y dy dz$ to obtain

$$A = \frac{\Lambda^2}{m^2} \int \frac{dx dy dz}{x^2 y} \frac{e^{-\frac{m^2}{\Lambda^2} x(1+y+z)}}{(1+z+y z)^2}$$

$$= 6 \int \frac{dt}{\Lambda^2} e^{-\psi(t)}. \quad (2.207)$$

$$\psi(t) = \int \frac{dy}{y} \int \frac{dz}{z} e^{-t(1+y)} \frac{e^{-t(1+y+z)}}{(1+z+y z)^2}$$

$$= \int \frac{dy}{y(y+1)} e^{-t \frac{y^2}{y+1}} \int \frac{dz}{z} e^{-t \frac{w^2}{y+1}}$$

$$= \int \frac{dy}{y(y+1)} e^{-t \frac{y^2}{y+1}} \left( e^{-\frac{y(y+2)}{y+1}} + \frac{yt}{y+1} \text{Ei}(-\frac{y(y+2)}{y+1}) \right). \quad (2.208)$$

The most important contribution in the limit $\Lambda \to \infty$ comes from the region $t \sim 0$. Thus near $t = 0$ we have

$$\psi(t) = \int \frac{dy}{y(y+1)} \left( e^{-2ty} \frac{y+2}{y} + \frac{yt}{y+1} e^{-t \frac{y^2}{y+1}} \left( C + \ln t + \ln \frac{y(y+2)}{y+1} + O(t) \right) \right)$$

$$= \psi_0 + \psi_1 \ln t + \psi_2 t + \ldots \quad (2.209)$$

$$\psi_0 = \int \frac{dy}{y(y+1)(y+2)} = \frac{1}{2} \ln \frac{4}{3}. \quad (2.210)$$
\[
\psi_1 = \int_1^\infty \frac{dy}{(y+1)^2} = \frac{1}{2}.
\tag{2.211}
\]
\[
\psi_2 = -2 \int_1^\infty \frac{dy}{(y+1)(y+2)} + C \int_1^\infty \frac{dy}{(y+1)^2} + \int_1^\infty \frac{dy}{(y+1)^2} \ln \frac{y(y+2)}{y+1} = 2(\ln 2 - \ln 3) + \frac{C}{2} + \int_2^\infty \frac{dy}{y^2} \ln \frac{y^2 - 1}{y} = 2(\ln 2 - \ln 3) + \frac{C}{2} - \frac{3}{2} \ln 3 - \frac{1}{2} \ln 2 = \frac{1}{2}(C - 1 - 3 \ln 2).
\tag{2.212}
\]

We have then
\[
A = 6\psi_0 \int \frac{dt}{m^2} e^{-t} + 6\psi_1 \int \frac{dt}{m^2} e^{-t} \ln t + 6\psi_2 \int \frac{dt}{m^2} e^{-t} + ...
= 6\psi_0 \frac{\Lambda^2}{m^2} + 6\psi_1 \int \frac{dt}{m^2} e^{-t} \ln t + 6(\psi_2 - \psi_0) \int \frac{dt}{m^2} e^{-t} + ...
= 6\psi_0 \frac{\Lambda^2}{m^2} + 6\psi_1 \int \frac{dt}{m^2} e^{-t} \ln t + 6(\psi_2 - \psi_0) \frac{m^2}{\Lambda^2} \int dt \ln t e^{-\frac{m^2}{\Lambda^2}} + ...
= 6\psi_0 \frac{\Lambda^2}{m^2} - 3\psi_1 \left( \ln \frac{\Lambda^2}{m^2} \right)^2 + 3\psi_1 \int \frac{dt}{m^2} e^{-t} (\ln t)^2 - 6(\psi_2 - \psi_0) \text{Ei}(-\frac{m^2}{\Lambda^2}) + ...
= 6\psi_0 \frac{\Lambda^2}{m^2} - 3\psi_1 \left( \ln \frac{\Lambda^2}{m^2} \right)^2 + 6(\psi_2 - \psi_0) \ln \left( \frac{\Lambda^2}{m^2} \right) + ...
\tag{2.213}
\]

Now we compute
\[
B = -\int \frac{x_1x_2x_3dx_1dx_2dx_3}{m^2\Lambda^3} \frac{e^{-x_1-x_2-x_3}}{(x_1x_2 + x_1x_3 + x_2x_3)^3}
= -\int \frac{x_1x_2x_3dx_1dx_2dx_3}{(x_2 + x_3 + x_2x_3)^3} e^{-\frac{m^2}{\Lambda^2}(x+x_2+x_3)}
= -6 \int_1^\infty dx \int_x^\infty x_2 dx_2 \int_2^\infty x_3 dx_3 \frac{e^{-\frac{m^2}{\Lambda^2}(x+x_2+x_3)}}{(x_2 + x_3 + x_2x_3)^3}
= -6 \int_1^\infty dx \int_x^\infty dy \int_1^\infty zdz e^{-\frac{m^2}{\Lambda^2}x(1+y+z)} \frac{1}{(1 + z + yz)^3}
= -6 \int_1^\infty \frac{dt}{m^2} e^{-t \psi(t)}. \tag{2.214}
\]
\[
\tilde{\psi}(t) = \int_1^\infty dy \int_1^\infty zdz e^{-ty(1+z)} \frac{1}{(1 + z + yz)^3}. \tag{2.215}
\]
It is not difficult to convince ourselves that only the constant part of \( \bar{\psi} \) leads to a divergence, i.e. \( \bar{\psi}(0) = \frac{1}{12} \). We get

\[
B = -6 \int \frac{dt}{m^2} t e^{-t} \bar{\psi}(0) = -6 \bar{\psi}(0) \ln \frac{\Lambda^2}{m^2}.
\]

Now we compute

\[
C = \frac{1}{2} \int \frac{m^2}{\Lambda^2} \int (x_1 x_2 x_3)^2 dx_1 dx_2 dx_3 \frac{e^{-x_1 - x_2 - x_3}}{(x_1 x_2 + x_1 x_3 + x_2 x_3)^4}
\]

\[
= \frac{m^2}{\Lambda^2} \int \frac{m^2}{\Lambda^2} (x x_2 x_3)^2 dx_2 dx_3 \frac{e^{-m^2 (x+x_2+x_3)}}{(x x_2 + x x_3 + x_2 x_3)^4}
\]

\[
= 6 \frac{m^2}{\Lambda^2} \int \frac{m^2}{\Lambda^2} \int_1^\infty dx_2 dx_3 \int_1^\infty x_2^2 dx_2 \int_1^\infty x_3^2 dx_3 \frac{e^{-\frac{m^2}{\Lambda^2} (x+x_2+x_3)}}{(x x_2 + x x_3 + x_2 x_3)^4}
\]

\[
= 6 \frac{m^2}{\Lambda^2} \int \frac{m^2}{\Lambda^2} \int_1^\infty dy \int_1^\infty z^2 dy \int_1^\infty z^2 d z \frac{e^{-\frac{m^2}{\Lambda^2} (y+z)}}{(1 + y + z)^4}
\]

\[
= 6 \int \frac{m^2}{\Lambda^2} \int \frac{m^2}{\Lambda^2} \int_1^\infty e^{-t} \int_1^\infty dy \int_1^\infty z^2 dy \frac{e^{-t(y+z)}}{(1 + y + z)^4}.
\]

(2.217)

This integral is well defined in the limit \( \Lambda \rightarrow \infty \). Furthermore it is positive definite.

In summary we have found that both \( K(0, m^2) \) and \( K'(0, m^2) \) are divergent in the limit \( \Lambda \rightarrow \infty \), i.e. \( K(p^2, m^2) - K(0, m^2) \) is divergent at the two-loop order. This means that \( \Gamma_{ij}^{(2)}(p) \) and \( d\Gamma_{ij}^{(2)}(p)/dp^2 \) are divergent at \( p^2 = 0 \) and hence in order to renormalize the 2–point proper vertex \( \Gamma_{ij}^{(2)}(p) \) at the two-loop order we must impose two conditions on it. The first condition is the same as before namely we require that the value of the 2–point proper vertex at zero momentum is precisely the physical or renormalized mass. The second condition is essentially a renormalization of the coefficient of the kinetic term, i.e. \( d\Gamma_{ij}^{(2)}(p)/dp^2 \). Before we can write these two conditions we introduce a renormalization of the scalar field \( \phi \) known also as wave function renormalization given by

\[
\phi = \sqrt{Z} \phi_R.
\]

(2.218)

This induces a renormalization of the \( n \)–point proper vertices. Indeed the effective action becomes

\[
\Gamma[\phi] = \sum_{n=0} \frac{1}{n!} (x_1, \ldots, x_n) \phi_1(x_1) \ldots \phi_n(x_n)
\]

\[
= \sum_{n=0} \frac{1}{n!} (x_1, \ldots, x_n) R(x_1) \ldots \phi_n R(x_n).
\]

(2.219)
The renormalized $n$–point proper vertex $\Gamma_{i_1...i_nR}^{(n)}$ is given in terms of the bare $n$–point proper vertex $\Gamma_{i_1...i_n}^{(n)}$ by

$$\Gamma_{i_1...i_nR}(x_1, ..., x_n) = Z^{2n}\Gamma_{i_1...i_n}^{(n)}(x_1, ..., x_n). \quad (2.220)$$

Thus the renormalized 2–point proper vertex $\Gamma_{ijR}^{(2)}(p)$ in momentum space is given by

$$\Gamma_{ijR}^{(2)}(p) = Z\Gamma_{ij}^{(2)}(p). \quad (2.221)$$

Now we impose on the renormalized 2–point proper vertex $\Gamma_{ijR}^{(2)}(p)$ the two conditions given by

$$\Gamma_{ijR}^{(2)}(p)|_{p=0} = Z\Gamma_{ij}^{(2)}(p)|_{p=0} = \delta_{ij}m^2. \quad (2.222)$$

$$\frac{d}{dp^2}\Gamma_{ijR}^{(2)}(p)|_{p=0} = Z\frac{d}{dp^2}\Gamma_{ij}^{(2)}(p)|_{p=0} = \delta_{ij}. \quad (2.223)$$

The second condition yields immediately

$$Z = \frac{1}{1 - \hbar^2\frac{2}{3}N + 2} \frac{2N + 2}{2} K'(0, m^2, \Lambda) = 1 + \hbar^2\left(\frac{\lambda}{3}\right)^2 \frac{2N + 2}{2} K'(0, m^2, \Lambda). \quad (2.224)$$

The first condition gives then

$$m^2 = m^2_R - h\frac{\lambda}{6}(N + 2)I(m^2, \Lambda) + \hbar^2\left(\frac{\lambda}{3}\right)^2 \frac{2N + 2}{2} \left[ - \frac{N + 2}{2} I(m^2, \Lambda)J(0, m^2, \Lambda) + K(0, m^2, \Lambda) 
- m^2 K'(0, m^2, \Lambda) \right]$$

$$= m^2_R - h\frac{\lambda_R}{6}(N + 2)I(m^2, \Lambda) + \hbar^2\left(\frac{\lambda_R}{3}\right)^2 \frac{2N + 2}{2} \left[ - \frac{N + 2}{2} I(m^2, \Lambda)J(0, m^2, \Lambda) + K(0, m^2, \Lambda) 
- m^2_R K'(0, m^2, \Lambda) \right]$$

$$= m^2_R - h\frac{\lambda_R}{6}(N + 2)I(m^2, \Lambda) + \hbar^2\left(\frac{\lambda_R}{3}\right)^2 \frac{2N + 2}{2} \left[ - \frac{N + 8}{2} I(m^2, \Lambda)J(0, m^2, \Lambda) + K(0, m^2, \Lambda) 
- m^2_R K'(0, m^2, \Lambda) \right]. \quad (2.225)$$

In above we have used the relation between the bare coupling constant $\lambda$ and the renormalized coupling constant $\lambda_R$ at one-loop given by equation (2.175). We have also used the relation $I(m^2, \Lambda) = I(m^2, \Lambda) + h\frac{\lambda}{6}(N + 2)I(m^2, \Lambda)J(0, m^2, \Lambda)$ where we have assumed that $m^2 = m^2_R - h\frac{\lambda}{6}(N + 2)I(m^2, \Lambda)$. We get therefore the 2–point proper vertex

$$\Gamma_{ijR}^{(2)}(p) = \delta_{ij}(p^2 + m^2_R) - \hbar^2\left(\frac{\lambda_R}{3}\right)^2 \frac{2N + 2}{2} \delta_{ij} \left[ K(p^2, m^2, \Lambda) - K(0, m^2, \Lambda) - p^2 K'(0, m^2, \Lambda) \right]. \quad (2.226)$$
2.5.4 The 2–Loop Renormalization of the 4–Point Proper Vertex

The Euclidean expression of the proper 4–point vertex at 2–loop is given by

\[
\Gamma^{(4)}_{i_1\ldots i_4}(p_1, \ldots, p_4) = \frac{\lambda}{3} \delta_{i_1i_2i_3i_4} - \hbar \left( \frac{\lambda}{3} \right)^2 \frac{1}{2} \left[ (N + 2) \delta_{i_1i_4} \delta_{i_2i_3} + 2 \delta_{i_1i_2i_3i_4} \right] J(p^2, m^2) \\
+ \text{2 permutations} \\
+ \hbar^2 \left( \frac{\lambda}{3} \right)^3 \frac{N + 2}{2} \left[ (N + 2) \delta_{i_1i_4} \delta_{i_2i_3} + 2 \delta_{i_1i_2i_3i_4} \right] I(m^2) L(p^2, m^2) \\
+ \text{2 permutations} \\
+ \hbar^2 \left( \frac{\lambda}{3} \right)^3 \frac{1}{4} \left[ (N + 2)(N + 4) \delta_{i_1i_4} \delta_{i_2i_3} + 4 \delta_{i_1i_2i_3i_4} \right] J(p^2, m^2)^2 \\
+ \text{2 permutations} \\
+ \hbar^2 \left( \frac{\lambda}{3} \right)^3 \frac{1}{2} \left[ 2(N + 2) \delta_{i_1i_4} \delta_{i_2i_3} + (N + 6) \delta_{i_1i_2i_3i_4} \right] M(p^2, p^2, m^2) \\
+ \text{5 permutations} .
\]

\[ L(p^2, m^2) = \int \frac{d^4k}{(2\pi)^4} \Delta(k)^2 \Delta(k - p_{14}) . \tag{2.227} \]

\[ M(p^2, p^2, m^2) = \int \frac{d^4l}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \Delta(l) \Delta(k) \Delta(l - p_{14}) \Delta(l - k + p_2) . \tag{2.229} \]

For simplicity we will not write explicitly the dependence on the cut-off \( \Lambda \) in the following. The renormalized 4–point proper vertex \( \Gamma^{(4)}_{i_1i_2i_3i_4R}(p_1, p_2, p_3, p_4) \) in momentum space is given by

\[ \Gamma^{(4)}_{i_1i_2i_3i_4R}(p_1, p_2, p_3, p_4) = Z^2 \Gamma^{(4)}_{i_1i_2i_3i_4}(p_1, p_2, p_3, p_4) . \tag{2.230} \]

We will impose the renormalization condition

\[ \Gamma^{(4)}_{i_1\ldots i_4R}(0, \ldots, 0) = \frac{\lambda_R}{3} \delta_{i_1i_2i_3i_4} . \tag{2.231} \]

We introduce a new renormalization constant \( Z_g \) defined by

\[ Z_g \Gamma^{(4)}_{i_1\ldots i_4}(0, \ldots, 0) = \frac{\lambda}{3} \delta_{i_1i_2i_3i_4} . \tag{2.232} \]
Equivalently this means
\[ \frac{Z_g}{Z^2} \lambda_R = \lambda. \] (2.233)

The constant \( Z \) is already known at two-loop. The constant \( Z_g \) at two-loop is computed to be
\[ Z_g = 1 + \hbar \lambda \frac{6(N + 8)}{N + 8} J(0, m^2) - \hbar^2 \left( \frac{\lambda}{3} \right)^2 \left[ \frac{(N + 2)(N + 8)}{2} I(m^2) L(0, m^2) \right] + \frac{(N + 2)(N + 4) + 12}{4} J(0, m^2)^2 + (5N + 22) M(0, 0, m^2) \] (2.234)

We compute
\[ \Gamma^{(4)}_{i_1 i_2 i_3 i_4 R} (p_1, p_2, p_3, p_4) = Z^2 \Gamma^{(4)}_{i_1 i_2 i_3 i_4} (p_1, p_2, p_3, p_4) \]
\[ = Z_g \frac{\lambda_R}{3} \delta_{i_1 i_2 i_3 i_4} + \Gamma^{(4)}_{i_1 i_2 i_3 i_4} (p_1, p_2, p_3, p_4)|_{1\text{-loop}} + \Gamma^{(4)}_{i_1 i_2 i_3 i_4} (p_1, p_2, p_3, p_4)|_{2\text{-loop}}, \] (2.235)

By using the relation \( J(p_{12}^2, m^2) = J(p_{12}^2, m_{R}^2) + \hbar \frac{\lambda_R}{3} (N + 2) I(m_R^2) L(p_{12}^2, m_R^2) \) we compute
\[ Z_g \frac{\lambda_R}{3} \delta_{i_1 i_2 i_3 i_4} = \frac{\lambda_R}{3} \delta_{i_1 i_2 i_3 i_4} + \hbar \left( \frac{\lambda_R}{3} \right)^2 \frac{N + 8}{2} \delta_{i_1 i_2 i_3 i_4} J(0, m_{R}^2) - \hbar^2 \left( \frac{\lambda_R}{3} \right)^3 \delta_{i_1 i_2 i_3 i_4} \left[ \frac{(N + 2)(N + 4) + 12}{4} J(0, m_{R}^2)^2 + (5N + 22) M(0, 0, m_{R}^2) \right]. \] (2.236)

\[ \Gamma^{(4)}_{i_1 i_2 i_3 i_4} (p_1, p_2, p_3, p_4)|_{1\text{-loop}} = -\hbar \left( \frac{\lambda_R}{3} \right)^2 \left[ \left( (N + 2) \delta_{i_1 i_2} \delta_{i_3 i_4} + 2 \delta_{i_1 i_2 i_3 i_4} \right) J(p_{12}^2, m_{R}^2) \right] + 2 \text{ permutations} \]
\[ - \hbar^2 \left( \frac{\lambda_R}{3} \right)^3 \frac{N + 8}{2} J(0, m_{R}^2) \left[ \left( (N + 2) \delta_{i_1 i_2} \delta_{i_3 i_4} + 2 \delta_{i_1 i_2 i_3 i_4} \right) J(p_{12}^2, m_{R}^2) \right] + 2 \text{ permutations} \]
\[ - \hbar^2 \left( \frac{\lambda_R}{3} \right)^3 \frac{N + 2}{2} I(m_{R}^2) \left[ \left( (N + 2) \delta_{i_1 i_2} \delta_{i_3 i_4} + 2 \delta_{i_1 i_2 i_3 i_4} \right) L(p_{12}^2, m_{R}^2) \right] + 2 \text{ permutations}. \] (2.237)
We then find
\[
\Gamma^{(4)}_{x_1,...,x_4}(p_1,...,p_4) = \frac{\lambda_R}{3} \delta_{i_1i_2i_3i_4} - \hbar \left( \frac{\lambda_R}{3} \right)^2 \left[ \frac{1}{2} \left( (N + 2) \delta_{i_1i_2} \delta_{i_3i_4} + 2 \delta_{i_1i_2i_3i_4} \right) \left( J(p_{14}^2, m_R^2) - J(0, m_R^2) \right) \right] + 2 \text{ permutations} \\
+ \hbar^2 \left( \frac{\lambda_R}{3} \right)^3 \left[ \frac{1}{4} \left( (N + 2)(N + 4) \delta_{i_1i_4} \delta_{i_2i_3} + 4 \delta_{i_1i_2i_3i_4} \right) \left( J(p_{14}^2, m_R^2) - J(0, m_R^2) \right)^2 \right] + 2 \text{ permutations} \\
- \hbar^2 \left( \frac{\lambda_R}{3} \right)^3 \left[ \frac{1}{2} \left( 2(N + 2) \delta_{i_1i_4} \delta_{i_2i_3} + (N + 6) \delta_{i_1i_2i_3i_4} \right) \left( J(p_{14}^2, m_R^2) - J(0, m_R^2) \right) \right] + 2 \text{ permutations} \\
+ \hbar^2 \left( \frac{\lambda_R}{3} \right)^3 \left[ \frac{1}{4} \left( 2(N + 2) \delta_{i_1i_4} \delta_{i_2i_3} + (N + 6) \delta_{i_1i_2i_3i_4} \right) \left( M(p_{14}^2, p_{12}^2, m_R^2) - M(0, 0, m_R^2) \right) \right] + 5 \text{ permutations}.
\]
(2.238)

In the above last equation the combination \(M(p_{14}^2, p_{12}^2, m_R^2) - M(0, 0, m_R^2) - J(0, m_R^2)(J(p_{14}^2, m_R^2) - J(0, m_R^2))\) must be finite in the limit \(\Lambda \rightarrow \infty\).

2.6 Renormalized Perturbation Theory

The \((\phi^2)^2\) theory with \(O(N)\) symmetry studied in this chapter is given by the action
\[
S = \int \! d^4x \left[ \frac{1}{2} \partial_{\mu} \phi_i \partial^{\mu} \phi_i - \frac{1}{2} m^2 \phi_i^2 - \frac{\lambda}{4!} (\phi_i^2)^2 \right].
\]
(2.239)

This is called a bare action, the fields \(\phi_i\) are the bare fields and the parameters \(m^2\) and \(\lambda\) are the bare coupling constants of the theory.

Let us recall that the free 2–point function \(< 0| T(\hat{\phi}_{i,in}(x)\hat{\phi}_{j,in}(y))|0 >\) is the probability amplitude for a free scalar particle to propagate from a spacetime point \(y\) to a spacetime \(x\). In the interacting theory the 2–point function is \(< \Omega| T(\hat{\phi}_i(x)\hat{\phi}_j(y))|\Omega >\) where \(|\Omega >| = |0 >/ \sqrt{< 0|0 >}\) is the ground state of the full Hamiltonian \(\hat{H}\). On general grounds we can verify that the 2–point function \(< \Omega| T(\hat{\phi}_i(x)\hat{\phi}_j(y))|\Omega >\) is given by
\[
\int \! d^4xe^{ip(x-y)} < \Omega| T(\hat{\phi}_i(x)\hat{\phi}_j(y))|\Omega > = \frac{iZ\delta_{ij}}{p^2 - m_R^2 + i\epsilon} + \ldots
\]
(2.240)

Exercise: show this result.
The dots stands for regular terms at \( p^2 = m_R^2 \), where \( m_R \) is the physical or renormalized mass. The residue or renormalization constant \( Z \) is called the wave function renormalization. Indeed the renormalized 2−point function < \( \Omega | T(\phi_R(x)\phi_R(y)) | \Omega \) > is given by

\[
\int d^4xe^{ip(x-y)} < \Omega | T(\phi_R(x)\phi_R(y)) | \Omega > = \frac{i\delta_{ij}}{p^2 - m_R^2 + i\epsilon} + .... \tag{2.241}
\]

The physical or renormalized field \( \phi_R \) is given by

\[
\phi = \sqrt{Z}\phi_R. \tag{2.242}
\]

As we have already discussed this induces a renormalization of the \( n \)−point proper vertices. Indeed the effective action becomes

\[
\Gamma[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1... \int d^4x_n \Gamma^{(n)}_{i_1...i_n}(x_1,...,x_n)\phi_{i_1}(x_1)...\phi_{i_n}(x_n)
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1... \int d^4x_n \Gamma^{(n)}_{i_1...i_nR}(x_1,...,x_n)\phi_{i_1R}(x_1)...\phi_{i_nR}(x_n). \tag{2.243}
\]

The renormalized \( n \)−point proper vertex \( \Gamma^{(n)}_{i_1...i_nR} \) is given in terms of the bare \( n \)−point proper vertex \( \Gamma^{(n)}_{i_1...i_n} \) by

\[
\Gamma^{(n)}_{i_1...i_nR}(x_1,...,x_n) = Z^n\Gamma^{(n)}_{i_1...i_n}(x_1,...,x_n). \tag{2.244}
\]

We introduce a renormalized coupling constant \( \lambda_R \) and a renormalization constant \( Z_g \) by

\[
Z_g\lambda_R = Z^2\lambda. \tag{2.245}
\]

The action takes the form

\[
S = \int d^4x \left[ \frac{Z}{2} \partial_\mu \phi_{iR} \partial^\mu \phi_{iR} - \frac{Z}{2} m_R^2 \phi_{iR}^2 - \frac{\lambda Z^2}{4!} (\phi_{iR}^2)^2 \right]
\]

\[
= S_R + \delta S. \tag{2.246}
\]

The renormalized action \( S_R \) is given by

\[
S_R = \int d^4x \left[ \frac{1}{2} \partial_\mu \phi_{iR} \partial^\mu \phi_{iR} - \frac{1}{2} m_R^2 \phi_{iR}^2 - \frac{\lambda_R}{4!} (\phi_{iR}^2)^2 \right]. \tag{2.247}
\]

The action \( \delta S \) is given by

\[
\delta S = \int d^4x \left[ \frac{\delta Z}{2} \partial_\mu \phi_{iR} \partial^\mu \phi_{iR} - \frac{1}{2} \delta m_R^2 \phi_{iR}^2 - \frac{\delta \lambda_R}{4!} (\phi_{iR}^2)^2 \right]. \tag{2.248}
\]

The counterterms \( \delta Z, \delta m \) and \( \delta \lambda \) are given by

\[
\delta Z = Z - 1, \quad \delta m = Zm^2 - m_R^2, \quad \delta \lambda = \lambda Z^2 - \lambda_R = (Z_g - 1)\lambda_R. \tag{2.249}
\]
The new Feynman rules derived from $S_R$ and $\delta S$ are shown on figure 7.

The so-called renormalized perturbation theory consists in the following. The renormalized or physical parameters of the theory $m_R$ and $\lambda_R$ are always assumed to be finite whereas the counterterms $\delta_Z$, $\delta_m$ and $\delta_\lambda$ will contain the unobservable infinite shifts between the bare parameters $m$ and $\lambda$ and the physical parameters $m_R$ and $\lambda_R$. The renormalized parameters are determined from imposing renormalization conditions on appropriate proper vertices. In this case we will impose on the 2–point proper vertex $\Gamma^{(2)}_{ijR}(p)$ and the 4–point proper vertex $\Gamma^{(4)}_{ijR}(p)$ the three conditions given by

$$\Gamma^{(2)}_{ijR}(p)|_{p=0} = \delta_{ij} m_R^2.$$  \hfill (2.250)

$$\frac{d}{dp^2} \Gamma^{(2)}_{ijR}(p)|_{p=0} = \delta_{ij}.$$ \hfill (2.251)

$$\Gamma^{(4)}_{ij...4R}(0,...,0) = -\frac{\lambda_R}{3} \delta_{i_1i_2i_3i_4}.$$ \hfill (2.252)

As an example let us consider the 2–point and 4–point functions up to the 1–loop order. We have immediately the results

$$\Gamma^{(2)}_{Rij}(p) = \left[ (p^2 - m_R^2) - \frac{\hbar \lambda_R}{i} (N + 2) I(m_R^2) + (\delta_Z p^2 - \delta_m) \right] \delta_{ij}.$$ \hfill (2.253)

$$\Gamma^{(4)}_{Ri...4}(p_1,...,p_4) = -\frac{\lambda_R}{3} \delta_{i_1i_2i_3i_4} + \frac{\hbar}{i} \left( \frac{\lambda_R}{3} \right)^2 \frac{1}{2} \left[ (N + 2) \delta_{i_1i_2} \delta_{i_3i_4} + 2 \delta_{i_1i_2i_3i_4} \right] J(p_{12}^2, m_R^2)$$

$$+ 2 \text{ permutations} \right] - \frac{\delta_\lambda}{3} \delta_{i_1i_2i_3i_4}.$$ \hfill (2.254)

The first two terms in both $\Gamma^{(2)}_R$ and $\Gamma^{(4)}_R$ come from the renormalized action $S_R$ and they are identical with the results obtained with the bare action $S$ with the substitutions $m \rightarrow m_R$ and $\lambda \rightarrow \lambda_R$. The last terms in $\Gamma^{(2)}_R$ and $\Gamma^{(4)}_R$ come from the action $\delta S$. By imposing renormalization conditions we get (including a cut-off $\Lambda$)

$$\delta_Z = 0, \delta_m = -\frac{\hbar \lambda_R}{i} (N + 2) I(0,m_R^2), \delta_\lambda = \frac{\hbar \lambda_R^2}{6} (N + 8) J(0,0,m_R^2).$$ \hfill (2.255)

In other words

$$\Gamma^{(2)}_{Rij}(p) = (p^2 - m_R^2) \delta_{ij}.$$ \hfill (2.256)

$$\Gamma^{(4)}_{Ri...4}(p_1,...,p_4) = -\frac{\lambda_R}{3} \delta_{i_1i_2i_3i_4} + \frac{\hbar}{i} \left( \frac{\lambda_R}{3} \right)^2 \frac{1}{2} \left[ (N + 2) \delta_{i_1i_2} \delta_{i_3i_4} + 2 \delta_{i_1i_2i_3i_4} \right] \left[ J(p_{12}^2, m_R^2) - J(0,0,m_R^2) \right]$$

$$+ 2 \text{ permutations} \right].$$ \hfill (2.257)
It is clear that the end result of renormalized perturbation theory up to 1–loop is the same as the somewhat "direct" renormalization employed in the previous sections to renormalize the perturbative expansion of $\Gamma_R^{(2)}$ and $\Gamma_R^{(4)}$ up to 1–loop. This result extends also to the 2–loop order\(^{13}\).

Let us note at the end of this section that renormalization of higher $n$–point vertices should proceed along the same lines discussed above for the 2–point and 4–point vertices. The detail of this exercise will be omitted at this stage.

### 2.7 Effective Potential and Dimensional Regularization

Let us go back to our original $O(N)$ action which is given by

$$S[\phi] = \int d^4x \left[ \frac{1}{2} \partial_\mu \phi_i \partial^\mu \phi_i + \frac{\mu^2}{2} \phi_i^2 - \frac{g}{4} (\phi_i^2)^2 + J_i \phi_i \right]. \quad (2.258)$$

Now we expand the field as $\phi_i = \phi_{ci} + \eta_i$ where $\phi_{ci}$ is the classical field. We can always choose $\phi_c$ to point in the $N$ direction, viz $\phi_c = (0, \ldots, 0, \phi_c)$. By translational invariance we may assume that $\phi_{ci}$ is a constant. The action becomes (where $V$ is the spacetime volume)

$$S[\phi_c, \eta] = V \left[ \frac{1}{2} \phi_{ci}^2 - \frac{g}{4} (\phi_{ci})^2 + J_i \phi_{ci} \right] + \int d^4x \left[ \frac{1}{2} \partial_\mu \eta_i \partial^\mu \eta_i + \frac{\mu^2}{2} \eta_i^2 + (\mu^2 - g\phi_{ci}^2) \phi_{ci} \eta_i + J_i \eta_i - \frac{g}{2} [\phi_{ci} \eta_j^2 + 2(\phi_{ci} \eta_i)] - g(\phi_{ci} \eta_i) \eta_j^2 - \frac{g}{4} (\eta_i^2)^2 \right]. \quad (2.259)$$

In the spirit of renormalized perturbation theory we will think of the parameters $\mu^2$ and $g$ as renormalized parameters and add the counterterms

$$\delta S[\phi] = \int d^4x \left[ \frac{1}{2} \delta \partial_\mu \phi_i \partial^\mu \phi_i + \frac{1}{2} \delta \mu \phi_i^2 - \frac{1}{4} \delta g (\phi_i^2)^2 + \delta J_i \phi_i \right]. \quad (2.260)$$

The counterterm $\delta J_i$ is chosen so that the 1–point vertex $\Gamma_{i_1}^{(1)}(x_1)$ is identically zero to all orders in perturbation theory. This is equivalent to the removal of all tadpole diagrams that contribute to $\langle \eta_i \rangle$.

Let us recall the form of the effective action up to 1–loop and the classical 2–point function. These are given by

$$\Gamma = S + \frac{1}{2} \hbar \ln \det G_0 + \ldots \quad (2.261)$$

$$G_0^{ij} = -S^{-1}_{ij} \big|_{\phi=\phi_c}. \quad (2.262)$$

The effective action can always be rewritten as the spacetime integral of an effective Lagrangian $L_{\text{eff}}$. For slowly varying fields the most important piece in this effective

\(^{13}\)Exercise: Try this explicitly.
Lagrangian is the so-called effective potential which is the term with no dependence on the derivatives of the field. The effective Lagrangian takes the generic form
\[
\mathcal{L}_{\text{eff}}(\phi_c, \partial \phi_c, \partial \partial \phi_c, ...) = -V(\phi_c) + Z(\phi_c) \partial_\mu \phi_c \partial^\mu \phi_c + ... \tag{2.263}
\]
For constant classical field we have
\[
\Gamma(\phi_c) = - \int d^4 x V(\phi_c) = - \left( \int d^4 x \right) V(\phi_c). \tag{2.264}
\]
We compute immediately
\[
\frac{\delta^2 S}{\delta \eta_i(x) \delta \eta_j(y)} |_{\eta=0} = \left[ - \delta^2 \delta_{ij} + \mu^2 \delta_{ij} - g [\phi_c^2 \delta_{ij} + 2 \phi_c \phi_{ij}] \right] \delta^4(x - y)
= \left[ - \delta^2 - m_i^2 \right] \delta_{ij} \delta^4(x - y). \tag{2.265}
\]
The masses \(m_i\) are given by
\[
m_i^2 = g \phi_c^2 - \mu^2, \ i, j \neq N \text{ and } m_i^2 = 3g \phi_c^2 - \mu^2, \ i = j = N. \tag{2.266}
\]
The above result can be put in the form
\[
\frac{\delta^2 S}{\delta \eta_i(x) \delta \eta_j(y)} |_{\eta=0} = \int \frac{d^4 p}{(2\pi)^d} \left[ p^2 - m_i^2 \right] \delta_{ij} e^{ip(x-y)}. \tag{2.267}
\]
We compute
\[
\frac{1}{2i} \frac{\hbar}{2} \ln \det G_0 = - \frac{1}{2i} \hbar \ln \det G_0^{-1}
= \frac{1}{2i} \hbar \ln \det \left( - \frac{\delta^2 S}{\delta \eta_i(x) \delta \eta_j(y)} |_{\eta=0} \right)
= \frac{1}{2i} \hbar Tr \ln \left( - \frac{\delta^2 S}{\delta \eta_i(x) \delta \eta_j(y)} |_{\eta=0} \right)
= \frac{1}{2i} \hbar \int d^4 x < x| \ln \left( - \frac{\delta^2 S}{\delta \eta_i(x) \delta \eta_j(y)} |_{\eta=0} \right) |x>
= \frac{1}{2i} \hbar V \int \frac{d^4 p}{(2\pi)^4} \ln \left( - p^2 + m_i^2 \right) \delta_{ij}
= \frac{1}{2i} \hbar V \left[ (N - 1) \int \frac{d^4 p}{(2\pi)^4} \ln \left( - p^2 - \mu^2 + g \phi_c^2 \right) + \int \frac{d^4 p}{(2\pi)^4} \ln \left( - p^2 - \mu^2 + 3g \phi_c^2 \right) \right]. \tag{2.268}
\]
The basic integral we need to compute is
\[
I(m^2) = \int \frac{d^4 p}{(2\pi)^4} \ln \left( - p^2 + m^2 \right). \tag{2.269}
\]
This is clearly divergent. We will use here the powerful method of dimensional regularization to calculate this integral. This consists in 1) performing a Wick rotation $k^0 \rightarrow k^4 = -ik^0$ and 2) continuing the number of dimensions from 4 to $d \neq 4$. We have then

$$I(m^2) = i \int \frac{d^dp_E}{(2\pi)^d} \ln \left( p_E^2 + m^2 \right). \quad (2.270)$$

We use the identity

$$\frac{\partial}{\partial \alpha} x^{-\alpha} \big|_{\alpha=0} = - \ln x. \quad (2.271)$$

We get then

$$I(m^2) = -i \frac{\partial}{\partial \alpha} \left( \int \frac{d^dp_E}{(2\pi)^d} \frac{1}{(p_E^2 + m^2)^\alpha} \right) \big|_{\alpha=0}$$

$$= -i \frac{\partial}{\partial \alpha} \left( \frac{\Omega_{d-1}}{(2\pi)^d} \int dp_E \frac{p_E^{d-1}}{(p_E^2 + m^2)^\alpha} \right) \big|_{\alpha=0}. \quad (2.272)$$

The $\Omega_{d-1}$ is the solid angle in $d$ dimensions, i.e. the area of a sphere $S^{d-1}$. It is given by 14

$$\Omega_{d-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma \left( \frac{d}{2} \right)}. \quad (2.273)$$

We make the change of variables $x = p_E^2$ then the change of variables $t = m^2/(x + m^2)$. We get

$$I(m^2) = -i \frac{\partial}{\partial \alpha} \left( \frac{\Omega_{d-1}}{2(2\pi)^d} \int_0^\infty dx \frac{x^{\frac{d}{2}-1}}{(x + m^2)^\alpha} \right) \big|_{\alpha=0}$$

$$= -i \frac{\partial}{\partial \alpha} \left( \Omega_{d-1}(m^2)^{\frac{d}{2}-\alpha} \frac{\Gamma(\frac{d}{2})}{\Gamma(\alpha)} \right) \bigg|_{\alpha=0}. \quad (2.274)$$

We use the result

$$\int_0^1 dt \ t^{\alpha-1}(1-t)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} \quad (2.275)$$

We get then

$$I(m^2) = -i \frac{\partial}{\partial \alpha} \left( \frac{\Omega_{d-1}(m^2)^{\frac{d}{2}-\alpha} \Gamma(\alpha - \frac{d}{2})\Gamma(\frac{d}{2})}{\Gamma(\alpha)} \right) \big|_{\alpha=0}$$

$$= -i \frac{\partial}{\partial \alpha} \left( \frac{1}{(4\pi)^\frac{d}{2}} (m^2)^{\frac{d}{2}-\alpha} \frac{\Gamma(\alpha - \frac{d}{2})\Gamma(\frac{d}{2})}{\Gamma(\alpha)} \right) \big|_{\alpha=0}. \quad (2.276)$$

14 Derive this result.
Now we use the result that
\[ \Gamma(\alpha) \longrightarrow \frac{1}{\alpha}, \ \alpha \longrightarrow 0. \] (2.277)

Thus
\[ I(m^2) = -i \frac{1}{(4\pi)^{\frac{d}{2}}} (m^2)^{\frac{d}{2}} \Gamma(-\frac{d}{2}). \] (2.278)

By using this result we have
\[ \frac{\hbar}{2} \ln \det G_0 = \frac{i\hbar}{2} V \left( -i \frac{1}{(4\pi)^{\frac{d}{2}}} \Gamma(-\frac{d}{2}) \right) \left[ (N-1)(-\mu^2 + g\phi_c^2)^{\frac{d}{2}} + (-\mu^2 + 3g\phi_c^2)^{\frac{d}{2}} \right] \]
\[ = \frac{\hbar}{2} V \frac{\Gamma(-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \left[ (N-1)(-\mu^2 + g\phi_c^2)^{\frac{d}{2}} + (-\mu^2 + 3g\phi_c^2)^{\frac{d}{2}} \right]. \] (2.279)

The effective potential including counterterms is given by
\[ V(\phi_c) = -\frac{\mu^2}{2} \phi_c^2 + \frac{g}{4}(\phi_c^2)^2 - \frac{\hbar}{2} \frac{\Gamma(-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \left[ (N-1)(-\mu^2 + g\phi_c^2)^{\frac{d}{2}} + (-\mu^2 + 3g\phi_c^2)^{\frac{d}{2}} \right] \]
\[ - \frac{1}{2} \delta\mu \phi_c^2 + \frac{1}{4} \delta g (\phi_c^2)^2. \] (2.280)

Near \( d = 4 \) we use the approximation given by (with \( \epsilon = 4 - d \) and \( \gamma = 0.5772 \) is Euler-Mascheroni constant)
\[ \Gamma(-\frac{d}{2}) \approx \frac{1}{d(\frac{d}{2} - 1)} \Gamma(\frac{\epsilon}{2}) \]
\[ = \frac{1}{2} \left[ \frac{2}{\epsilon} - \gamma + \frac{3}{2} + O(\epsilon) \right]. \] (2.281)

This divergence can be absorbed by using appropriate renormalization conditions. We remark that the classical minimum is given by \( \phi_c = v = \sqrt{\mu^2/g} \). We will demand that the value of the minimum of \( V_{\text{eff}} \) remains given by \( \phi_c = v \) at the one-loop order by imposing the condition
\[ \frac{\partial}{\partial \phi_c} V(\phi_c)|_{\phi_c = v} = 0. \] (2.282)

As we will see in the next section this is equivalent to saying that the sum of all tadpole diagrams is 0. This condition leads immediately to \( \delta_\mu - \delta g v^2 = \hbar g \frac{\Gamma(1 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}} (2\mu^2)^{1-\frac{d}{2}}} \). (2.283)

\( ^{15} \)Exercise: Verify explicitly.
The second renormalization condition is naturally chosen to be given by
\[ \frac{\partial^4}{\partial \phi_c^4} V(\phi_c)|_{\phi_c=v} = \frac{g}{4} 4! . \] (2.284)

This leads to the result
\[ \delta g = \hbar g^2 (N + 8) \frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}} . \] (2.285)

As a consequence we obtain
\[ \delta \mu = \hbar g \mu^2 (N + 2) \frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}} . \] (2.286)

After substituting back in the potential we get
\[ V(\phi_c) = -\frac{\mu^2}{2} \phi_c^2 + \frac{g}{4} (\phi_c^2)^2 + \frac{\hbar}{4(4\pi)^2} \left[ (N - 1)(-\mu^2 + g\phi_c^2)^2 \left( \ln(-\mu^2 + g\phi_c^2) - \frac{3}{2} \right) \right. \]
\[ + \left. (-\mu^2 + 3g\phi_c^2)^2 \left( \ln(-\mu^2 + 3g\phi_c^2) - \frac{3}{2} \right) \right] . \] (2.287)

In deriving this result we have used in particular the equation
\[ \Gamma\left(-\frac{d}{2}\right) \left( \frac{m^2}{4\pi} \right)^{\frac{d}{2}} = \frac{m^4}{2(4\pi)^2} \left[ \frac{2}{\epsilon} + \ln 4\pi - \ln m^2 - \gamma + \frac{3}{2} + O(\epsilon) \right] . \] (2.288)

### 2.8 Spontaneous Symmetry Breaking

#### 2.8.1 Example: The $O(N)$ Model

We are still interested in the $(\phi^2)^2$ theory with $O(N)$ symmetry in $d$ dimensions ($d = 4$ is of primary importance but other dimensions are important as well) given by the classical action (with the replacements $m^2 = -\mu^2$ and $\lambda/4! = g/4$)
\[ S[\phi] = \int d^4x \left[ \frac{1}{2} \partial_{\mu} \phi_i \partial^{\mu} \phi_i + \frac{1}{2} \mu^2 \phi_i^2 - \frac{g}{4} (\phi_i^2)^2 \right] . \] (2.289)

This scalar field can be in two different phases depending on the value of $m^2$. The "symmetric phase" is characterized by the "order parameter" $\phi_{ic}(J = 0) \equiv \phi_i > 0$ and the "broken phase" with $\phi_{ic} \neq 0$. This corresponds to the spontaneous symmetry breaking of $O(N)$ down to $O(N - 1)$ and the appearance of massless particles called Goldstone bosons in $d \geq 3$. For $N = 1$, it is the $Z_2$ symmetry $\phi \rightarrow -\phi$ which is broken.

---

Exercise: Verify explicitly.
Exercise: Verify explicitly.
Exercise: Verify explicitly.
spontaneously. This is a very concrete instance of Goldstone theorem. In "local" scalar field theory in \(d \leq 2\) there can be no spontaneous symmetry breaking according to the Wagner-Mermin-Coleman theorem. To illustrate these points we start from the classical potential

\[
V[\phi] = \int d^d x \left[ -\frac{1}{2} \mu^2 \phi_i^2 + \frac{g}{4} (\phi_i^2)^2 \right].
\]

This has a Mexican-hat shape. The minimum of the system is a configuration which must minimize the potential and also is uniform so that it minimizes also the Hamiltonian. The equation of motion is

\[
\phi_j (\mu^2 + g\phi^2) = 0.
\]

For \(\mu^2 < 0\) the minimum is unique given by the vector \(\phi_i = 0\) whereas for \(\mu^2 > 0\) we can have as solution either the vector \(\phi_i < 0\) (which in fact is not a minimum) or any vector \(\phi_i\) such that

\[
\phi_i^2 = \frac{\mu^2}{g}.
\]

As one may check any of these vectors is a minimum. In other words we have an infinitely degenerate ground state given by the sphere \(S^{N-1}\). The ground state is conventionally chosen to point in the \(N\) direction by adding to the action a symmetry breaking term of the form

\[
\Delta S = \epsilon \int d^d x \phi_N, \quad \epsilon > 0.
\]

\[
(-\mu^2 + g\phi^2)\phi_j = \epsilon \delta_{jN}.
\]

The solution is clearly of the form

\[
\phi_i = v \delta_{iN}.
\]

The coefficient \(v\) is given by

\[
(-\mu^2 + g\nu^2) v = \epsilon \Rightarrow v = \sqrt{\frac{\mu^2}{g}}, \quad \epsilon \rightarrow 0.
\]

We expand around this solution by writing

\[
\phi_k = \pi_k, \quad k = 1, \ldots, N - 1, \quad \phi_N = v + \sigma.
\]

By expanding the potential around this solution we get

\[
V[\phi] = \int d^d x \left[ \frac{1}{2} (-\mu^2 + g\nu^2) \pi_k^2 + \frac{1}{2} (-\mu^2 + 3g\nu^2) \sigma^2 + \nu(-\mu^2 + g\nu^2) \sigma + g\nu \sigma^3 + g\nu \sigma \pi_k^2 + \frac{g}{2} \sigma^2 \pi_k^2 + \frac{g}{4} \sigma^4 + \frac{g}{4} (\pi_k^2) \right].
\]
We have therefore one massive field (the $\sigma$) and $N - 1$ massless fields (the pions $\pi_k$) for $\mu^2 > 0$. Indeed

$$m^2_\pi = -\mu^2 + g v^2 \equiv 0, \quad m^2_\sigma = -\mu^2 + 3 g v^2 \equiv 2 \mu^2.$$ \hspace{1cm} (2.299)

For $\mu^2 < 0$ we must have $v = 0$ and thus $m^2_\pi = m^2_\sigma = -\mu^2$.

It is well known that the $O(4)$ model provides a very good approximation to the dynamics of the real world pions with masses $m_+ = m_- = 139.6$ Mev, $m_0 = 135$ Mev which are indeed much less than the mass of the 4th particle (the sigma particle) which has mass $m_\sigma = 900$ Mev. The $O(4)$ model can also be identified with the Higgs sector of the standard model.

The action around the "broken phase" solution is given by

$$S[\phi] = \int d^4x \left[ \frac{1}{2} \partial_\mu \pi_i \partial^\mu \pi_i + \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma - \mu^2 \sigma^2 - g v \sigma^3 - g v \sigma \pi_i^2 - \frac{g}{2} \sigma^2 \pi_i^2 - \frac{g}{4} \sigma^4 - \frac{g}{4} \pi_i^2 \right].$$ \hspace{1cm} (2.300)

We use the counterterms

$$\delta S[\phi] = \int d^4x \left[ \frac{1}{2} \delta Z \partial_\mu \phi_i \partial^\mu \phi_i + \frac{1}{2} \delta \mu \phi_i^2 - \frac{1}{4} \delta g (\phi_i^2)^2 \right]$$

$$\quad = \int d^4x \left[ \frac{1}{2} \delta Z \partial_\mu \pi_i \partial^\mu \pi_i - \frac{1}{2} \delta \mu \pi_i^2 + \frac{1}{2} \delta Z \partial_\mu \sigma \partial^\mu \sigma - \frac{1}{4} \delta g \sigma^2 \pi_i^2 - \frac{1}{2} \delta g v \pi_i^2 - \frac{1}{4} \delta g \sigma^2 \pi_i^2 - \frac{1}{4} \delta g \sigma v \pi_i^2 - \frac{3}{4} \delta g \sigma v \pi_i^2 \right].$$ \hspace{1cm} (2.301)

We compute the 1–point proper vertex of the sigma field. We start from the result

$$\Gamma_{ij} = S_{ij} + \frac{1}{2} \text{i} \eta_{ijk} G_{ij} G_{jk} + ...$$ \hspace{1cm} (2.302)

$$G_{ij}^{\phi} = -S^{-1}_{ij} |_{\phi=\phi_c}.$$ \hspace{1cm} (2.303)

We compute immediately

$$\delta \Gamma |_{\sigma=\pi_i=0} = 0 + \frac{1}{2} \text{i} \eta_{ijk} \left[ G_{0}^{\sigma \sigma} S_{\sigma \sigma} + G_{0}^{\pi \pi} S_{\pi \pi} \right]$$

$$= \frac{1}{2} \text{i} \eta_{ijk} \left[ \int \frac{d^d k}{(2\pi)^d} \frac{1}{-k^2 + 2 \mu^2} \left( -3 g v + \frac{1}{2} \delta_{ij} \right) \right]$$

$$= -3 i g v h \int \frac{d^d k}{(2\pi)^d} \frac{1}{-k^2 - 2 \mu^2} - i g v (N - 1) h \int \frac{d^d k}{(2\pi)^d} \frac{1}{-k^2 - \xi^2}.$$ \hspace{1cm} (2.304)
In the above equation we have added a small mass $\xi^2$ for the pions to control the infrared behavior. We need to compute

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - m^2} = -i \int \frac{d^d k_E}{(2\pi)^d} \frac{1}{k_E^2 + m^2}$$

$$= -i \frac{\Omega_{d-1}}{2(2\pi)^d} \int \frac{d x^{d-1}}{x + m^2} dx$$

$$= -i \frac{\Omega_{d-1}}{2(2\pi)^d} (m^2)^{\frac{d}{2} - 1} \int_0^1 t^{-\frac{d}{2}} (1 - t)^{\frac{d}{2} - 1} dt$$

$$= -i \frac{1}{(4\pi)^{\frac{d}{2}}} (m^2)^{\frac{d}{2} - 1} \Gamma(1 - \frac{d}{2}).$$

We get

$$\frac{\delta \Gamma}{\delta \sigma} \bigg|_{\sigma = \pi, \xi = 0} = -g v \hbar \frac{\Gamma(1 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \left( \frac{3}{(2\mu^2)^{1-\frac{d}{2}}} + \frac{N - 1}{(\xi^2)^{1-\frac{d}{2}}} \right).$$

By adding the contribution of the counterterms we get

$$\frac{\delta \Gamma}{\delta \sigma} \bigg|_{\sigma = \pi, \xi = 0} = -(-\delta \mu v + \delta g v^2) - g v \hbar \frac{\Gamma(1 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \left( \frac{3}{(2\mu^2)^{1-\frac{d}{2}}} + \frac{N - 1}{(\xi^2)^{1-\frac{d}{2}}} \right).$$

The corresponding Feynman diagrams are shown on figure 8. We will impose the renormalization condition

$$\frac{\delta \Gamma}{\delta \sigma} = 0.$$ (2.308)

This is equivalent to the statement that the sum of all tadpole diagrams giving the 1–point proper vertex for the $\sigma$ field vanishes. In other words we do not allow any quantum shifts in the vacuum expectation value of $\phi_N$ which is given by $< \phi_N > = v$.

We get then

$$(-\delta \mu + \delta g v^2) = -g v \frac{\Gamma(1 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \left( \frac{3}{(2\mu^2)^{1-\frac{d}{2}}} + \frac{N - 1}{(\xi^2)^{1-\frac{d}{2}}} \right).$$ (2.309)

Next we consider the $\pi \pi$ amplitude. We use the result

$$\Gamma_{j_0 k_0} = S_{j_0 k_0} + \frac{\hbar}{i} \left[ \frac{1}{2} c_{0m}^{\bar{m}n} S[\phi]_{j_0 k_0 m n} + \frac{1}{2} c_{0m}^{\bar{m}n} c_{0n}^{\bar{n}m} S[\phi]_{j_0 \bar{k}_0 m n} S[\phi]_{k_0 \bar{m} n \bar{m} n} \right].$$

We compute immediately (including again a small mass $\xi^2$ for the pions)

$$S_{j_0 k_0} = -\delta_{j_0 k_0} (\Delta + \xi^2) \delta^d(x - y).$$ (2.311)
\[\frac{1}{2} G_{\mu\nu}^{mn} S[\phi]_{j_{0}k_{0}mn} = \frac{1}{2} \int d^d z d^d w [G_{\phi}^{\sigma\sigma}(z, w)][-2g\delta_{j_{0}k_{0}}\delta^d(x-y)\delta^d(x-z)\delta^d(x-w)] + \frac{1}{2} \int d^d z d^d w [\delta_{mn} G_{\phi}^{\sigma\sigma}(z, w)][-3\frac{2}{3} g\delta_{j_{0}k_{0}mn}\delta^d(x-y)\delta^d(x-z)\delta^d(x-w)] = -g\delta_{j_{0}k_{0}} G_{\phi}^{\sigma\sigma}(x, x)\delta^d(x-y) - (N + 1)g\delta_{j_{0}k_{0}} G_{\phi}^{\sigma\sigma}(x, x)\delta^d(x-y). \tag{2.312}\]

\[\frac{1}{2} G_{\mu\nu}^{mno} G_{\phi}^{\sigma\sigma} S[\phi]_{j_{0}mnS[\phi]_{k_{0}mona}} = \left(\frac{2}{\pi}\right) \int d^d z \int d^d w \int d^d u_0 [\delta_{mn} G_{\phi}^{\sigma\sigma}(z, w)][G_{\phi}^{\sigma\sigma}(w, u_0)] x \times [-2g\nu\delta_{j_{0}mn} \delta^d(x-z)\delta^d(x-w)] - 2g\nu\delta_{k_{0}mon} \delta^d(y-z_0)\delta^d(y-w_0)] = 4g^2 v^2 G_{\phi}^{\sigma\sigma}(x, x)G_{\phi}^{\sigma\sigma}(x, y). \tag{2.313}\]

Thus we get
\[\Gamma_{j_{0}k_{0}}^{\pi\pi}(x, y) = -\delta_{j_{0}k_{0}}(\Delta + \xi^2)\delta^d(x-y) + \frac{h}{i} \left[-g\delta_{j_{0}k_{0}} G_{\phi}^{\sigma\sigma}(x, x)\delta^d(x-y) - (N + 1)g\delta_{j_{0}k_{0}} G_{\phi}^{\sigma\sigma}(x, x)\delta^d(x-y) + 4g^2 v^2 \delta_{j_{0}k_{0}} G_{\phi}^{\sigma\sigma}(x, y)G_{\phi}^{\sigma\sigma}(x, y)\right]. \tag{2.314}\]

Recall also that
\[G_{\phi}^{\sigma\sigma}(x, y) = \int \frac{d^d p}{(2\pi)^d} \frac{1}{-p^2 + \xi^2} e^{ip(x-y)}, \quad G_{\phi}^{\sigma\sigma}(x, y) = \int \frac{d^d p}{(2\pi)^d} \frac{1}{-p^2 + 2m^2} e^{ip(x-y)}. \tag{2.315}\]

The Fourier transform is defined by
\[\int d^d x \int d^d y \Gamma_{j_{0}k_{0}}^{\pi\pi}(x, y)e^{ipx}e^{iky} = (2\pi)^d\delta^d(p + k)\Gamma_{j_{0}k_{0}}^{\pi\pi}(p). \tag{2.316}\]

We compute then
\[\Gamma_{j_{0}k_{0}}^{\pi\pi}(p) = \delta_{j_{0}k_{0}}(p^2 - \xi^2) + \frac{h}{i} \delta_{j_{0}k_{0}} \left[-g \int \frac{d^d k}{(2\pi)^d} \frac{1}{-k^2 + 2m^2} - (N + 1)g \int \frac{d^d k}{(2\pi)^d} \frac{1}{-k^2 + \xi^2}\right] + 4g^2 v^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{-k^2 + \xi^2} \frac{1}{-(k + p)^2 + 2m^2}\tag{2.317}\].

By adding the contribution of the counterterms we get
\[\Gamma_{j_{0}k_{0}}^{\pi\pi}(p) = \delta_{j_{0}k_{0}}(p^2 - \xi^2) + \frac{h}{i} \delta_{j_{0}k_{0}} \left[-g \int \frac{d^d k}{(2\pi)^d} \frac{1}{-k^2 + 2m^2} - (N + 1)g \int \frac{d^d k}{(2\pi)^d} \frac{1}{-k^2 + \xi^2}\right] + 4g^2 v^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{-k^2 + \xi^2} \frac{1}{-(k + p)^2 + 2m^2} + (\delta z p^2 + \delta \mu - \delta g v^2)\delta_{j_{0}k_{0}}. \tag{2.318}\]

The corresponding Feynman diagrams are shown on figure 8. After some calculation we obtain
\[\Gamma_{j_{0}k_{0}}^{\pi\pi}(p) = \delta_{j_{0}k_{0}}(p^2 - \xi^2) - 2h g \delta_{j_{0}k_{0}} \frac{\Gamma(1 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \left[(\xi^2)^{\frac{d}{2} - 1} - (2m^2)^{\frac{d}{2} - 1}\right] + \frac{h}{i} \delta_{j_{0}k_{0}} \left[4g^2 v^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{-k^2 + \xi^2} \frac{1}{-(k + p)^2 + 2m^2} + \delta z p^2 \delta_{j_{0}k_{0}}\right]. \tag{2.319}\]
The last integral can be computed using Feynman parameters $x_1$, $x_2$ introduced by the identity
\[
\frac{1}{A_1 A_2} = \int_0^1 dx_1 \int_0^1 dx_2 \frac{1}{(x_1 A_1 + x_2 A_2)^2} \delta(x_1 + x_2 - 1). \tag{2.320}
\]
We have then (with $s = 2$, $l = k + (1 - x_1)p$ and $M^2 = \xi^2 x_1 + 2 \mu^2 (1 - x_1) - p^2 x_1 (1 - x_1)$ and after a Wick rotation)
\[
\int \frac{d^d k}{(2\pi)^d} \frac{1}{-k^2 + \xi^2 - (k + p)^2 + 2\mu^2} = \int \frac{d^d k}{(2\pi)^d} \frac{1}{\int_0^1 dx_1 \int_0^1 dx_2 \frac{1}{[x_1 (k^2 - \xi^2) + x_2 ((k + p)^2 - 2\mu^2)]^s} \delta(x_1 + x_2 - 1)} \\
= \int_0^1 dx_1 \int_0^1 dx_2 \frac{1}{(2\pi)^d} \frac{1}{(l^2 - M^2)^s} \delta(x_1 + x_2 - 1) \\
= i(-1)^s \int_0^1 dx_1 \int_0^1 dx_2 \delta(x_1 + x_2 - 1) \int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2 - M^2)^s} \\
= i(-1)^s \Omega_{d-1} \int_0^1 dx_1 \int_0^1 dx_2 \delta(x_1 + x_2 - 1) \int \frac{x^{\frac{d-4}{2}} dx}{(x + M^2)^s} \\
= i(-1)^s \Omega_{d-1} \frac{1}{2(2\pi)^d} \frac{1}{(M^2)^{\frac{d}{2}-s}} \int_0^1 dx_1 \int_0^1 dx_2 \delta(x_1 + x_2 - 1) \int_0^1 (1-t)^{\frac{d}{2}-1} t^{s-\frac{d}{2}} \\
= i(-1)^s \Omega_{d-1} \frac{1}{2(2\pi)^d} \frac{1}{(M^2)^{\frac{d}{2}-s}} \int_0^1 dx_1 \int_0^1 dx_2 \delta(x_1 + x_2 - 1) \frac{\Gamma(s-\frac{d}{2})\Gamma(\frac{d}{2})}{\Gamma(s)} \\
= \int_0^1 dx_1 \int_0^1 dx_2 \delta(x_1 + x_2 - 1) \frac{i(-1)^s}{(4\pi)^{\frac{d}{2}}} (M^2)^{\frac{d}{2}-s} \frac{\Gamma(s-\frac{d}{2})}{\Gamma(s)}. \tag{2.321}
\]

Using this result we have
\[
\Gamma_{\mu\nu}^{\pi\pi}(p) = \delta_{j_0 k_0} (p^2 - \xi^2) - 2hg\delta_{j_0 k_0} \frac{\Gamma(1 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}} [(\xi^2)^{\frac{d}{2}-1} - (2\mu^2)^{\frac{d}{2}-1}] \\
+ 4hg^2 v^2 \delta_{j_0 k_0} \frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \int_0^1 dx_1 [\xi^2 x_1 + 2\mu^2 (1 - x_1) - p^2 x_1 (1 - x_1)]^{\frac{d}{2}-2} + \delta Z p^2 \delta_{j_0 k_0}. \tag{2.322}
\]

By studying the amplitudes $\sigma\sigma$, $\sigma\sigma\pi\pi$ and $\pi\pi\pi\pi$ we can determine that the counterterm $\delta_Z$ is finite at one-loop whereas the counterterm $\delta_g$ is divergent.\(^{19}\) This means in particular that the divergent part of the above remaining integral does not depend on $p$.

---

\(^{19}\)Exercise: Show this result explicitly. You need to figure out and then use two more renormalization conditions.
We simply set $p^2 = 0$ and study

$$
\Gamma_{j_0 k_0}^{\pi\pi}(0) = \delta_{j_0 k_0}(-\xi^2) - 2\hbar g \delta_{j_0 k_0} \frac{\Gamma(1 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \left[ (\xi^2)^{\frac{d}{2} - 1} - (2\mu^2)^{\frac{d}{2} - 1} \right] + 4\hbar g^2 v^2 \delta_{j_0 k_0} \frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \int_0^1 dx_1
\times \left[ \xi^2 x_1 + 2\mu^2(1 - x_1) \right]^{\frac{d}{2} - 2}.
$$

We get (using $gv^2 = \mu^2$)

$$
\Gamma_{j_0 k_0}^{\pi\pi}(0) = \delta_{j_0 k_0}(-\xi^2) - 2\hbar g \delta_{j_0 k_0} \frac{\Gamma(1 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \left[ (\xi^2)^{\frac{d}{2} - 1} - (2\mu^2)^{\frac{d}{2} - 1} \right] + 2\hbar g \delta_{j_0 k_0} \frac{\Gamma(1 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \left( \frac{2\mu^2}{2\mu^2 - \xi^2} \right) \left[ (\xi^2)^{\frac{d}{2} - 1} - (2\mu^2)^{\frac{d}{2} - 1} \right].
$$

This vanishes exactly in the limit $\xi \to 0$ and therefore the pions remain massless at one-loop$^{20}$. This is a manifestation of the Goldstone’s theorem which states that there must exist $N - 1$ massless particles associated with the $N - 1$ broken symmetries of the breaking pattern $O(N) \to O(N - 1)$.

### 2.8.2 Goldstone’s Theorem

Spontaneous symmetry breaking of a continuous symmetry leads always to massless particles called Goldstone bosons. The number of massless Goldstone bosons which appear is precisely equal to the number of symmetry generators broken spontaneously. This is a general result known as Goldstone’s theorem. For example in the case of the $O(N)$ model studied in the previous sections the continuous symmetries are precisely $O(N)$ transformations, i.e. rotations in $N$ dimensions which rotate the different components of the scalar field into each other. There are in this case $N(N - 1)/2$ independent rotations and hence $N(N - 1)/2$ generators of the group $O(N)$. Under the symmetry breaking pattern $O(N) \to O(N - 1)$ the number of broken symmetries is exactly $N(N - 1)/2 - (N - 1)(N - 2)/2 = N - 1$ and hence there must appear $N - 1$ massless Goldstone bosons in the low energy spectrum of the theory which have been already verified explicitly up to the one-loop order. This holds also true at any arbitrary order in perturbation theory. Remark that for $N = 1$ there is no continuous symmetry and therefore there are no massless Goldstone particles associated to the symmetry breaking pattern $\phi \to -\phi$. We sketch now a general proof of Goldstone’s theorem.

A typical Lagrangian density of interest is of the form

$$
\mathcal{L}(\phi) = \text{terms with derivatives}(\phi) - V(\phi).
$$

The minimum of $V$ is denoted $\phi_0$ and satisfies

$$
\frac{\partial}{\partial \phi_a} V(\phi)|_{\phi = \phi_0} = 0.
$$

---

$^{20}$Exercise: Show this result directly. Start by showing that

$$
\frac{\hbar}{4} \left[ g^2 v^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{-k^2 + \xi^2 - k^2 + 2\mu^2} \right] = 2\hbar g |I(\xi^2) - I(2\mu^2)|, \quad I(m^2) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - m^2}.
$$
Now we expand $V$ around the minimum $\phi_0$ up to the second order in the fields. We get

$$V(\phi) = V(\phi_0) + \frac{1}{2}(\phi - \phi_0)\frac{\partial^2}{\partial \phi_a \partial \phi_b}V(\phi)|_{\phi=\phi_0} + ...$$

$$= V(\phi_0) + \frac{1}{2}(\phi - \phi_0)\frac{\partial^2}{\partial \phi_a \partial \phi_b}V(\phi)|_{\phi=\phi_0} + ...$$

(2.327)

The matrix $m^2_{ab}(\phi_0)$ called the mass matrix is clearly a symmetric matrix which is also positive since $\phi_0$ is assumed to be a minimum configuration.

A general continuous symmetry will transform the scalar field $\phi$ infinitesimally according to the generic law

$$\phi_a \rightarrow \phi'_a = \phi_a + \alpha \Delta_a(\phi).$$

(2.328)

The parameter $\alpha$ is infinitesimal and $\Delta_a$ are some functions of $\phi$. The invariance of the Lagrangian density is given by the condition

$$\text{terms with derivatives}(\phi) - V(\phi) = \text{terms with derivatives}(\phi + \alpha \Delta(\phi)) - V(\phi + \alpha \Delta(\phi)).$$

(2.329)

For constant fields this condition reduces to

$$V(\phi) = V(\phi + \alpha \Delta(\phi)).$$

(2.330)

Equivalently

$$\Delta_a(\phi) \frac{\partial}{\partial \phi_a}V(\phi) = 0.$$  

(2.331)

By differentiating with respect to $\phi_b$ and setting $\phi = \phi_0$ we get

$$m^2_{ab}(\phi_0)\Delta_b(\phi_0) = 0.$$  

(2.332)

The symmetry transformations, as we have seen, leave always the Lagrangian density invariant which was actually our starting point. In the case that the above symmetry transformation leaves also the ground state configuration $\phi_0$ invariant we must have $\Delta(\phi_0) = 0$ and thus the above equation becomes trivial. However, in the case that the symmetry transformation does not leave the ground state configuration $\phi_0$ invariant, which is precisely the case of a spontaneously broken symmetry, $\Delta_b(\phi_0)$ is an eigenstate of the mass matrix $m^2_{ab}(\phi_0)$ with 0 eigenvalue which is exactly the massless Goldstone particle.
### 3.1 Free Dirac Field

#### 3.1.1 Canonical Quantization

The Dirac field $\psi$ describes particles of spin $\hbar/2$. The Dirac field $\psi$ is a 4–component object which transforms as spinor under the action of the Lorentz group. The classical equation of motion of a free Dirac field is the Dirac equation. This is given by

$$ (i\hbar \gamma^\mu \partial_\mu - mc)\psi = 0. \quad (3.1) $$

Equivalently the complex conjugate field $\bar{\psi} = \psi^+ \gamma^0$ obeys the equation

$$ \bar{\psi}(i\hbar \gamma^\mu \partial_\mu + mc) = 0. \quad (3.2) $$

These two equations are the Euler-Lagrange equations derived from the action

$$ S = \int d^4x \bar{\psi}(i\hbar \gamma^\mu \partial_\mu - mc^2)\psi. \quad (3.3) $$

The Dirac matrices $\gamma^\mu$ satisfy the usual Dirac algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. The Dirac equation admits positive-energy solutions (associated with particles) denoted by spinors $u^i(p)$ and negative-energy solutions (associated with antiparticles) denoted by spinors $v^i(p)$.

The spinor field can be put in the form (with $\omega(\vec{p}) = E/\hbar = \sqrt{\vec{p}^2 c^2 + m^2 c^4}/\hbar$)

$$ \psi(x) = \frac{1}{\hbar} \int \frac{d^3p}{(2\pi \hbar)^3} \sqrt{\frac{c}{2\omega(\vec{p})}} \sum_i \left( e^{-\frac{i}{\hbar} \hat{p}_x u^i(\vec{p})} b(\vec{p}, i) + e^{\frac{i}{\hbar} \hat{p}_x v^i(\vec{p})} d(\vec{p}, i)^+ \right). \quad (3.4) $$
The time-ordering operator is defined by
\[ \{ \hat{\psi}_\alpha(x^0, \vec{x}), \hat{\Pi}_\beta(x^0, \vec{y}) \} = i\hbar \delta_{\alpha\beta} \delta^3(\vec{x} - \vec{y}). \] (3.5)

Equivalently
\[
\begin{align*}
\{ \hat{b}(\vec{p}, i), \hat{\psi}(\vec{q}, j)^+ \} &= \hbar \delta_{ij} (2\pi \hbar)^3 \delta^3(\vec{p} - \vec{q}) \\
\{ \hat{d}(\vec{p}, i)^+, \hat{\psi}(\vec{q}, j) \} &= \hbar \delta_{ij} (2\pi \hbar)^3 \delta^3(\vec{p} - \vec{q}) \\
\{ \hat{b}(\vec{p}, i), \hat{d}(\vec{q}, j) \} &= \{ \hat{d}(\vec{q}, j)^+, \hat{b}(\vec{p}, i) \} = 0.
\end{align*}
\] (3.6)

We find that excited particle states are obtained by acting with \( \hat{b}(\vec{p}, i)^+ \) on the vacuum \( |0\rangle \) whereas excited antiparticle states are obtained by acting with \( \hat{d}(\vec{p}, i)^+ \). The vacuum state \( |0\rangle \) is the eigenstate with energy equal 0 of the Hamiltonian
\[ \hat{H} = \int \frac{d^3p}{(2\pi \hbar)^3} \omega(p) \sum_i \left( \hat{b}(\vec{p}, i)^+ \hat{b}(\vec{p}, i) + \hat{d}(\vec{p}, i)^+ \hat{d}(\vec{p}, i) \right). \] (3.7)

The Feynman propagator for a Dirac spinor field is defined by
\[ (S_F)_{ab}(x - y) = \langle 0| T \hat{\psi}_a(x) \hat{\psi}_b(y) |0\rangle. \] (3.8)

The time-ordering operator is defined by
\[
\begin{align*}
T \hat{\psi}(x) \hat{\psi}(y) &= +\hat{\psi}(x) \hat{\psi}(y) \ , \ x^0 > y^0 \\
T \hat{\psi}(x) \hat{\psi}(y) &= -\hat{\psi}(y) \hat{\psi}(x) \ , \ x^0 < y^0.
\end{align*}
\] (3.9)

Explicitly we have \(^1\)
\[ (S_F)_{ab}(x - y) = \frac{\hbar}{c} \int \frac{d^4p}{(2\pi \hbar)^4} \frac{i(\gamma^\mu p_\mu + mc)_{ab}}{p^2 - mc^2 + i\epsilon} e^{-i\hat{p}(x - y)}. \] (3.10)

### 3.1.2 Fermionic Path Integral and Grassmann Numbers

Let us now expand the spinor field as
\[ \psi(x^0, \vec{x}) = \frac{1}{\hbar} \int \frac{d^3p}{(2\pi \hbar)^3} \chi(x^0, \vec{p}) e^{\frac{i}{\hbar} \hat{p} \hat{x}}. \] (3.11)

\(^1\)There was a serious (well not really serious) error in our computation of the scalar propagator in the first semester which propagated to an error in the Dirac propagator. This must be corrected there and also in the previous chapter of this current semester in which we did not follow the factors of \( \hbar \) and \( c \) properly. In any case the coefficient \( \hbar/c \) appearing in front of this propagator is now correct.
The Lagrangian in terms of $\chi$ and $\chi^+$ is given by

$$
L = \int d^3x L
= \int d^3x \bar{\psi}(ihc\gamma^\mu \partial_\mu - mc^2)\psi
= \frac{c}{\hbar^2} \int \frac{d^3p}{(2\pi\hbar)^3} \bar{\chi}(x^0, \vec{p})(ihc\gamma^0 \partial_0 - \gamma^i p^i - mc)\chi(x^0, \vec{p}).
$$
(3.12)

We use the identity

$$
\gamma^0(\gamma^i p^i + mc)\chi(x^0, \vec{p}) = \frac{\hbar \omega(\vec{p})}{c} \chi(x^0, \vec{p}).
$$
(3.13)

We get then

$$
L = \frac{1}{\hbar} \int \frac{d^3p}{(2\pi\hbar)^3} \bar{\chi}^+(x^0, \vec{p})(i\partial_t - \omega(\vec{p}))\chi(x^0, \vec{p}).
$$
(3.14)

Using the box normalization the momenta become discrete and the measure $\int d^3\vec{p}/(2\pi\hbar)^3$ becomes the sum $\sum_{\vec{p}}/V$. Thus the Lagrangian becomes with $\theta_p(t) = \chi(x^0, \vec{p})/\sqrt{\hbar V}$ given by

$$
L = \sum_{\vec{p}} \theta_p^+(t)(i\partial_t - \omega(\vec{p}))\theta_p(t).
$$
(3.15)

For a single momentum $\vec{p}$ the Lagrangian of the theory simplifies to the single term

$$
L_p = \theta_p^+(t)(i\partial_t - \omega(\vec{p}))\theta_p(t).
$$
(3.16)

We will simplify further by thinking of $\theta_p(t)$ as a single component field. The conjugate variable is $\pi_p(t) = i\theta^+_p(t)$. In the quantum theory we replace $\theta_p$ and $\pi_p$ with operators $\hat{\theta}_p$ and $\hat{\pi}_p$. The canonical commutation relations are

$$
\{\hat{\theta}_p, \hat{\pi}_p\} = i\hbar, \\{\hat{\theta}_p, \hat{\theta}_p\} = \{\hat{\pi}_p, \hat{\pi}_p\} = 0.
$$
(3.17)

There several remarks here:

• In the limit $\hbar \rightarrow 0$, the operators reduce to fields which are anticommuting classical functions. In other words even classical fermion fields must be represented by anticommuting numbers which are known as Grassmann numbers.

• There is no eigenvalues of the operators $\hat{\theta}_p$ and $\hat{\pi}_p$ in the set of complex numbers except 0. The non-zero eigenvalues must be therefore anticommuting Grassmann numbers.

• Obviously given two anticommuting Grassmann numbers $\alpha$ and $\beta$ we have immediately the following fundamental properties

$$
\alpha\beta = -\beta\alpha, \ \alpha^2 = \beta^2 = 0.
$$
(3.18)
The classical equation of motion following from the Lagrangian $L_p$ is $i\partial_t \theta_p = \omega(\vec{p}) \theta_p$. An immediate solution is given by

$$\hat{\theta}_p(t) = \hat{\theta}_p \exp(-i\omega(\vec{p})t).$$

(3.19)

Thus

$$\{\hat{b}_p, \hat{b}_p^+\} = \hbar, \{\hat{b}_p, \hat{b}_p\} = \{\hat{b}_p^+, \hat{b}_p^+\} = 0.$$

(3.20)

The Hilbert space contains two states $|0\rangle$ (the vacuum) and $|1\rangle = \hat{b}_p^+ |0\rangle$ (the only excited state). Indeed we clearly have $\hat{b}_p^+ |1\rangle = 0$ and $\hat{b}_p |1\rangle = \hbar |0\rangle$. We define the (coherent) states at time $t = 0$ by

$$|\theta_p(0)\rangle = e^{\hat{b}_p^+ \theta_p(0)} |0\rangle, \quad <\theta_p(0)| = e^{\theta_p^-(0) \hat{b}_p}.$$

(3.21)

The number $\theta_p(0)$ must be anticommuting Grassmann number, i.e. it must satisfy $\theta_p(0)^2 = 0$ whereas the number $\theta^+(0)$ is the complex conjugate of $\theta_p(0)$ which should be taken as independent and hence $(\theta^+_p(0))^2 = 0$ and $\theta^+_p(0)\theta_p(0) = -\theta_p(0)\theta^+_p(0)$. We compute immediately that

$$\hat{\theta}_p(0)|\theta_p(0)\rangle = \theta_p(0)|\theta_p(0)\rangle, \quad <\theta_p(0)|\hat{\theta}_p(0)^+ = <\theta_p(0)|\theta^+_p(0).$$

(3.22)

The Feynman propagator for the field $\theta_p(t)$ is defined by

$$S(t - t') = <0|T(\hat{\theta}_p(t)\hat{\theta}_p^+(t'))|0\rangle.$$

(3.23)

We compute immediately (with $\epsilon > 0$)

$$S(t - t') = \hbar e^{-i\omega(\vec{p})(t-t')} \equiv \hbar^2 \int \frac{dp^0}{2\pi \hbar} \frac{i}{p^0 - \hbar \omega(\vec{p}) + i\epsilon} e^{-\frac{i}{\hbar} p^0(t-t')}, \quad t > t'.$$

(3.24)

$$S(t - t') = 0 \equiv \hbar^2 \int \frac{dp^0}{2\pi \hbar} \frac{i}{p^0 - \hbar \omega(\vec{p}) + i\epsilon} e^{-\frac{i}{\hbar} p^0(t-t')}, \quad t < t'.$$

(3.25)

The anticommuting Grassmann numbers have the following properties:

- A general function $f(\theta)$ of a single anticommuting Grassmann number can be expanded as

$$f(\theta) = A + B\theta.$$

(3.26)

- The integral of $f(\theta)$ is therefore

$$\int d\theta f(\theta) = \int d\theta (A + B\theta).$$

(3.27)

We demand that this integral is invariant under the shift $\theta \rightarrow \theta + \eta$. This leads immediately to the so-called Berezin integration rules

$$\int d\theta = 0, \quad \int d\theta = 1.$$

(3.28)

Exercise: Verify this result using the residue theorem.
• The differential $d\theta$ anticommutes with $\theta$, viz
  \[ d\theta \theta = -\theta d\theta. \]  
  (3.29)

• We have immediately
  \[ \int d\theta d\eta \eta \theta = 1. \]  
  (3.30)

• The most general function of two anticommuting Grassmann numbers $\theta$ and $\theta^+$ is
  \[ f(\theta, \theta^+) = A + B\theta + C\theta^+ + D\theta^+ \theta. \]  
  (3.31)

• Given two anticommuting Grassmann numbers $\theta$ and $\eta$ we have
  \[ (\theta \eta)^+ = \eta^+ \theta^+. \]  
  (3.32)

• We compute the integrals
  \[ \int d\theta^+ d\theta e^{-\theta^+ b\theta} = \int d\theta^+ d\theta (1 - \theta^+ b\theta) = b. \]  
  (3.33)

\[ \int d\theta^+ d\theta^+ e^{-\theta^+ b\theta} = \int d\theta^+ d\theta^+ (1 - \theta^+ b\theta) = 1. \]  
  (3.34)

It is instructive to compare the first integral with the bosonic integral
\[ \int dz^+ dz e^{-z^+ bz} = \frac{2\pi}{b}. \]  
  (3.35)

• We consider now a general integral of the form
  \[ \int \prod_i d\theta_i^+ d\theta_i f(\theta^+, \theta). \]  
  (3.36)

Consider the unitary transformation $\theta_i \rightarrow \theta_i' = U_{ij} \theta_j$ where $U^+ U = 1$. It is rather obvious that
\[ \prod_i d\theta_i' = \det U \prod_i d\theta_i. \]  
  (3.37)

Hence $\prod_i d\theta_i' + d\theta_i' = \prod_i d\theta_i^+ d\theta_i$ since $U^+ U = 1$. On the other hand, by expanding the function $f(\theta^+, \theta)$ and integrating out we immediately see that the only non-zero term will be exactly of the form $\prod_i \theta_i^+ \theta_i$ which is also invariant under the unitary transformation $U$. Hence
\[ \int \prod_i d\theta_i' + d\theta_i' f(\theta^+, \theta') = \int \prod_i d\theta_i^+ d\theta_i f(\theta^+, \theta). \]  
  (3.38)

\textsuperscript{3}Exercise: Verify this fact.
Consider the above integral for

$$f(\theta^+, \theta) = e^{-\theta^+ M \theta}. \quad (3.39)$$

$M$ is a Hermitian matrix. By using the invariance under $U(N)$ we can diagonalize the matrix $M$ without changing the value of the integral. The eigenvalues of $M$ are denoted $m_i$. The integral becomes

$$\int \prod_i d\theta_i^+ d\theta_i e^{-\theta^+ M \theta} = \int \prod_i d\theta_i^+ d\theta_i e^{-\theta^+ m_i \theta_i} = \prod_i m_i = \text{det} M. \quad (3.40)$$

Again it is instructive to compare with the bosonic integral

$$\int \prod_i dz_i^+ dz_i e^{-z^+ Mz} = (2\pi)^n \text{det} M. \quad (3.41)$$

We consider now the integral

$$\int \prod_i d\theta_i^+ d\theta_i e^{-\theta^+ M \theta - \theta - \eta - \eta^+} = \int \prod_i d\theta_i^+ d\theta_i e^{-\theta^+ (\eta^+ + M^{-1}) M (\theta + M^{-1} \eta) + (\theta^+ + M^{-1}) \eta} = \text{det} M e^{\eta^+ M^{-1} \eta}. \quad (3.42)$$

Let us consider now the integral

$$\int \prod_i d\theta_i^+ d\theta_i e^{-\theta^+ M \theta} = \frac{\delta}{\delta \eta_k} \frac{\delta}{\delta \eta_l} \left( \int \prod_i d\theta_i^+ d\theta_i e^{-\theta^+ M \theta - \theta - \eta - \eta^+} \right)_{\eta = \eta^+ = 0} = \text{det} M (M^{-1})_{kl}. \quad (3.43)$$

In the above equation we have always to remember that the order of differentials and variables is very important since they are anticommuting objects.

In the above equation we observe that if the matrix $M$ has eigenvalue 0 then the result is 0 since the determinant vanishes in this case.

We go back now to our original problem. We want to express the propagator $S(t-t') = \langle 0 | T(\hat{\theta}_p(t) \hat{\theta}_p^+(t')) | 0 \rangle$ as a path integral over the classical fields $\theta_p(t)$ and $\theta_p^+(t)$ which must be complex anticommuting Grassmann numbers. By analogy with what happens in scalar field theory we expect the path integral to be a functional integral of the probability amplitude $\exp(i S_p/\hbar)$ where $S_p$ is the action $S_p = \int dt L_p$ over the classical fields $\theta_p(t)$ and $\theta_p^+(t)$ (which are taken to be complex anticommuting Grassmann numbers instead of ordinary complex numbers). In the presence of sources $\eta_p(t)$ and $\eta_p^+(t)$ this path integral reads

$$Z[\eta_p, \eta_p^+] = \int \mathcal{D}\theta_p^+ \mathcal{D}\theta_p \exp \left( \frac{i}{\hbar} \int dt \theta_p^+ (i \partial_t - \omega(\vec{p})) \theta_p + \frac{i}{\hbar} \int dt \eta_p^+ \theta_p + \frac{i}{\hbar} \int dt \theta_p^+ \eta_p \right) \quad (3.44)$$
By using the result (3.42) we know immediately that

\[
Z[\eta_p, \eta_p^+] = \det M e^{\eta^+ M^{-1} \eta}, \quad M = \frac{i}{\hbar} (i\partial_t - \omega(\vec{p})) \ , \ \eta = \frac{i}{\hbar} \eta_p, \ \eta^+ = \frac{i}{\hbar} \eta_p^+. \quad (3.45)
\]

In other words

\[
Z[\eta_p, \eta_p^+] = \det M e^{-\frac{1}{\hbar^2} \int dt \int dt' \eta_p^+(t) M^{-1}(t,t') \eta_p(t')}.
\]  
\(3.46\)

From one hand we have

\[
\left( \frac{\hbar}{i} \right)^2 \left( \frac{1}{Z} \frac{\delta^2}{\delta \eta_p(t') \delta \eta^+_p(t)} Z \right)_{\eta_p=\eta^+_p=0} = \frac{\int D\theta^+_p D\theta_p(t)\theta^+_p(t') \exp \left( \frac{i}{\hbar} \int dt \theta^+_p (i\partial_t - \omega(\vec{p})) \theta_p \right)}{\int D\theta^+_p D\theta_p \exp \left( \frac{i}{\hbar} \int dt \theta^+_p (i\partial_t - \omega(\vec{p})) \theta_p \right)}
\]

\[
\equiv < \theta^+_p(t) \theta^+_p(t') > . \quad (3.47)
\]

From the other hand

\[
\left( \frac{\hbar}{i} \right)^2 \left( \frac{1}{Z} \frac{\delta^2}{\delta \eta_p(t') \delta \eta^+_p(t)} Z \right)_{\eta_p=\eta^+_p=0} = M^{-1}(t,t'). \quad (3.48)
\]

Therefore

\[
< \theta^+_p(t) \theta^+_p(t') > = M^{-1}(t,t'). \quad (3.49)
\]

We have

\[
M(t,t') = -\frac{i}{\hbar} (i\partial_t - \omega(\vec{p})) \delta(t-t')
\]

\[
= \frac{1}{\hbar^2} \int \frac{dp^0}{2\pi \hbar} \frac{p^0 - \omega(\vec{p})}{i} e^{-i\vec{p} \cdot (t-t')}.
\]  
\(3.50\)

The inverse is therefore given by

\[
< \theta^+_p(t) \theta^+_p(t') > = M^{-1}(t,t') = \frac{1}{\hbar^2} \int \frac{dp^0}{2\pi \hbar} \frac{i}{p^0 - \omega(\vec{p}) + i\epsilon} e^{-i\vec{p} \cdot (t-t')}.
\]  
\(3.51\)

We conclude therefore that

\[
<0 | T(\hat{\theta}_p(t) \hat{\theta}^+_p(t')) | 0 > = \frac{\int D\theta^+_p D\theta_p \theta^+_p(t) \theta^+_p(t') \exp \left( \frac{i}{\hbar} \int dt \theta^+_p (i\partial_t - \omega(\vec{p})) \theta_p \right)}{\int D\theta^+_p D\theta_p \exp \left( \frac{i}{\hbar} \int dt \theta^+_p (i\partial_t - \omega(\vec{p})) \theta_p \right)}.
\]  
\(3.52\)
3.1.3 The Electron Propagator

We are now ready to state our main punch line. The path integral of a free Dirac field in the presence of non-zero sources must be given by the functional integral

\[ Z[\eta, \bar{\eta}] = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left( \frac{i}{\hbar} S_0[\psi, \bar{\psi}] + \frac{i}{\hbar} \int d^4x \bar{\psi} \gamma^\mu \partial_\mu \psi + \frac{i}{\hbar} \int d^4x \bar{\psi} \gamma^0 \right). \]  

The path integral for a free Dirac field is given by

\[ S_0[\psi, \bar{\psi}] = \int d^4x \bar{\psi} (i\hbar c \gamma^\mu \partial_\mu - mc^2) \psi. \]  

The Dirac spinor \( \psi \) and its Dirac conjugate spinor \( \bar{\psi} = \psi^+ \gamma^0 \) must be treated as independent complex spinors with components which are Grassmann-valued functions of \( x \). Indeed by taking \( \chi_i(x) \) to be an orthonormal basis of 4-component Dirac spinors (for example it can be constructed out of the \( u^i(p) \) and \( v^i(p) \) in an obvious way) we can expand \( \psi \) and \( \bar{\psi} \) as \( \psi = \sum_i \theta_i \chi_i(x) \) and \( \bar{\psi} = \sum_i \theta_i^\dagger \bar{\chi}_i \) respectively. The coefficients \( \theta_i \) and \( \theta_i^\dagger \) must then be complex Grassmann numbers. The measure appearing in the above integral is therefore

\[ \mathcal{D}\bar{\psi} \mathcal{D}\psi = \prod_i \mathcal{D}\theta_i^\dagger \mathcal{D}\theta_i. \]  

The path integral \( Z[\eta, \bar{\eta}] \) is the generating functional of all correlation functions of the fields \( \psi \) and \( \bar{\psi} \). Indeed we have

\[ < \psi_{\alpha_1}(x_1) ... \psi_{\alpha_n}(x_n) \bar{\psi}_{\beta_1}(y_1) ... \bar{\psi}_{\beta_n}(y_n) > = \left( \frac{\int \mathcal{D}\bar{\psi} \mathcal{D}\psi \psi_{\alpha_1}(x_1) ... \psi_{\alpha_n}(x_n) \bar{\psi}_{\beta_1}(y_1) ... \bar{\psi}_{\beta_n}(y_n) \exp \frac{i}{\hbar} S_0[\psi, \bar{\psi}]}{\int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \frac{i}{\hbar} S_0[\psi, \bar{\psi}] \delta^{2n} Z[\eta, \bar{\eta}]} \right)_{\eta = \bar{\eta} = 0}. \]  

For example the 2-point function is given by

\[ < \psi_{\alpha}(x) \bar{\psi}_{\beta}(y) > = \left( \frac{\hbar^2}{Z[\eta, \bar{\eta}]} \delta^{2} Z[\eta, \bar{\eta}] \right)_{\eta = \bar{\eta} = 0}. \]  

However by comparing the path integral \( Z[\eta, \bar{\eta}] \) with the path integral (3.42) we can make the identification

\[ M_{ij} \rightarrow -\frac{i}{\hbar} (i\hbar c \gamma^\mu \partial_\mu - mc^2)_{\alpha\beta} \delta^4(x - y) \ , \ \eta_i \rightarrow -\frac{i}{\hbar}\eta_i \ , \ \eta_i^\dagger \rightarrow -\frac{\hbar}{i}\bar{\eta}_i. \]  

We define

\[ M_{\alpha\beta}(x, y) = -\frac{i}{\hbar} (i\hbar c \gamma^\mu \partial_\mu - mc^2)_{\alpha\beta} \delta^4(x - y) \]

\[ = -\frac{i c}{\hbar} \int \frac{d^4p}{(2\pi)^4} (\gamma^\mu p_\mu - mc)_{\alpha\beta} e^{-\frac{i}{\hbar} p(x - y)} \]

\[ = \frac{e}{i\hbar} \int \frac{d^4p}{(2\pi)^4} \left( \frac{p^2 - m^2c^2}{\gamma^\mu p_\mu + mc} \right)_{\alpha\beta} e^{-\frac{i}{\hbar} p(x - y)}. \]
By using equation (3.42) we can deduce immediately the value of the path integral $Z[\eta, \bar{\eta}]$. We find

$$Z[\eta, \bar{\eta}] = \det M \exp \left( -\frac{1}{\hbar^2} \int \! d^4x \int \! d^4y \bar{\eta}_\alpha(x) M^{-1}_{\alpha\beta}(x,y) \eta_\beta(y) \right). \quad (3.60)$$

Hence the electron propagator is

$$\langle \psi_\alpha(x) \psi_\beta(y) \rangle = M^{-1\dagger}_{\alpha\beta}(x,y). \quad (3.61)$$

From the form of the Laplacian (3.59) we get immediately the propagator (including also an appropriate Feynman prescription)

$$\langle \psi_\alpha(x) \psi_\beta(y) \rangle = \frac{i\hbar}{c} \int \! \frac{d^4p}{(2\pi\hbar)^4} \frac{(\gamma^\mu p_\mu + mc)_{\alpha\beta}}{p^2 - m^2c^2 + i\epsilon} e^{-\frac{i}{\hbar} p(x-y)}. \quad (3.62)$$

## 3.2 Free Abelian Vector Field

### 3.2.1 Maxwell’s Action

The electric and magnetic fields $\vec{E}$ and $\vec{B}$ generated by a charge density $\rho$ and a current density $\vec{J}$ are given by the Maxwell’s equations written in the Heaviside-Lorentz system as

$$\nabla \vec{E} = \rho, \text{ Gauss’ Law.} \quad (3.63)$$

$$\nabla \vec{B} = 0, \text{ No – Magnetic Monopole Law.} \quad (3.64)$$

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}, \text{ Faraday’s Law.} \quad (3.65)$$

$$\nabla \times \vec{B} = \frac{1}{c} (\vec{J} + \frac{\partial \vec{E}}{\partial t}), \text{ Ampere – Maxwell’ s Law.} \quad (3.66)$$

The Lorentz force law expresses the force exerted on a charge $q$ moving with a velocity $\vec{u}$ in the presence of an electric and magnetic fields $\vec{E}$ and $\vec{B}$. This is given by

$$\vec{F} = q(\vec{E} + \frac{1}{c} \vec{u} \times \vec{B}). \quad (3.67)$$

The continuity equation expresses local conservation of the electric charge. It reads

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0. \quad (3.68)$$
The so-called field strength tensor is a second-rank antisymmetric tensor $F_{\mu\nu}$ defined by
\[
F_{\mu\nu} = \begin{pmatrix}
0 & -E_x & -E_y & -E_z \\
E_x & 0 & -B_z & B_y \\
E_y & B_z & 0 & -B_x \\
E_z & -B_y & B_x & 0
\end{pmatrix}.
\] (3.69)

The dual field strength tensor is also a second-rank antisymmetric tensor $\tilde{F}_{\mu\nu}$ defined by
\[
\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}.
\] (3.70)

In terms of $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ Maxwell’s equations will take the form
\[
\partial_\mu F^{\mu\nu} = \frac{1}{c} J^\nu, \quad \partial_\mu \tilde{F}^{\mu\nu} = 0.
\] (3.71)

The 4-vector current density $J^\mu$ is given by $J^\mu = (c\rho, J_x, J_y, J_z)$. The first equation yields Gaussian’s and Ampere-Maxwell’s laws whereas the second equation yields Maxwell’s third equation $\nabla \times \vec{B} = 0$ and Faraday’s law. The continuity equation and the Lorentz force law respectively can be rewritten in the covariant forms
\[
\partial_\mu J^\mu = 0.
\] (3.72)

\[
K^\mu = \frac{q}{c} \frac{dx^\mu}{d\tau} F^{\mu\nu}.
\] (3.73)

The electric and magnetic fields $\vec{E}$ and $\vec{B}$ can be expressed in terms of a scalar potential $V$ and a vector potential $\vec{A}$ as
\[
\vec{B} = \nabla \times \vec{A}.
\] (3.74)

\[
\vec{E} = -\frac{1}{c} (\nabla V + \frac{\partial \vec{A}}{\partial t}).
\] (3.75)

We construct the 4-vector potential $A^\mu$ as
\[
A^\mu = (V/c, \vec{A}).
\] (3.76)

The field tensor $F_{\mu\nu}$ can be rewritten in terms of $A_\mu$ as
\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.
\] (3.77)
This equation is actually equivalent to the two equations (3.74) and (3.75). The homogeneous Maxwell’s equation \( \partial_\mu \tilde{F}^{\mu\nu} = 0 \) is automatically solved by this ansatz. The inhomogeneous Maxwell’s equation \( \partial_\mu F^{\mu\nu} = J^\nu / c \) becomes

\[
\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = \frac{1}{c} J^\nu.
\] (3.78)

These equations of motion should be derived from a local Lagrangian density \( \mathcal{L} \), i.e. a Lagrangian which depends only on the fields and their first derivatives at the point \( \vec{x} \). Indeed it can be easily proven that the above equations of motion are the Euler-Lagrange equations of motion corresponding to the Lagrangian density

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} J_\mu A^\mu.
\] (3.79)

The free Maxwell’s action is

\[
S_0[A] = -\frac{1}{4} \int d^4 x F_{\mu\nu} F^{\mu\nu}.
\] (3.80)

The total Maxwell’s action will include a non-zero source and is given by

\[
S[A] = -\frac{1}{4} \int d^4 x F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} \int d^4 x J_\mu A^\mu.
\] (3.81)

### 3.2.2 Gauge Invariance and Canonical Quantization

We have a gauge freedom in choosing \( A^\mu \) given by local gauge transformations of the form (with \( \lambda \) any scalar function)

\[
A^\mu \rightarrow A'^\mu = A^\mu + \partial^\mu \Lambda.
\] (3.82)

Indeed under this transformation we have

\[
F'^{\mu\nu} = F^{\mu\nu}.
\] (3.83)

These local gauge transformations form a (gauge) group. In this case the group is just the abelian \( U(1) \) unitary group. The invariance of the theory under these transformations is termed a gauge invariance. The 4-vector potential \( A^\mu \) is called a gauge potential or a gauge field. We make use of the invariance under gauge transformations by working with a gauge potential \( A^\mu \) which satisfies some extra conditions. This procedure is known as gauge fixing. Some of the gauge conditions so often used are

\[
\partial_\mu A^\mu = 0, \text{ Lorentz Gauge.}
\] (3.84)

\[
\partial_i A^i = 0, \text{ Coulomb Gauge.}
\] (3.85)

\[
A^0 = 0, \text{ Temporal Gauge.}
\] (3.86)
\[ A^3 = 0 \text{, Axial Gauge.} \] (3.87)

\[ A^0 + A^1 = 0 \text{, Light Cone Gauge.} \] (3.88)

The form of the equations of motion (3.78) strongly suggest we impose the Lorentz condition. In the Lorentz gauge the equations of motion (3.78) become

\[ \partial_{\mu} \partial^{\mu} A^\nu = \frac{1}{c} J^\nu. \] (3.89)

Clearly we still have a gauge freedom \[ A^\mu \rightarrow A'^\mu = A^\mu + \partial^\mu \phi \] where \[ \partial^\mu \phi = 0. \] In other words if \( A^\mu \) satisfies the Lorentz gauge \( \partial_{\mu} A^\mu = 0 \) then \( A'^\mu \) will also satisfy the Lorentz gauge, i.e. \( \partial_{\mu} A'^\mu = 0 \) iff \( \partial_{\mu} \partial^\mu \phi = 0 \). This residual gauge symmetry can be fixed by imposing another condition such as the temporal gauge \( A^0 = 0 \). We have therefore 2 constraints imposed on the components of the gauge potential \( A^\mu \) which means that only two of them are really independent. The underlying mechanism for the reduction of the number of degrees of freedom is actually more complicated than this simple counting.

We incorporate the Lorentz condition via a Lagrange multiplier \( \zeta \), i.e. we add to the Maxwell’s Lagrangian density a term proportional to \( (\partial_\mu A_\mu)^2 \) in order to obtain a gauge-fixed Lagrangian density, viz

\[ \mathcal{L}_\zeta = -\frac{1}{4} F^\mu\nu F_{\mu\nu} - \frac{1}{2} \zeta (\partial_\mu A_\mu)^2 - \frac{1}{c} J_\mu A^\mu. \] (3.90)

The added extra term is known as a gauge-fixing term. The conjugate fields are

\[ \pi_0 = \frac{\delta \mathcal{L}_\zeta}{\delta \partial_t A_0} = -\frac{\zeta}{c} (\partial_0 A_0 - \partial_i A_i). \] (3.91)

\[ \pi_i = \frac{\delta \mathcal{L}_\zeta}{\delta \partial_t A_i} = \frac{1}{c} (\partial_0 A_i - \partial_i A_0). \] (3.92)

We remark that in the limit \( \zeta \rightarrow 0 \) the conjugate field \( \pi_0 \) vanishes and as a consequence canonical quantization becomes impossible. The source of the problem is gauge invariance which characterize the limit \( \zeta \rightarrow 0 \). For \( \zeta \neq 0 \) canonical quantization (although a very involved exercise) can be carried out consistently. We will not do this exercise here but only quote the result for the 2–point function. The propagator of the photon field in a general gauge \( \zeta \) is given by the formula (with \( \hbar = c = 1 \))

\[ iD_F^{\mu\nu}(x - y) = < 0 | T \left( \hat{A}_\mu^\mu(x) \hat{A}_\nu^\nu(y) \right) | 0 > = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 + i\epsilon} \left( -\eta_{\mu\nu} + (1 - \frac{1}{\zeta} \frac{p^\mu p^\nu}{p^2}) \right) \exp(-ip(x - y)). \] (3.93)

In the following we will give a derivation of this fundamental result based on the path integral formalism.
3.2.3 Path Integral Quantization and the Faddeev-Popov Method

The starting point is to posit that the path integral of an Abelian vector field $A^\mu$ in the presence of a source $J^\mu$ is given by analogy with the scalar field by the functional integral (we set $\hbar = c = 1$)

$$Z[J] = \int \prod_\mu \mathcal{D}A_\mu \exp iS[A]$$

$$= \int \prod_\mu \mathcal{D}A_\mu \exp \left(-\frac{i}{4} \int d^4xF_{\mu\nu}F^{\mu\nu} - i \int d^4xJ_\mu A_\mu \right).$$

This is the generating functional of all correlation functions of the field $A^\mu(x)$. This is clear from the result

$$<A^{\mu_1}(x_1) A^{\mu_2}(x_2)> \equiv \frac{\int \prod_\mu \mathcal{D}A_\mu A^{\mu_1}(x_1) A^{\mu_2}(x_2) \exp iS_0[A]}{\int \prod_\mu \mathcal{D}A_\mu \exp iS_0[A]}$$

$$= \left(\frac{i^n Z[J]}{Z[J] \delta J_{\mu_1}(x_1) \cdots \delta J_{\mu_n}(x_n)}\right)_{J=0}. \quad (3.95)$$

The Maxwell's action can be rewritten as

$$S_0[A] = -\frac{1}{4} \int d^4xF_{\mu\nu}F^{\mu\nu}$$

$$= \frac{1}{2} \int d^4x A_\nu (\partial_\mu \eta^{\nu\lambda} - \partial^{\nu} \partial_\lambda) A_\lambda. \quad (3.96)$$

We Fourier transform $A_\mu(x)$ as

$$A_\mu(x) = \int \frac{d^4k}{(2\pi)^4} \hat{A}_\mu(k)e^{ikx}. \quad (3.97)$$

Then

$$S_0[A] = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \hat{A}_\nu(k)(-k^2\eta^{\nu\lambda} + k^{\nu}k^{\lambda}) \hat{A}_\lambda(-k). \quad (3.98)$$

We observe that the action is 0 for any configuration of the form $\hat{A}_\mu(k) = k_\mu f(k)$. Thus we conclude that the so-called pure gauge configurations given by $A_\mu(x) = \partial_\mu \Lambda(x)$ are zero modes of the Laplacian which means in particular that the Laplacian can not be inverted. More importantly this means that in the path integral $Z[J]$ these zero modes (which are equivalent to $A_\mu = 0$) are not damped and thus the path integral is divergent. This happens for any other configuration $A_\mu$. Indeed all gauge equivalent configurations $A^A_\mu = A_\mu + \partial_\mu \Lambda$ have the same probability amplitude and as a consequence the sum of their contributions to the path integral will be proportional to the divergent integral over the Abelian $U(1)$ gauge group which is here the integral over $\Lambda$. The problem lies therefore in gauge invariance which must be fixed in the path integral. This entails the
selection of a single gauge configuration from each gauge orbit \( A^\Lambda_\mu = A_\mu + \partial_\mu \Lambda \) as a representative and using it to compute the contribution of the orbit to the path integral.

In path integral quantization gauge fixing is done in an elegant and efficient way via the method of Faddeev and Popov. Let us say that we want to gauge fix by imposing the Lorentz condition \( G(A) = \partial_\mu A^\mu - \omega = 0 \). Clearly \( G(A^\Lambda) = \partial_\mu A^\mu - \omega + \partial_\mu \partial^\mu \Lambda \) and thus

\[
\int \mathcal{D}\Lambda \delta(G(A^\Lambda)) = \int \mathcal{D}\Lambda \delta(\partial_\mu A^\mu - \omega + \partial_\mu \partial^\mu \Lambda). \tag{3.99}
\]

By performing the change of variables \( \Lambda \rightarrow \Lambda' = \partial_\mu \partial^\mu \Lambda \) and using the fact that \( \mathcal{D}\Lambda' = |(\partial\Lambda'/\partial\Lambda)| \mathcal{D}\Lambda = \det(\partial_\mu \partial^\mu) \mathcal{D}\Lambda \) we get

\[
\int \mathcal{D}\Lambda \delta(G(A^\Lambda)) = \int \frac{\mathcal{D}\Lambda'}{\det(\partial_\mu \partial^\mu)} \delta(\partial_\mu A^\mu - \omega + \Lambda') = \frac{1}{\det(\partial_\mu \partial^\mu)}. \tag{3.100}
\]

This can be put in the form

\[
\int \mathcal{D}\Lambda \delta(G(A^\Lambda)) \det\left(\frac{\delta G(A^\Lambda)}{\delta \Lambda}\right) = 1. \tag{3.101}
\]

This is the generalization of

\[
\int \prod_
u da_\nu \delta^{(n)}(\bar{g}(\bar{a})) \det\left(\frac{\partial \bar{g}_i}{\partial a_j}\right) = 1. \tag{3.102}
\]

We insert 1 in the form (3.101) in the path integral as follows

\[
Z[J] = \int \prod_\mu \mathcal{D}A_\mu \int \mathcal{D}\Lambda \delta(G(A^\Lambda)) \det\left(\frac{\delta G(A^\Lambda)}{\delta \Lambda}\right) \exp iS[A] = \det(\partial_\mu \partial^\mu) \int \mathcal{D}\Lambda \int \prod_\mu \mathcal{D}A_\mu \delta(G(A^\Lambda)) \exp iS[A] = \det(\partial_\mu \partial^\mu) \int \mathcal{D}\Lambda \int \prod_\mu \mathcal{D}A^\Lambda_\mu \delta(G(A^\Lambda)) \exp iS[A^\Lambda]. \tag{3.103}
\]

Now we shift the integration variable as \( A^\Lambda_\mu \rightarrow A_\mu \). We observe immediately that the integral over the \( U(1) \) gauge group decouples, viz

\[
Z[J] = \det(\partial_\mu \partial^\mu) \left( \int \mathcal{D}\Lambda \right) \int \prod_\mu \mathcal{D}A_\mu \delta(G(A)) \exp iS[A] = \det(\partial_\mu \partial^\mu) \left( \int \mathcal{D}\Lambda \right) \int \prod_\mu \mathcal{D}A_\mu \delta(\partial_\mu A^\mu - \omega) \exp iS[A]. \tag{3.104}
\]
Next we want to set $\omega = 0$. We do this in a smooth way by integrating both sides of the above equation against a Gaussian weighting function centered around $\omega = 0$, viz
\[
\int \mathcal{D}\omega \exp\left(-i \int d^4x \frac{\omega^2}{2\xi}\right) Z[J] = \text{det}(\partial_\mu \partial_\nu) \left( \int \mathcal{D}\Lambda \int \mathcal{D}\omega \exp\left(-i \int d^4x \frac{\omega^2}{2\xi}\right) \delta\left(\partial_\mu A_\nu - \omega\right) \exp iS[A] \right) = \text{det}(\partial_\mu \partial_\nu) \left( \int \mathcal{D}\Lambda \int \mathcal{D}A_\mu \exp\left(-i \int d^4x \frac{(\partial_\mu A_\nu)^2}{2\xi}\right) \exp iS[A] \right).
\]

Hence
\[
Z[J] = \mathcal{N} \int \mathcal{D}A_\mu \exp\left(-i \int d^4x \frac{(\partial_\mu A_\nu)^2}{2\xi}\right) \exp iS[A] = \mathcal{N} \int \mathcal{D}A_\mu \exp\left(-i \int d^4x \frac{(\partial_\mu A_\nu)^2}{2\xi} - \frac{i}{4} \int d^4x F_\mu\nu F^{\mu\nu} - i \int d^4x J_\mu A_\mu\right).
\]

Therefore the end result is the addition of a term proportional to $(\partial_\mu A_\nu)^2$ to the action which fixes gauge invariance to a sufficient degree.

### 3.2.4 The Photon Propagator

The above path integral can also be put in the form
\[
Z[J] = \mathcal{N} \int \mathcal{D}A_\mu \exp\left(\frac{i}{2} \int d^4x A_\mu \left(\partial_\mu \partial_\nu \eta^{\nu\lambda} + \left(\frac{1}{\xi} - 1\right)\partial_\nu \partial_\lambda\right) A_\lambda - i \int d^4x J_\mu A_\mu\right).
\]

We use the result
\[
\int \prod_{i=1}^{n} dz_i e^{-z_i M_{ij} z_j - z_j} = e^{\frac{i}{2} \beta_i M_{ij}^{-1} \beta_j} \int \prod_{i=1}^{n} dz_i e^{-\left(z_i + \frac{1}{\xi} \beta_j M_{ij}^{-1}\right) z_j + \frac{1}{2} \beta_j M_{ij}^{-1} \beta_i}
\]
\[
= e^{\frac{i}{2} \beta_i M_{ij}^{-1} \beta_j} \int \prod_{i=1}^{n} dy_i e^{-\gamma_i M_{ij} y_j}
\]
\[
= e^{\frac{i}{2} \beta_i M_{ij}^{-1} \beta_j} \int \prod_{i=1}^{n} dx_j e^{-x_m x_j}
\]
\[
= e^{\frac{i}{2} \beta_i M_{ij}^{-1} \beta_j} \prod_{i=1}^{n} \sqrt{\frac{\pi}{m_i}}
\]
\[
= e^{\frac{i}{2} \beta_i M_{ij}^{-1} \beta_j} \pi^{\frac{n}{2}} (\det M)^{-\frac{1}{2}}.
\]

By comparison we have
\[
M_{ij} \rightarrow -\frac{i}{2} \left(\partial_\mu \partial_\nu \eta^{\nu\lambda} + \left(\frac{1}{\xi} - 1\right)\partial_\nu \partial_\lambda\right) \delta^4(x - y), \quad j_i \rightarrow iJ_\mu.
\]
We define
\[ M^{\nu\lambda}(x, y) = -\frac{i}{2} \left( \partial_{\mu} \partial^{\mu} \eta^{\nu\lambda} + \frac{1}{\xi} - 1 \right) \partial^{\nu} \partial^{\lambda} \right) \delta^4(x - y) \]
\[ = \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \left( k^2 \eta^{\nu\lambda} + \frac{1}{\xi} - 1 \right) k^\nu k^\lambda e^{ik(x-y)}. \tag{3.110} \]

Hence our path integral is actually given by
\[ Z[J] = N \pi^{\frac{n}{2}} (\det M)^{-\frac{1}{2}} \exp \left( -\frac{1}{4} \int d^4x \int d^4y J^\mu(x) M^{-1}_{\mu\nu}(x, y) J^\nu(y) \right) \]
\[ = N' \exp \left( -\frac{1}{4} \int d^4x \int d^4y J^\mu(x) M^{-1}_{\mu\nu}(x, y) J^\nu(y) \right). \tag{3.111} \]

The inverse of the Laplacian is defined by
\[ \int d^4y M^{\nu\lambda}(x, y) M^{-1}_{\mu\nu}(y, z) = \eta^{\nu\lambda}(x - y). \tag{3.112} \]

For example the 2-point function is given by
\[ <A_\mu(x)A_\nu(y)> = \left( \frac{i^2}{Z[J]} \frac{\delta^2 Z[J]}{\delta J^\mu(x) \delta J^\nu(y)} \right)_{J=0} \]
\[ = \frac{1}{2} M^{-1}_{\mu\nu}(x, y). \tag{3.113} \]

It is not difficult to check that the inverse is given by
\[ M^{-1}_{\mu\nu}(x, y) = -2i \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + i\epsilon} (\eta_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2}) e^{ik(x-y)}. \tag{3.114} \]

Hence the propagator is
\[ <A_\mu(x)A_\nu(y)> = \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2 + i\epsilon} (\eta_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2}) e^{ik(x-y)}. \tag{3.115} \]

The most important gauges we will make use of are the Feynman gauge \( \xi = 1 \) and the Landau gauge \( \xi = 0 \).

### 3.3 Gauge Interactions

#### 3.3.1 Spinor and Scalar Electrodynamics: Minimal Coupling

The actions of a free Dirac field and a free Abelian vector field in the presence of sources are given by (with \( \hbar = c = 1 \))
\[ S[\psi, \bar{\psi}] = \int d^4x \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi + \int d^4x (\bar{\psi} \eta + \bar{\eta} \psi). \tag{3.116} \]
\[ S[A] = \frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu} - \int d^4x J_\mu A^\mu. \] (3.117)

The action \( S[A] \) gives Maxwell’s equations with a vector current source equal to the external vector current \( J^\mu \). As we have already discussed the Maxwell’s action \((J^\mu = 0)\) is invariant under the gauge symmetry transformations

\[ A_\mu \longrightarrow A_\mu^\Lambda = A_\mu + \partial_\mu \Lambda. \] (3.118)

The action \( S[A] \) is also invariant under these gauge transformations provided the vector current \( J^\mu \) is conserved, viz \( \partial_\mu J^\mu = 0 \).

The action describing the interaction of a photon which is described by the Abelian vector field \( A^\mu \) and an electron described by the Dirac field \( \psi \) must be given by

\[ S[\psi, \bar{\psi}, A] = S[\psi, \bar{\psi}] + S[A] - \int d^4x j_\mu A^\mu. \] (3.119)

The interaction term \( -j_\mu A^\mu \) is dictated by the requirement that this action must also give Maxwell’s equations with a vector current source equal now to the sum of the external vector current \( J^\mu \) and the internal vector current \( j^\mu \). The internal vector current \( j^\mu \) must clearly depend on the spinor fields \( \psi \) and \( \bar{\psi} \) and furthermore it must be conserved.

In order to ensure that \( j^\mu \) is conserved we will identify it with the Noether’s current associated with the local symmetry transformations

\[ \psi \longrightarrow \psi^\Lambda = \exp(-ie\Lambda) \psi, \quad \bar{\psi} \longrightarrow \bar{\psi}^\Lambda = \bar{\psi} \exp(i e\Lambda). \] (3.120)

Indeed under these local transformations the Dirac action transforms as

\[ S[\psi, \bar{\psi}] \longrightarrow S[\psi^\Lambda, \bar{\psi}^\Lambda] = S[\psi, \bar{\psi}] - e \int d^4x \Lambda \partial_\mu (\bar{\psi} \gamma^\mu \psi). \] (3.121)

The internal current \( j^\mu \) will be identified with \( e \bar{\psi} \gamma^\mu \psi \), viz

\[ j^\mu = e \bar{\psi} \gamma^\mu \psi. \] (3.122)

This current is clearly invariant under the local transformations (3.120). By performing the local transformations (3.118) and (3.120) simultaneously, i.e. by considering the transformations (3.120) a part of gauge symmetry, we obtain the invariance of the action \( S[\psi, \bar{\psi}, A] \). The action remains invariant under the combined transformations (3.118) and (3.120) if we also include a conserved external vector current source \( J^\mu \) and Grassmann spinor sources \( \eta \) and \( \bar{\eta} \) which transform under gauge transformations as the dynamical Dirac spinors, viz \( \eta \longrightarrow \eta^\Lambda = \exp(-i e\Lambda) \eta \) and \( \bar{\eta} \longrightarrow \bar{\eta}^\Lambda = \bar{\eta} \exp(i e\Lambda) \). We write this result as (with \( S_\eta, \bar{\eta}, J[\psi, \bar{\psi}, A] \equiv S[\psi, \bar{\psi}, A] \))

\[ S_\eta, \bar{\eta}, J[\psi, \bar{\psi}, A] \longrightarrow S_\eta^\Lambda, \bar{\eta}^\Lambda, J[\psi^\Lambda, \bar{\psi}^\Lambda, A^\Lambda] = S_\eta, \bar{\eta}, J[\psi, \bar{\psi}, A]. \] (3.123)
The action $S[\psi, \bar{\psi}, A]$ with the corresponding path integral define quantum spinor electrodynamics which is the simplest and most important gauge interaction. The action $S[\psi, \bar{\psi}, A]$ can also be put in the form

$$S[\psi, \bar{\psi}, A] = \int d^4 x \bar{\psi} \left( i \gamma^\mu \nabla_\mu - m \right) \psi - \frac{1}{4} \int d^4 x F_{\mu\nu} F^{\mu\nu} + \int d^4 x (\bar{\psi} \eta + \bar{\eta} \psi) - \int d^4 x J_\mu A^\mu. \quad (3.124)$$

The derivative operator $\nabla_\mu$ which is called the covariant derivative is given by

$$\nabla_\mu = \partial_\mu + ie A_\mu. \quad (3.125)$$

The action $S[\psi, \bar{\psi}, A]$ could have been obtained from the free action $S[\psi, \bar{\psi}] + S[A]$ by making the simple replacement $\partial_\mu \rightarrow \nabla_\mu$ which is known as the principle of minimal coupling. In flat Minkowski spacetime this prescription always works and it allows us to obtain the most minimal consistent interaction starting from a free theory.

As another example consider complex quartic scalar field given by the action

$$S[\phi, \phi^+] = \int d^4 x \left( \partial_\mu \phi^+ \partial^\mu \phi - m^2 \phi^+ \phi - \frac{g}{4} (\phi^+ \phi)^2 \right). \quad (3.126)$$

By applying the principle of minimal coupling we replace the ordinary $\partial_\mu$ by the covariant derivative $\nabla_\mu = \partial_\mu + ie A_\mu$ and then add the Maxwell’s action. We get immediately the gauge invariant action

$$S[\phi, \phi^+, A] = \int d^4 x \left( \nabla_\mu \phi^+ \nabla^\mu \phi - m^2 \phi^+ \phi - \frac{g}{4} (\phi^+ \phi)^2 \right) - \frac{1}{4} \int d^4 x F_{\mu\nu} F^{\mu\nu}. \quad (3.127)$$

This is indeed invariant under the local gauge symmetry transformations acting on $A^\mu$, $\phi$ and $\phi^+$ as

$$A_\mu \rightarrow A'^\mu_\mu = A_\mu + \partial_\mu \Lambda, \quad \phi \rightarrow \exp(-ie\Lambda) \phi, \quad \phi^+ \rightarrow \phi^+ \exp(ie\Lambda). \quad (3.128)$$

It is not difficult to add vector and scalar sources to the action (3.126) without spoiling gauge invariance. The action (3.126) with the corresponding path integral define quantum scalar electrodynamics which describes the interaction of the photon $A^\mu$ with a charged scalar particle $\phi$ whose electric charge is $q = -e$.

### 3.3.2 The Geometry of $U(1)$ Gauge Invariance

The set of all gauge transformations which leave invariant the actions of spinor and scalar electrodynamics form a group called $U(1)$ and as a consequence spinor and scalar electrodynamics are said to be invariant under local $U(1)$ gauge symmetry. The group $U(1)$ is the group of $1 \times 1$ unitary matrices given by

$$U(1) = \{ g = \exp(-ie\Lambda), \forall \Lambda \}. \quad (3.129)$$
In order to be able to generalize the local $U(1)$ gauge symmetry to local gauge symmetries based on other groups we will exhibit in this section the geometrical content of the gauge invariance of spinor electrodynamics. The starting point is the free Dirac action given by

$$S = \int d^4x \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi. \quad (3.130)$$

This is invariant under the global transformations

$$\psi \rightarrow e^{-ie\Lambda} \psi, \quad \bar{\psi} \rightarrow \bar{\psi} e^{ie\Lambda}. \quad (3.131)$$

We demand next that the theory must be invariant under the local transformations obtained by allowing $\Lambda$ to be a function of $x$ in the above equations, viz

$$\psi \rightarrow \psi^g = g(x) \psi, \quad \bar{\psi} \rightarrow \bar{\psi}^g = \bar{\psi}g^+(x). \quad (3.132)$$

The fermion mass term is trivially still invariant under these local $U(1)$ gauge transformations, i.e.

$$\bar{\psi} \psi \rightarrow \bar{\psi}^g \psi^g = \bar{\psi} \psi. \quad (3.133)$$

The kinetic term is not so easy. The difficulty clearly lies with the derivative of the field which transforms under the local $U(1)$ gauge transformations in a complicated way. To appreciate more this difficulty let us consider the derivative of the field $\psi$ in the direction defined by the vector $n^\mu$ which is given by

$$n^\mu \partial_\mu \psi = \lim_{\epsilon \rightarrow 0} \frac{[\psi(x + \epsilon n) - \psi(x)]}{\epsilon}. \quad (3.134)$$

The two fields $\psi(x + \epsilon n)$ and $\psi(x)$ transform under the local $U(1)$ symmetry with different phases given by $g(x + \epsilon n)$ and $g(x)$ respectively. The point is that the fields $\psi(x + \epsilon n)$ and $\psi(x)$ since they are evaluated at different spacetime points $x + \epsilon n$ and $x$ they transform independently under the local $U(1)$ symmetry. As a consequence the derivative $n^\mu \partial_\mu \psi$ has no intrinsic geometrical meaning since it involves the comparison of fields at different spacetime points which transform independently of each other under $U(1)$.

In order to be able to compare fields $\psi(y)$ and $\psi(x)$ at different spacetime points $y$ and $x$ we need to introduce a new object which connects the two points $y$ and $x$ and which allows a meaningful comparison between $\psi(y)$ and $\psi(x)$. We introduce a comparator field $U(y, x)$ which connects the points $y$ and $x$ along a particular path with the properties:

- The comparator field $U(y, x)$ must be an element of the gauge group $U(1)$ and thus $U(y, x)$ is a pure phase, viz
  $$U(y, x) = \exp(-ie\phi(y, x)) \in U(1). \quad (3.135)$$

- Clearly we must have
  $$U(x, x) = 1 \Leftrightarrow \phi(x, x) = 0. \quad (3.136)$$
Under the $U(1)$ gauge transformations $\psi(x) \rightarrow \psi^g(x) = g(x)\psi(x)$ and $\psi(y) \rightarrow \psi^g(y) = g(y)\psi(y)$ the comparator field transforms as

$$U(y, x) \rightarrow U^g(y, x) = g(y)U(y, x)g^+(x).$$  \hspace{1cm} (3.137)

We impose the restriction

$$U(y, x) + = U(x, y).$$ \hspace{1cm} (3.138)

Thus $U(y, x)\psi(x)$ transforms under the $U(1)$ gauge group with the same group element as the field $\psi(y)$. This means in particular that the comparison between $U(y, x)\psi(x)$ and $\psi(y)$ is meaningful. We are led therefore to define a new derivative of the field $\psi$ in the direction defined by the vector $n^\mu$ by

$$n^\mu \nabla_\mu \psi = \lim_{\epsilon \rightarrow 0} \frac{[\psi(x + \epsilon n) - U(x + \epsilon n, x)\psi(x)]}{\epsilon},$$  \hspace{1cm} (3.140)

This is known as the covariant derivative of $\psi$ in the direction $n^\mu$.

The second property $U(x, x) = 1$ allows us to conclude that if the point $y$ is infinitesimally close to the point $x$ then we can expand $U(y, x)$ around 1. We can write for $y = x + \epsilon n$ the expansion

$$U(x + \epsilon n, x) = 1 - i\epsilon n_\mu A^\mu(x) + O(\epsilon^2).$$ \hspace{1cm} (3.141)

The coefficient of the displacement vector $y_\mu - x_\mu = \epsilon n_\mu$ is a new vector field $A^\mu$ which is precisely, as we will see shortly, the electromagnetic vector potential. The coupling $e$ will, on the other hand, play the role of the electric charge. We compute immediately

$$\nabla_\mu \psi = (\partial_\mu + i e A^\mu)\psi.$$ \hspace{1cm} (3.142)

Thus $\nabla_\mu$ is indeed the covariant derivative introduced in the previous section.

By using the language of differential geometry we say that the vector field $A_\mu$ is a connection on a $U(1)$ fiber bundle over spacetime which defines the parallel transport of the field $\psi$ from $x$ to $y$. The parallel transported field $\psi_\parallel$ is defined by

$$\psi_\parallel(y) = U(y, x)\psi(x).$$ \hspace{1cm} (3.143)

The third property with a comparator $U(y, x)$ with $y$ infinitesimally close to $x$, for example $y = x + \epsilon n$, reads explicitly

$$1 - i\epsilon n_\mu A_\mu(x) \rightarrow 1 - i\epsilon n_\mu A^\mu_\parallel(x) = g(y)\left(1 - i\epsilon n_\mu A_\mu(x)\right)g^+(x).$$ \hspace{1cm} (3.144)
Equivalently we have
\[ A^g_\mu = gA_\mu g^+ + \frac{i}{e} \partial_\mu g g^+ \Leftrightarrow A^g_\mu = A_\mu + \partial_\mu \Lambda. \] (3.145)

Again we find the gauge field transformation law considered in the previous section. For completeness we find the transformation law of the covariant derivative of the field \( \psi \).

We have
\[
\nabla_\mu \psi = (\partial_\mu + ieA_\mu)\psi \longrightarrow (\nabla_\mu \psi)^g = (\partial_\mu + ieA^g_\mu)\psi^g = (\partial_\mu + ieA_\mu + ie\partial_\mu \Lambda)(e^{-ie\Lambda}\psi) = e^{-ie\Lambda}(\partial_\mu + ieA_\mu)\psi = g(x)\nabla_\mu \psi. \] (3.146)

Thus the covariant derivative of the field transforms exactly in the same way as the field itself. This means in particular that the combination \( \bar{\psi}i\gamma^\mu \nabla_\mu \psi \) is gauge invariant. In summary given the free Dirac action we can obtain a gauge invariant Dirac action by the simple substitution \( \partial_\mu \longrightarrow \nabla_\mu \). This is the principle of minimal coupling discussed in the previous section. The gauge invariant Dirac action is
\[
S = \int d^4x \bar{\psi}(i\gamma^\mu \nabla_\mu - m)\psi. \] (3.147)

We need finally to construct a gauge invariant action which provides a kinetic term for the vector field \( A^\mu \). This can be done by integrating the comparator \( U(y,x) \) along a closed loop. For \( y = x + e n \) we write \( U(y,x) \) up to the order \( e^2 \) as
\[
U(y, x) = 1 - iene_\mu A^\mu + ie^2 X + O(e^3). \] (3.148)

The fourth fundamental property of \( U(y, x) \) restricts the comparator so that \( U(y, x)^+ = U(x, y) \). This leads immediately to the solution \( X = -ene_\mu e_\nu \partial^\nu A^\mu /2 \). Thus
\[
U(y, x) = 1 - iene_\mu A^\mu - \frac{ie^2}{2} e_\mu e_\nu \partial^\nu A^\mu + O(e^3) = 1 - iene_\mu A^\mu (x + \frac{e}{2} n) + O(e^3) = \exp(-iene_\mu A^\mu (x + \frac{e}{2} n)). \] (3.149)

We consider now the group element \( U(x) \) given by the product of the four comparators associated with the four sides of a small square in the \((1, 2)\)–plane. This is given by
\[
U(x) = \text{tr} U(x, x + e\hat{1}) U(x + e\hat{1}, x + e\hat{1} + e\hat{2}) U(x + e\hat{1} + e\hat{2}, x + e\hat{2}) U(x + e\hat{2}, x). \] (3.150)

This is called the Wilson loop associated with the square in question. The trace \( \text{tr} \) is of course trivial for a \( U(1) \) gauge group. The Wilson loop \( U(x) \) is locally invariant under the
gauge group $U(1)$, i.e. under $U(1)$ gauge transformations the Wilson loop $U(x)$ behaves as

$$U(x) \rightarrow U^g(x) = U(x).$$

(3.151)

The Wilson loop is the phase accumulated if we parallel transport the spinor field $\psi$ from the point $x$ around the square and back to the point $x$. This phase can be computed explicitly. Indeed we have

$$U(x) = \exp \left( i e \epsilon \left[ A^1(x + \frac{\epsilon}{2} \hat{1}) + A^2(x + \epsilon \hat{1} + \frac{\epsilon}{2} \hat{2}) - A^1(x + \frac{\epsilon}{2} \hat{1} + \epsilon \hat{2}) - A^2(x + \frac{\epsilon}{2} \hat{2}) \right] \right)$$

$$= \exp(-i e \epsilon^2 F_{12})$$

$$= 1 - i e \epsilon^2 F_{12} - \frac{e^2 \epsilon^4}{2} F_{12}^2 + ...$$

(3.152)

In the above equation $F_{12} = \partial_1 A_2 - \partial_2 A_1$. We conclude that the field strength tensor $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is locally gauge invariant under $U(1)$ transformations. This is precisely the electromagnetic field strength tensor considered in the previous section.

The field strength tensor $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ can also be obtained from the commutator of the two covariant derivatives $\nabla_\mu$ and $\nabla_\nu$ acting on the spinor field $\psi$. Indeed we have

$$[\nabla_\mu, \nabla_\nu] \psi = i e (\partial_\mu A_\nu - \partial_\nu A_\mu) \psi.$$  

(3.153)

Thus under $U(1)$ gauge transformations we have the behavior

$$[\nabla_\mu, \nabla_\nu] \psi \rightarrow g(x)[\nabla_\mu, \nabla_\nu] \psi.$$  

(3.154)

In other words $[\nabla_\mu, \nabla_\nu]$ is not a differential operator and furthermore it is locally invariant under $U(1)$ gauge transformations. This shows in a slightly different way that the field strength tensor $F_{\mu \nu}$ is the fundamental structure which is locally invariant under $U(1)$ gauge transformations. The field strength tensor $F_{\mu \nu}$ can be given by the expressions

$$F_{\mu \nu} = \frac{1}{ie} [\nabla_\mu, \nabla_\nu] = (\partial_\mu A_\nu - \partial_\nu A_\mu).$$  

(3.155)

In summary we can conclude that any function of the vector field $A^\mu$ which depends on the vector field only through the field strength tensor $F_{\mu \nu}$ will be locally invariant under $U(1)$ gauge transformations and thus can serve as an action functional. By appealing to the requirement of renormalizability the only renormalizable $U(1)$ gauge action in four dimensions (which also preserves $P$ and $T$ symmetries) is Maxwell’s action which is quadratic in $F_{\mu \nu}$ and also quadratic in $A^\mu$. We get then the pure gauge action

$$S = -\frac{1}{4} \int d^4x F_{\mu \nu} F^{\mu \nu}.$$  

(3.156)

The total action of spinor electrodynamics is therefore given by

$$S = \int d^4x \bar{\psi} (i \gamma^\mu \nabla_\mu - m) \psi - \frac{1}{4} \int d^4x F_{\mu \nu} F^{\mu \nu}.$$  

(3.157)
3.3.3 Generalization: $SU(N)$ Yang-Mills Theory

We can now immediately generalize the previous construction by replacing the Abelian gauge group $U(1)$ by a different gauge group $G$ which will generically be non-Abelian, i.e. the generators of the corresponding Lie algebra will not commute. In this chapter we will be interested in the gauge groups $G = SU(N)$. Naturally we will start with the first non-trivial non-Abelian gauge group $G = SU(2)$ which is the case considered originally by Yang and Mills.

The group $SU(2)$ is the group of $2 \times 2$ unitary matrices which have determinant equal 1. This is given by

$$SU(2) = \{ u_{ab}, a, b = 1, \ldots, 2 : u^+ u = 1, \det u = 1 \}. \quad (3.158)$$

The generators of $SU(2)$ are given by Pauli matrices given by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.159)$$

Thus any element of $SU(2)$ can be rewritten as

$$u = \exp(-ig\Lambda), \quad \Lambda = \sum_A A^A \sigma^A. \quad (3.160)$$

The group $SU(2)$ has therefore 3 gauge parameters $\Lambda^A$ in contrast with the group $U(1)$ which has only a single parameter. These 3 gauge parameters correspond to three orthogonal symmetry motions which do not commute with each other. Equivalently the generators of the Lie algebra $su(2)$ of $SU(2)$ (consisting of the Pauli matrices) do not commute which is the reason why we say that the group $SU(2)$ is non-Abelian. The Pauli matrices satisfy the commutation relations

$$\left[ \frac{\sigma^A}{2}, \frac{\sigma^B}{2} \right] = if_{ABC} \frac{\sigma^C}{2}, \quad f_{ABC} = \epsilon_{ABC}. \quad (3.161)$$

The $SU(2)$ group element $u$ will act on the Dirac spinor field $\psi$. Since $u$ is a $2 \times 2$ matrix the spinor $\psi$ must necessarily be a doublet with components $\psi^a$, $a = 1, 2$. The extra label $a$ will be called the color index. We write

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \quad (3.162)$$

We say that $\psi$ is in the fundamental representation of the group $SU(2)$. The action of an element $u \in SU(2)$ is given by

$$\psi^a \rightarrow (\psi^u)^a = \sum_B u^{ab}\psi^b. \quad (3.163)$$

We start from the free Dirac action

$$S = \int d^4x \sum_a \bar{\psi}^a(i\gamma^\mu \partial_\mu - m)\psi^a. \quad (3.164)$$
Clearly this is invariant under global $SU(2)$ transformations, i.e. transformations $g$ which do not depend on $x$. Local $SU(2)$ gauge transformations are obtained by letting $g$ depend on $x$. Under local $SU(2)$ gauge transformations the mass term remains invariant whereas the kinetic term transforms in a complicated fashion as in the case of local $U(1)$ gauge transformations. Hence as in the $U(1)$ case we appeal to the principle of minimal coupling and replace the ordinary derivative $n^\mu \partial_\mu$ with the covariant derivative $n^\mu \nabla_\mu$ which is defined by

$$n^\mu \nabla_\mu \psi = \lim_{\epsilon \to 0} \frac{[\psi(x + \epsilon n) - U(x + \epsilon n, x)\psi(x)]}{\epsilon}, \quad \epsilon \to 0. \quad (3.165)$$

Since the spinor field $\psi$ is a 2-component object the comparator $U(y, x)$ must be a $2 \times 2$ matrix which transforms under local $SU(2)$ gauge transformations as

$$U(y, x) \longrightarrow U(y, x) = u(y) U(y, x) u^+(x). \quad (3.166)$$

In fact $U(y, x)$ is an element of $SU(2)$. We must again impose the condition that $U(x, x) = 1$. Hence for an infinitesimal separation $y - x = \epsilon n$ we can expand $U(y, x)$ as

$$U(x + \epsilon n, x) = 1 - ig\epsilon n^\mu A_\mu(x) \frac{\sigma^A}{2} + O(\epsilon^2). \quad (3.167)$$

In other words we have three vector fields $A_\mu(x)$. They can be unified in a single object $A_{\mu}(x)$ defined by

$$A_{\mu}(x) = A^A_{\mu}(x) \frac{\sigma^A}{2}. \quad (3.168)$$

We will call $A_{\mu}(x)$ the $SU(2)$ gauge field whereas we will refer to $A^A_{\mu}(x)$ as the components of the $SU(2)$ gauge field. Since $A^{\mu}(x)$ is $2 \times 2$ matrix it will carry two color indices $a$ and $b$ in an obvious way. The components of the $SU(2)$ gauge field in the fundamental representation of $SU(2)$ are given by $A_{ab}^\mu(x)$. The color index is called the $SU(2)$ fundamental index whereas the index $A$ carried by the components $A^A_{\mu}(x)$ is called the $SU(2)$ adjoint index. In fact $A^A_{\mu}(x)$ are called the components of the $SU(2)$ gauge field in the adjoint representation of $SU(2)$.

First by inserting the expansion $U(x + \epsilon n, x) = 1 - ig\epsilon n^\mu A_\mu(x) \frac{\sigma^A}{2} + O(\epsilon^2)$ in the definition of the covariant derivative we obtain the result

$$\nabla_\mu \psi = (\partial_\mu + igA^A_\mu \frac{\sigma^A}{2}) \psi. \quad (3.169)$$

The spinor $U(x + \epsilon n, x) \psi(x)$ is the parallel transport of the spinor $\psi$ from the point $x$ to the point $x + \epsilon n$ and thus by construction it must transform under local $SU(2)$ gauge transformations in the same way as the spinor $\psi(x + \epsilon n)$. Hence under local $SU(2)$ gauge transformations the covariant derivative is indeed covariant, viz

$$\nabla_\mu \psi \longrightarrow u(x) \nabla_\mu \psi. \quad (3.170)$$
Next by inserting the expansion $U(x + \epsilon n, x) = 1 - ig\epsilon^\mu A^A_\mu(x)\sigma^A/2 + O(\epsilon^2)$ in the transformation law $U(y, x) \rightarrow U^g(y, x) = u(y)U(y, x)u^+(x)$ we obtain the transformation law

$$A_\mu \rightarrow A^u_\mu = u A_\mu u^+ + \frac{i}{g} \partial_\mu u. u^+.$$  \hspace{0.5cm} (3.171)

For infinitesimal $SU(2)$ transformations we have $u = 1 - ig\Lambda$. We get

$$A_\mu \rightarrow A^u_\mu = A_\mu + \partial_\mu \Lambda + ig[A_\mu, \Lambda].$$  \hspace{0.5cm} (3.172)

In terms of components we have

$$A^C A^C \rightarrow A^u_\mu A^C = A^C A^C + \partial_\mu \Lambda C + ig[A^C A^A A^B] \frac{\sigma^C}{2}.$$  \hspace{0.5cm} (3.173)

In other words

$$A^u_\mu A^C = A^C + \partial_\mu \Lambda C - g f_{ABC} A^A_A B.$$  \hspace{0.5cm} (3.174)

The spinor field transforms under infinitesimal $SU(2)$ transformations as

$$\psi \rightarrow \psi^u = \psi - ig\Lambda \psi.$$  \hspace{0.5cm} (3.175)

We can now check explicitly that the covariant derivative is indeed covariant, viz

$$\nabla_\mu \psi \rightarrow (\nabla_\mu \psi)^u = \nabla_\mu \psi - ig\Lambda \nabla_\mu \psi.$$  \hspace{0.5cm} (3.176)

By applying the principle of minimal coupling to the free Dirac action (3.164) we replace the ordinary derivative $\partial_\mu \psi^a$ by the covariant derivative $(\nabla_\mu)_{ab} \psi^b$. We obtain the interacting action

$$S = \int d^4 x \sum_{a,b} \bar{\psi}^a (i\gamma^n (\nabla_\mu)_{ab} - m\delta_{ab}) \psi^b.$$  \hspace{0.5cm} (3.177)

Clearly

$$(\nabla_\mu)_{ab} = \partial_\mu \delta_{ab} + ig A^A_\mu \frac{\sigma^A}{2}_{ab}.$$  \hspace{0.5cm} (3.178)

This action is by construction invariant under local $SU(2)$ gauge transformations. It provides obviously the free term for the Dirac field $\psi$ as well as the interaction term between the $SU(2)$ gauge field $A^\mu$ and the Dirac field $\psi$. There remains therefore to find an action which will provide the free term for the $SU(2)$ gauge field $A^\mu$. As opposed to the $U(1)$ case the action which will provide a free term for the $SU(2)$ gauge field $A^\mu$ will
also provide extra interaction terms (cubic and quartic) which involve only $A^\mu$. This is another manifestation of the non-Abelian structure of the $SU(2)$ gauge group and it is generic to all other non-Abelian groups.

By analogy with the $U(1)$ case a gauge invariant action which depends only on $A^\mu$ can only depend on $A^\mu$ through the field strength tensor $F_{\mu\nu}$. This in turn can be constructed from the commutator of two covariant derivatives. We have then

$$F_{\mu\nu} = \frac{1}{ig} \left[ \nabla_\mu, \nabla_\nu \right]$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]. \quad (3.179)$$

$F_{\mu\nu}$ is also a $2 \times 2$ matrix. In terms of components the above equation reads

$$F_{\mu\nu}^{C} \sigma^{C} = \frac{\partial_\mu A_\nu^{C} \sigma^{C}}{2} - \frac{\partial_\nu A_\mu^{C} \sigma^{C}}{2} + ig[A_\mu^{A} \sigma^{A}, A_\nu^{B} \sigma^{B}]{\sigma^{C}}{2}$$

$$= \left( \partial_\mu A_\nu^{C} - \partial_\nu A_\mu^{C} + igA_\mu^{A}A_\nu^{B}i f_{ABC} \right){\sigma^{C}}{2}. \quad (3.180)$$

Equivalently

$$F_{\mu\nu}^{C} = \partial_\mu A_\nu^{C} - \partial_\nu A_\mu^{C} - gf_{ABC}A_\mu^{A}A_\nu^{B}. \quad (3.181)$$

The last term in the above three formulas is of course absent in the case of $U(1)$ gauge theory. This is the term that will lead to novel cubic and quartic interaction vertices which involve only the gauge field $A^\mu$. We remark also that although $F_{\mu\nu}$ is the commutator of two covariant derivatives it is not a differential operator. Since $\nabla_\mu \psi$ transforms as $\nabla_\mu \psi \rightarrow u \nabla_\mu \psi$ we conclude that $\nabla_\mu \nabla_\nu \psi \rightarrow u \nabla_\mu \nabla_\nu \psi$ and hence

$$F_{\mu\nu} \psi \rightarrow u F_{\mu\nu} \psi. \quad (3.182)$$

This means in particular that

$$F_{\mu\nu} \rightarrow F_{\mu\nu}^u = u F_{\mu\nu} u^+ . \quad (3.183)$$

This can be verified explicitly by using the finite and infinitesimal transformation laws $A_\mu \rightarrow u A_\mu u^+ + i \partial_\mu u u^+ / g$ and $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda + ig[A_\mu, \Lambda]^4$. The infinitesimal form of the above transformation law is

$$F_{\mu\nu} \rightarrow F_{\mu\nu}^u = F_{\mu\nu} + ig[F_{\mu\nu}, \Lambda]. \quad (3.184)$$

In terms of components this reads

$$F_{\mu\nu}^{C} \rightarrow F_{\mu\nu}^{uC} = F_{\mu\nu}^{C} - gf_{ABC}F_{\mu\nu}^{A}A^{B}. \quad (3.185)$$

Exercise: Verify this.
Although the field strength tensor $F_{\mu\nu}$ is not gauge invariant its gauge transformation $F_{\mu\nu} \to u F_{\mu\nu} u^+$ is very simple. Any function of $F_{\mu\nu}$ will therefore transform in the same way as $F_{\mu\nu}$ and as a consequence its trace is gauge invariant under local $SU(2)$ transformations. For example $\text{tr} F_{\mu\nu} F^{\mu\nu}$ is clearly gauge invariant. By appealing again to the requirement of renormalizability the only renormalizable $SU(2)$ gauge action in four dimensions (which also preserves $P$ and $T$ symmetries) must be quadratic in $F_{\mu\nu}$. The only candidate is $\text{tr} F_{\mu\nu} F^{\mu\nu}$. We get then the pure gauge action

$$S = -\frac{1}{2} \int d^4 x \text{tr} F_{\mu\nu} F^{\mu\nu}. \quad (3.186)$$

We note that Pauli matrices satisfy

$$\text{tr} \frac{\sigma^A \sigma^B}{2} = \frac{1}{2} \delta^{AB}. \quad (3.187)$$

Thus the above pure action becomes

$$S = -\frac{1}{4} \int d^4 x F^C_{\mu\nu} F^{\mu\nu C}. \quad (3.188)$$

This action provides as promised the free term for the $SU(2)$ gauge field $A^\mu$ but also it will provide extra cubic and quartic interaction vertices for the gauge field $A^\mu$. In other words this action is not free in contrast with the $U(1)$ case. This interacting pure gauge theory is in fact highly non-trivial and strictly speaking this is what we should call Yang-Mills theory.

The total action is the sum of the gauge invariant Dirac action and the Yang-Mills action. This is given by

$$S = \int d^4 x \sum_{a,b} \bar{\psi}^a (i \gamma^\mu (\nabla_\mu)_{ab} - m \delta_{ab}) \psi^b - \frac{1}{4} \int d^4 x F^C_{\mu\nu} F^{\mu\nu C}. \quad (3.189)$$

The final step is to generalize further to $SU(N)$ gauge theory which is quite straightforward. The group $SU(N)$ is the group of $N \times N$ unitary matrices which have determinant equal 1. This is given by

$$SU(N) = \{ u_{ab}, a, b = 1, ..., N : u^\dagger u = 1, \det u = 1 \}. \quad (3.190)$$

The generators of $SU(N)$ can be given by the so-called Gell-Mann matrices $t^A = \lambda^A / 2$. They are traceless Hermitian matrices which generate the Lie algebra $su(N)$ of $SU(N)$. There are $N^2 - 1$ generators and hence $su(N)$ is an $(N^2 - 1)$-dimensional vector space. They satisfy the commutation relations

$$[t^A, t^B] = i f_{ABC} t^C. \quad (3.191)$$

Exercise: Derive the equations of motion which follow from this action.
The non-trivial coefficients $f_{ABC}$ are called the structure constants. The Gell-Mann generators $t_a$ can be chosen such that
\[ \text{tr} t_a t_b = \frac{1}{2} \delta_{ab}. \] (3.192)

They also satisfy
\[ t_a t_b = \frac{1}{2N} \delta_{ab} + \frac{1}{2} (d_{ABC} + if_{ABC}) t^C. \] (3.193)

The coefficients $d_{ABC}$ are symmetric in all indices. They can be given by
\[ d_{ABC} = \frac{2\text{tr} t_A \{ t_B, t_C \}}{} \] and they satisfy for example
\[ d_{ABC} d_{ABD} = \frac{N^2 - 4}{N} \delta_{CD}. \] (3.194)

For example the group SU(3) is generated by the 8 Gell-Mann $3 \times 3$ matrices $t^A = \lambda^A/2$ given by

\[
\begin{align*}
\lambda^1 & = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda^4 & = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},
\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},
\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\lambda^7 & = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},
\lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{align*}
\] (3.195)

The structure constants $f_{ABC}$ and the totally symmetric coefficients $d_{ABC}$ are given in the case of the group SU(3) by

\[ f_{123} = 1, f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = \frac{1}{2}, f_{458} = f_{678} = \frac{\sqrt{3}}{2} \] (3.196)

\[ d_{118} = d_{228} = d_{338} = -d_{888} = \frac{1}{\sqrt{3}} \]
\[ d_{448} = d_{558} = d_{668} = d_{778} = -\frac{1}{2\sqrt{3}} \]
\[ d_{146} = d_{157} = -d_{247} = d_{256} = d_{344} = d_{355} = -d_{366} = -d_{377} = \frac{1}{2}. \] (3.197)

Thus any finite element of the group SU(N) can be rewritten in terms of the Gell-Mann matrices $t^A = \lambda^A/2$ as
\[ u = \exp(-ig\Lambda), \Lambda = \sum_A \Lambda^A \lambda^A \] (3.198)
The spinor field $\psi$ will be an $N$–component object. The $SU(N)$ group element $u$ will act on the Dirac spinor field $\psi$ in the obvious way $\psi \rightarrow u\psi$. We say that the spinor field transforms in the fundamental representation of the $SU(N)$ gauge group. The covariant derivative will be defined by the same formula found in the $SU(2)$ case after making the replacement $\sigma^A \rightarrow \lambda^A$, viz $(\nabla_\mu)_{ab} = \partial_\mu \delta_{ab} + ig A^A_\mu (t^A)_{ab}$ (recall also that the range of the fundamental index $a$ changes from 2 to $N$). The covariant derivative will transform covariantly under the $SU(N)$ gauge group. There are clearly $N^2 - 1$ components $A^A_\mu$ of the $SU(N)$ gauge field, i.e. $A_\mu = A^A_\mu t^A$. The transformation laws of $A_\mu$ and $A^A_\mu$ remain unchanged (only remember that the structure constants differ for different gauge groups). The field strength tensor $F_{\mu\nu}$ will be given, as before, by the commutator of two covariant derivatives. All results concerning $F_{\mu\nu}$ will remain intact with minimal changes involving the replacements $\sigma^A \rightarrow \lambda^A$, $\epsilon_{ABC} \rightarrow f_{ABC}$ (recall also that the range of the adjoint index changes from 3 to $N^2 - 1$). The total action will therefore be given by the same formula (3.189). We will refer to this theory as quantum chromodynamics (QCD) with $SU(N)$ gauge group whereas we will refer to the pure gauge action as $SU(N)$ Yang-Mills theory.

3.4 Quantization and Renormalization at 1–Loop

3.4.1 The Faddeev-Popov Gauge Fixing and Ghost Fields

We will be interested first in the $SU(N)$ Yang-Mills theory given by the action

$$S[A] = -\frac{1}{2} \int d^4x \text{tr} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} \int d^4x F_{\mu\nu} A^{\mu\nu}.$$

(3.199)

The corresponding path integral is given by

$$Z = \int \prod_{\mu,A} DA^A_\mu \exp(iS[A])$$

(3.200)

This path integral is invariant under finite $SU(N)$ gauge transformations given explicitly by

$$A_\mu \rightarrow A^u_\mu = u A_\mu u^+ + \frac{i}{g} \partial_\mu u u^+.$$

(3.201)

Also it is invariant under infinitesimal $SU(N)$ gauge transformations given explicitly by

$$A_\mu \rightarrow A^A_\mu = A_\mu + \partial_\mu \Lambda + ig [A_\mu, \Lambda] \equiv A_\mu + [\nabla_\mu, \Lambda].$$

(3.202)
Equivalently

\[ A^C_\mu \longrightarrow A^{\Lambda C}_\mu = A^C_\mu + \partial_\mu \Lambda^C - g f_{ABC} A^A_\mu A^B \equiv A^C_\mu + [\nabla_\mu, \Lambda]^C. \]  

(3.203)

As in the case of electromagnetism we must fix the gauge before we can proceed any further since the path integral is ill defined as it stands. We want to gauge fix by imposing the Lorentz condition \( G(A) = \partial_\mu A^\mu - \omega = 0 \). Clearly under infinitesimal \( SU(N) \) gauge transformations we have \( G(A^\Lambda) = \partial_\mu A^\mu - \omega + \partial_\mu [\nabla_\mu, \Lambda] \) and thus

\[
\int \mathcal{D} \Lambda \delta(G(A^\Lambda)) \det \left( \frac{\delta G(A^\Lambda)}{\delta \Lambda} \right) = \int \mathcal{D} \Lambda \delta(\partial_\mu A^\mu - \omega + \partial_\mu [\nabla_\mu, \Lambda]) \det \partial_\mu [\nabla_\mu, ..] .
\]

(3.204)

By performing the change of variables \( \Lambda \longrightarrow \Lambda' = \partial_\mu [\nabla_\mu, \Lambda] \) and using the fact that \( \mathcal{D} \Lambda' = |(\partial \Lambda' / \partial \Lambda)| \mathcal{D} \Lambda = \det(\partial_\mu [\nabla_\mu, ..]) \mathcal{D} \Lambda \) we get

\[
\int \mathcal{D} \Lambda \delta(G(A^\Lambda)) \det \left( \frac{\delta G(A^\Lambda)}{\delta \Lambda} \right) = \int \frac{\mathcal{D} \Lambda'}{\det(\partial_\mu [\nabla_\mu, ..])} \delta(\partial_\mu A^\mu - \omega + \Lambda') \det(\partial_\mu [\nabla_\mu, ..]) = 1.
\]

(3.205)

This can also be put in the form (with \( u \) near the identity)

\[
\int \mathcal{D} u \delta(G(A^u)) \det \left( \frac{\delta G(A^u)}{\delta u} \right) = 1 , \quad \frac{\delta G(A^u)}{\delta u} = \partial_\mu [\nabla_\mu, ..].
\]

(3.206)

For a given gauge configuration \( A^\mu \) we define

\[
\Delta^{-1}(A) = \int \mathcal{D} u \delta(G(A^u)).
\]

(3.207)

Under a gauge transformation \( A_\mu \longrightarrow A^v_\mu = v A_\mu v^+ + i \partial_\mu (v v^+) / g \) we have \( A^u_\mu \longrightarrow A^{uv}_\mu = u v A_\mu (u v)^+ + i \partial_\mu (u v (u v)^+) / g \) and thus

\[
\Delta^{-1}(A^v) = \int \mathcal{D} u \delta(G(A^{uv})) = \int \mathcal{D} (u v) \delta(G(A^{uv})) = \int \mathcal{D} u ^v \delta(G(A^v)) = \Delta^{-1}(A). \]

(3.208)

In other words \( \Delta^{-1} \) is gauge invariant. Further we can write

\[
1 = \int \mathcal{D} u \delta(G(A^u)) \Delta(A).
\]

(3.209)

As we will see shortly we are interested in configurations \( A_\mu \) which lie on the surface \( G(A) = \partial^\mu A_\mu - \omega = 0 \). Thus only \( SU(N) \) gauge transformations \( u \) which are near the identity are relevant in the above integral. Hence we conclude that (with \( u \) near the identity)

\[
\Delta(A) = \det \left( \frac{\delta G(A^u)}{\delta u} \right).
\]

(3.210)
The determinant \( \det(\delta G(A^u)/\delta u) \) is gauge invariant and as a consequence is independent of \( u \). The fact that this determinant is independent of \( u \) is also obvious from equation (3.206).

We insert 1 in the form (3.209) in the path integral as follows

\[
Z = \int \prod_{\mu,A} \mathcal{D}A^A_\mu \int \mathcal{D}u \delta(G(A^u)) \Delta(A) \exp iS[A]
\]

\[
= \int \mathcal{D}u \int \prod_{\mu,A} \mathcal{D}A^A_\mu \delta(G(A^u)) \Delta(A) \exp iS[A]
\]

\[
= \int \mathcal{D}u \int \prod_{\mu,A} \mathcal{D}A^A_\mu \delta(G(A^u)) \Delta(A^u) \exp iS[A^u]. \quad (3.211)
\]

Now we shift the integration variable as \( A^u_\mu \rightarrow A_\mu \). The integral over the \( SU(N) \) gauge group decouples and we end up with

\[
Z = (\int \mathcal{D}u) \int \prod_{\mu,A} \mathcal{D}A^A_\mu \delta(G(A)) \Delta(A) \exp iS[A]. \quad (3.212)
\]

Because of the delta function we are interested in knowing \( \Delta(A) \) only for configurations \( A^\mu \) which lie on the surface \( G(A) = 0 \). This means in particular that the gauge transformations \( u \) appearing in (3.209) must be close to the identity so that we do not go far from the surface \( G(A) = 0 \). As a consequence \( \Delta(A) \) can be equated with the determinant \( \det(\delta G(A^u)/\delta u) \), viz

\[
\Delta(A) = \det \left( \frac{\delta G(A^u)}{\delta u} \right) = \det \partial_\mu [\nabla^\mu, \ldots]. \quad (3.213)
\]

In contrast with the case of \( U(1) \) gauge theory, here the determinant \( \det(\delta G(A^u)/\delta u) \) actually depends on the \( SU(N) \) gauge field and hence it can not be taken out of the path integral. We have then the result

\[
Z = (\int \mathcal{D}u) \int \prod_{\mu,A} \mathcal{D}A^A_\mu \delta(\partial_\mu A^\mu - \omega) \det \partial_\mu [\nabla^\mu, \ldots] \exp iS[A]. \quad (3.214)
\]

Clearly \( \omega \) must be an \( N \times N \) matrix since \( A^\mu \) is an \( N \times N \) matrix. We want to set \( \omega = 0 \) by integrating both sides of the above equation against a Gaussian weighting function centered around \( \omega = 0 \), viz

\[
\int \mathcal{D}\omega \exp(-i \int d^4x \frac{\omega^2}{\xi}) Z = (\int \mathcal{D}u) \int \prod_{\mu,A} \mathcal{D}A^A_\mu \int \mathcal{D}\omega \exp(-i \int d^4x \frac{\omega^2}{\xi}) \delta(\partial_\mu A^\mu - \omega) \det \partial_\mu [\nabla^\mu, \ldots] \exp iS[A]
\]

\[
= (\int \mathcal{D}u) \int \prod_{\mu,A} \mathcal{D}A^A_\mu \exp(-i \int d^4x \frac{(\partial_\mu A^\mu)^2}{\xi}) \det \partial_\mu [\nabla^\mu, \ldots] \exp iS[A]. \quad (3.215)
\]
The path integral of $SU(N)$ Yang-Mills theory is therefore given by

$$Z = N \int \prod_{\mu, A} DA_\mu A \exp(-i \int d^4x \text{tr} \left( \frac{\partial_{\mu} A^\mu}{\xi} \right) \det \partial_{\mu} [\nabla^\mu, ..] \exp iS[A]. \quad (3.216)$$

Let us recall that for Grassmann variables we have the identity

$$\det M = \int \prod_i d\theta_i d\bar{\theta}_i e^{-\theta_i^+ M_{ij} \theta_j}. \quad (3.217)$$

Thus we can express the determinant $\det \partial_{\mu} [\nabla^\mu, ..]$ as a path integral over Grassmann fields $\bar{c}$ and $c$ as follows

$$\det \partial_{\mu} [\nabla^\mu, ..] = \int D\bar{c} Dc \exp(-i \int d^4x \text{tr} \bar{c} \partial_{\mu} [\nabla^\mu, c]). \quad (3.218)$$

The fields $\bar{c}$ and $c$ are clearly scalar under Lorentz transformations (their spin is 0) but they are anti-commuting Grassmann-valued fields and hence they cannot describe physical propagating particles (they simply have the wrong relation between spin and statistics). These fields are called Fadeev-Popov ghosts and they clearly carry two $SU(N)$ indices. More precisely since the covariant derivative is acting on them by commutator these fields must be $N \times N$ matrices and thus they can be rewritten as $c = c^A t^A$. We say that the ghost fields transform in the adjoint representation of the $SU(N)$ gauge group, i.e. as $c \to u c u^+$ and $\bar{c} \to u \bar{c} u^+$ which ensures global invariance. In terms of $c^A$ the determinant reads

$$\det \partial_{\mu} [\nabla^\mu, ..] = \int \prod_A D\bar{c}^A Dc^A \exp \left( i \int d^4x \bar{c}^A \left( -\partial_{\mu} \partial_{\mu}^A \delta^{AB} - g f_{ABC} \partial_{\mu}^A \partial_{\mu}^C \right) c^B \right). \quad (3.219)$$

The path integral of $SU(N)$ Yang-Mills theory becomes

$$Z = N \int \prod_{\mu, A} DA_\mu A \int \prod_A D\bar{c}^A Dc^A \exp(-i \int d^4x \text{tr} \left( \frac{\partial_{\mu} A^\mu}{\xi} \right) \exp(-i \int d^4x \text{tr} \bar{c} \partial_{\mu} [\nabla^\mu, c]) \exp iS[A]$$

$$= N \int \prod_{\mu, A} DA_\mu A \int \prod_A D\bar{c}^A Dc^A \exp iS_{FP}[A, c, \bar{c}]. \quad (3.220)$$

$$S_{FP}[A, c, \bar{c}] = S[A] - \int d^4x \text{tr} \left( \frac{\partial_{\mu} A^\mu}{\xi} \right) - \int d^4x \text{tr} \bar{c} \partial_{\mu} [\nabla^\mu, c]. \quad (3.221)$$

The second term is called the gauge fixing term whereas the third term is called the Fadeev-Popov ghost term. We add sources to obtain the path integral

$$Z[J, b, \bar{b}] = N \int \prod_{\mu, A} DA_\mu A \int \prod_A D\bar{c}^A Dc^A \exp \left( iS_{FP}[A, c, \bar{c}] - i \int d^4x J_\mu^A A^\mu A^A + i \int d^4x (bc + \bar{b} \bar{c}) \right). \quad (3.222)$$
In order to compute propagators we drop all interactions terms. We end up with the partition function
\[ Z[J, b, \bar{b}] = \mathcal{N} \int \prod_{\mu} DA_{\mu} \int \prod A \mathcal{D} \bar{A} \mathcal{D} c \exp \left( \frac{i}{2} \int d^4x A_{\mu} \left( \partial_\mu \partial_\nu \eta^{\nu\lambda} + \left( \frac{1}{\xi} - 1 \right) \partial_\nu \partial_\lambda \right) A_\lambda - i \int d^4x \bar{c} \partial_\mu \partial^\mu c \right). \]

(3.223)

The free SU(N) gauge part is \( N^2 - 1 \) copies of U(1) gauge theory. Thus without any further computation the SU(N) vector gauge field propagator is given by
\[ < A^A_\mu(x) A^B_\nu(y) > = \int \frac{d^4k}{(2\pi)^4} \frac{i\delta^{AB}}{k^2 + i\epsilon} (\eta_{\mu\nu} - \frac{1}{\xi} \frac{k_\mu k_\nu}{k^2}) e^{ik(x-y)}. \]

(3.224)

The propagator of the ghost field can be computed along the same lines used for the propagator of the Dirac field. We obtain \(^6\)
\[ < c^A(x) c^B(y) > = \int \frac{d^4k}{(2\pi)^4} \frac{i\delta^{AB}}{k^2 + i\epsilon} e^{ik(x-y)}. \]

(3.225)

3.4.2 Perturbative Renormalization and Feynman Rules

The QCD action with SU(N) gauge group is given by
\[ S_{QCD}[\psi, \bar{\psi}, A, c] = S_0[\psi, \bar{\psi}, A, c] + S_1[\psi, \bar{\psi}, A, c] \]

(3.226)

\[ S_0[\psi, \bar{\psi}, A, c] = \int d^4x \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - \frac{1}{4} \int d^4x (\partial_\mu A^A_\nu - \partial_\nu A^A_\mu)(\partial^\mu A^{A^\nu}_\nu - \partial^\nu A^{A^\mu}_\mu) - \frac{1}{2\xi} \int d^4x (\partial_\mu A^A_\nu)^2 - \int d^4x \bar{c} \partial_\mu \partial^\mu c. \]

(3.227)

\[ S_1[\psi, \bar{\psi}, A, c] = -gf_{ab} \int d^4x \sum_a \bar{\psi}^a \gamma^\mu \psi^b A^A_\mu + gf_{ABC} \int d^4x \partial_\mu A^C_\nu A^A_\mu A^B_\nu - \frac{g^2}{4} f_{ABC} f_{DEC} \int d^4x A^A_\mu A^B_\nu A^D_\rho A^E_\sigma - gf_{ABC} \int d^4x (\bar{c} A^A_\mu \partial_\mu A^C_\nu + \bar{c} A^A_\mu \partial^\mu A^C_\nu). \]

(3.228)

We introduce the renormalization constants \( Z_3, Z_2 \) and \( Z_2^c \) by introducing the renormalized fields \( A^\mu_R, \psi_R \) and \( c_R \) which are defined in terms of the bare fields \( A^\mu, \psi \) and \( c \) respectively by the equations
\[ A^\mu_R = \frac{A^\mu}{\sqrt{Z_3}}, \quad \psi_R = \frac{\psi}{\sqrt{Z_2}}, \quad c_R = \frac{c}{\sqrt{Z_2}}. \]

(3.229)

\(^6\)Exercise: Perform this calculation explicitly.
The renormalization constants $Z_3$, $Z_2$ and $Z_2^2$ can be expanded in terms of the counter terms $\delta_3$, $\delta_2$ and $\delta_2^2$ as

\[ Z_3 = 1 + \delta_3, \quad Z_2 = 1 + \delta_2, \quad Z_2^2 = 1 + \delta_2^2. \]  

(3.230)

Furthermore we relate the bare coupling constants $g$ and $m$ to the renormalized coupling $g_R$ and $m_R$ through the counter terms $\delta_1$ and $\delta_m$ by

\[ gZ_2\sqrt{Z_3} = g_R(1 + \delta_1), \quad Z_2m = m_R + \delta_m. \]  

(3.231)

Since we have also $AAA$, $AAAA$ and $ccA$ vertices we need more counter terms $\delta_1^A$, $\delta_1^B$ and $\delta_1^C$ which we define by

\[ gZ_3^2 = g_R(1 + \delta_1^A), \quad g^2Z_3^2 = g_R^2(1 + \delta_1^B), \quad gZ_3^3\sqrt{Z_3} = g_R(1 + \delta_1^C). \]  

(3.232)

We will also define a "renormalized gauge fixing parameter" $\xi_R$ by

\[ \frac{1}{\xi_R} = \frac{Z_3}{\xi}. \]  

(3.233)

As we will see shortly this is physically equivalent to imposing the gauge fixing condition on the renormalized gauge field $A^\mu_R$ instead of the bare gauge field $A^\mu$.

The action divides therefore as

\[ S = S_R + S_{\text{count \ term}}. \]  

(3.234)

The action $S_R$ is given by the same formula as $S$ with the replacement of all fields and coupling constants by the renormalized fields and renormalized coupling constants and also replacing $\xi$ by $\xi_R$. The counter term action $S_{\text{count \ term}}$ is given explicitly by

\[
S_{\text{count \ term}} = \delta_2 \int d^4x \sum_a \bar{\psi}_R^a \gamma_\mu \psi_R^a \rho_2 - \delta_m \int d^4x \sum_a \bar{\psi}_R^a \psi_R^a - \frac{\delta_3}{4} \int d^4x (\partial_\mu A^A_{\nu R} - \partial_\nu A^A_{\mu R})(\partial^\mu A^{A^T}_{\nu R} - \partial^\nu A^{A^T}_{\mu R})
\]

\[ - \delta_5^2 \int d^4x \bar{c}_R^a \partial_\mu c_R^a - g_R \delta_1^A \int d^4x \sum_{a,b} \bar{\psi}_R^a \gamma_\mu \psi_R^b A^A_{\mu R} + g_R \delta_1^B \int d^4x A^A_{\mu R} A^B_{\mu R} + g_R \delta_1^C \int d^4x A^A_{\mu R} A^C_{\mu R} A^D_{\mu R} A^E_{\mu R} - g_R \delta_1^f \int d^4x (\bar{c}_R^E A^B_{\mu R} A^C_{\mu R} + \bar{c}_R^B A^C_{\mu R} A^E_{\mu R}).
\]  

(3.235)

From the above discussion we see that we have eight counter terms $\delta_1$, $\delta_2$, $\delta_3$, $\delta_2^2$, $\delta_m$, $\delta_1^A$, $\delta_1^B$ and $\delta_1^C$ and five coupling constants $g$, $m$, $Z_2$, $Z_3$ and $Z_2^2$. The counter terms will be determined in terms of the coupling constants and hence there must be only five of them which are completely independent. The fact that only five counter terms are independent means that we need five renormalization conditions to fix them. This also means that the counter term must be related by three independent equations. It is not difficult to discover that these equations are

\[ \frac{g_R}{g} = \frac{Z_2\sqrt{Z_3}}{1 + \delta_1} = \frac{Z_3^3}{1 + \delta_1^A}. \]  

(3.236)
\[
\frac{g_R}{g} = \frac{Z_2 \sqrt{Z_3}}{1 + \delta_1} = \frac{Z_3}{\sqrt{1 + \delta_1^2}}.
\]  
(3.237)

\[
\frac{g_R}{g} = \frac{Z_2 \sqrt{Z_3}}{1 + \delta_1} = \frac{Z_3 \sqrt{Z_3}}{1 + \delta_1^2}.
\]  
(3.238)

At the one-loop order we can expand \(Z_3 = 1 + \delta_3\), \(Z_2 = 1 + \delta_2\) and \(Z_2^c = 1 + \delta_2^c\) where \(\delta_3\), \(\delta_2\) and \(\delta_2^c\) as well as \(\delta_1\), \(\delta_1^2\), \(\delta_1^4\) and \(\delta_1^c\) are all of order \(\hbar\) and hence the above equations become

\[
\delta_1^3 = \delta_3 + \delta_1 - \delta_2.
\]  
(3.239)

\[
\delta_1^4 = \delta_3 + 2\delta_1 - 2\delta_2.
\]  
(3.240)

\[
\delta_1^c = \delta_2^c + \delta_1 - \delta_2.
\]  
(3.241)

The independent counter terms are taken to be \(\delta_1\), \(\delta_2\), \(\delta_3\), \(\delta_2^c\), \(\delta_m\) which correspond respectively to the coupling constants \(g\), \(Z_2\), \(Z_3\), \(Z_2^c\) and \(m\). The counter term \(\delta_3\) will be determined in the following from the gluon self-energy, the counter terms \(\delta_2\) and \(\delta_m\) will be determined from the quark self-energy whereas the counter term \(\delta_1\) will be determined from the vertex. The counter term \(\delta_2^c\) should be determined from the ghost self-energy 7.

For ease of writing we will drop in the following the subscript \(R\) on renormalized quantities and when we need to refer to the bare quantities we will use the subscript 0 to distinguish them from their renormalized counterparts.

We write next the corresponding Feynman rules in momentum space. These are shown in figure 11. In the next two sections we will derive these rules from first principle, i.e. starting from the formula (2.136). The Feynman rules corresponding to the bare action are summarized as follows:

- The quark propagator is

\[
<\psi^b_\beta(p)\bar{\psi}_\alpha(-p)> = \delta^{ab} \left( \frac{\gamma^\mu p_\mu + m}{p^2 - m^2} \right) \delta_\alpha\beta.
\]  
(3.242)

- The gluon propagator is

\[
<A^A_\mu(k)A^B_\nu(-k)> = \delta^{AB} \frac{1}{k^2} \left[ \eta_{\mu\nu} + (\xi - 1) \frac{k_\mu k_\nu}{k^2} \right].
\]  
(3.243)

- The ghost propagator is

\[
<\bar{c}^B(p)c^A(-p)> = \delta^{AB} \frac{1}{p^2}.
\]  
(3.244)

Exercise: Compute \(\delta_2^c\) following the same steps taken for the other counter terms.
• The quartic vertex is
\[
< A^A_{\mu} A^B_{\nu} A^D_{\rho} A^E_{\sigma} > = -g^2 \left[ f_{ABC} f_{DEC} (\eta^{\mu \nu} \eta^{\rho \sigma} - \eta^{\mu \rho} \eta^{\nu \sigma}) + f_{BDC} f_{AEC} (\eta^{\rho \mu} \eta^{\nu \sigma} - \eta^{\rho \nu} \eta^{\mu \sigma}) \right].
\]
(3.245)

• The cubic vertex is
\[
< A^A_{\mu} (k) A^B_{\nu} (p) A^C_{\rho} (q) > = ig f_{ABC} \left[ (2p + k)^\mu \eta^{\rho \nu} - (p + 2k)^\rho \eta^{\mu \nu} + (k - p)^\rho \eta^{\mu \nu} \right], \quad q = -p - k.
\]
(3.246)

• The quark-gluon vertex is
\[
< A^A_{\mu} \bar{c}^A_{\alpha} (k) c^B_{\beta} > = g (t^A)_{ab} (\gamma^\mu)_{\alpha \beta}.
\]
(3.247)

• The ghost-gluon vertex is
\[
< A^C_{\mu} \bar{\psi}^a_{\alpha} \psi^b_{\beta} > = -ig f_{ABC} k^\mu.
\]
(3.248)

• Impose energy-momentum conservation at all vertices.

• Integrate over internal momenta.

• Symmetry factor. For example if the diagram is invariant under the permutation of two lines we should divide by $1/2$.

• Each fermion line must be multiplied by $-1$.

• All one-loop diagrams should be multiplied by $\hbar/i$.

3.4.3 The Gluon Field Self-Energy at 1–Loop

We are interested in computing the proper $n$–point vertices of this theory which are connected 1–particle irreducible $n$–point functions from which all external legs are amputated. The generating functional of the corresponding Feynman diagrams is of course the effective action. We recall the formal definition of the proper $n$–point vertices given by
\[
\Gamma^{(n)}(x_1, ..., x_n) = \Gamma_{i_1...i_n} = \frac{\delta^n \Gamma[\phi_c]}{\delta \phi_c(x_1) ... \delta \phi_c(x_n)}|_{\phi = 0}.
\]
(3.249)

The effective action up to the 1–loop order is
\[
\Gamma = S + \frac{1}{i} \Gamma_1 , \quad \Gamma_1 = \ln \det G_0 , \quad G_0^{-1} = -S_0^{-1}.
\]
(3.250)
As our first example we consider the proper 2-point vertex of the non-Abelian vector field $A_{\mu}$. This is defined by

$$\Gamma_{\mu\nu}^{AB}(x, y) = \frac{\delta^2 \Gamma}{\delta A^{\mu}A(x)\delta A^{\nu}B(y)}|_{A,\psi, c=0}. \quad (3.251)$$

We use the powerful formula (2.136) which we copy here for convenience

$$\Gamma_1[\phi]_{j_0k_0} = \frac{1}{2}G_{0}^{mn}S[\phi]_{j_0k_0mn} + \frac{1}{2}G_{0}^{mmq}G_{0}^{mnq}S[\phi]_{j_0mnq}S[\phi]_{k_0mnq0}. \quad (3.252)$$

We have then immediately four terms contributing to the gluon propagator at 1-loop. These are given by (with $j_0 = (x, \mu, A)$ and $k_0 = (y, \nu, B)$)

$$\Gamma_{\mu\nu}^{AB}(x, y) = \frac{\delta^2 S}{\delta A^{\mu}A(x)\delta A^{\nu}B(y)}|_{A,\psi, c=0} + \frac{i}{2} \left[ \frac{1}{2}G_{0}^{A_{m}A_{n}}S_{A_{j_0}A_{k_0}A_{m}A_{n}} + \frac{1}{2}G_{0}^{A_{m}A_{nq}}G_{0}^{A_{n}A_{nq}}S_{A_{j_0}A_{m}A_{nq}S_{A_{k_0}A_{m}A_{nq}A_{n}}} \right. $$

$$\left. + (-1) \times G_{0}^{m}G_{0}^{m}G_{0}^{m}G_{0}^{m}G_{0}^{m}G_{0}^{m}G_{0}^{m}G_{0}^{m}G_{0}^{m} \right] . \quad (3.253)$$

The corresponding Feynman diagrams are shown on figure 9. The minus signs in the last two diagrams are the famous fermion loops minus sign. To see how they actually originate we should go back to the derivation of (3.252) and see what happens if the fields are Grassmann valued. We start from the first derivative of the effective action $\Gamma_1$ which is given by the unambiguous equation (2.112), viz

$$\Gamma_{1,j} = \frac{1}{2}G_{0}^{mn}S_{jmn}. \quad (3.254)$$

Taking the second derivative we obtain

$$\Gamma_{1,ij} = \frac{1}{2}G_{0}^{mn}S_{ijmn} + \frac{1}{2} \frac{\delta G_{0}^{mn}}{\delta \phi^i}S_{jmn}. \quad (3.255)$$

The first term is correct. The second term can be computed using the identity $G_{0}^{mmq}S_{mon} = -\delta_{m}^{n}$ which can be rewritten as

$$\frac{\delta G_{0}^{mn}}{\delta \phi^i} = G_{0}^{mmq}S_{imq0}G_{0}^{n0}. \quad (3.256)$$

We have then

$$\Gamma_{1,ij} = \frac{1}{2}G_{0}^{mn}S_{ijmn} + \frac{1}{2}G_{0}^{mmq}G_{0}^{n0}S_{jmnq0}S_{jmn} $$

$$= \frac{1}{2}G_{0}^{mn}S_{j0k0mn} + \frac{1}{2}G_{0}^{mmq}G_{0}^{n0}S_{j0mnq0}S_{j0mnq0}. \quad (3.257)$$

Only the propagator $G_{0}^{mn}$ has reversed indices compared to (3.252) which is irrelevant for bosonic fields but reproduces a minus sign for fermionic fields.
The classical term in the gluon self-energy is given by

\[
S_{j_0k_0} = \frac{\delta^2 S}{\delta A^\mu A(x)\delta A^\nu B(y)}|_{A,\psi,\epsilon=0} = \left[ \partial_\mu \partial_\rho \eta^{\mu\nu} + (\frac{1}{\xi} - 1) \partial_\mu \partial_\nu \right] \delta^{AB} \delta^4(x - y)
\]

\[
= - \int \frac{d^4k}{(2\pi)^4} \left[ k^2 \eta^{\mu\nu} + (\frac{1}{\xi} - 1) k^\mu k^\nu \right] \delta_{AB} e^{ik(x - y)}.
\]

We compute

\[
G_0^{j_0k_0} = \delta_{AB} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + i\epsilon} \left[ \eta^{\mu\nu} + (\xi - 1) \frac{k^\mu k^\nu}{k^2} \right] e^{ik(x - y)}
\]

\[
= \delta_{AB} \int \frac{d^4k}{(2\pi)^4} G_0^{\mu\nu}(k) e^{ik(x - y)}
\]

\[
= \delta_{AB} G_0^{\mu\nu}(x, y).
\]

The quartic vertex can be put into the fully symmetric form

\[
-\frac{g^2}{4} f_{ABC} f_{DEC} \int d^4x A_\mu^A A_\nu^B A_\rho^D A_\sigma^E = -\frac{g^2}{8} f_{ABC} f_{DEC} \int d^4x \int d^4y \int d^4z \int d^4w A_\mu^A(x) A_\nu^B(y) A_\rho^D(z) A_\sigma^E(w)
\]

\[
\times (x - y) \delta^4(x - z) \delta^4(x - w) \left[ \eta^{\mu\nu} - \eta^{\mu\rho} - \eta^{\mu\sigma} \right]
\]

\[
= -\frac{g^2}{4!} \int d^4x \int d^4y \int d^4z \int d^4w A_\mu^A(x) A_\nu^B(y) A_\rho^D(z) A_\sigma^E(w) \delta^4(x - y)
\]

\[
\times (x - w) \left[ f_{ABC} f_{DEC} (\eta^{\rho\nu} \eta^{\sigma\mu} - \eta^{\rho\mu} \eta^{\sigma\nu}) + f_{BDC} f_{AEC} (\eta^{\sigma\rho} \eta^{\mu\nu} - \eta^{\mu\rho} \eta^{\sigma\nu}) + f_{DAC} f_{BEC} (\eta^{\rho\sigma} \eta^{\mu\nu} - \eta^{\mu\sigma} \eta^{\rho\nu}) \right].
\]

In other words (with \( j_0 = (x, \mu, A), k_0 = (y, \nu, B), m = (\rho, D) \) and \( n = (w, \sigma, E) \))

\[
S_{j_0k_0A_mB_n} = -g^2 \delta^4(x - y) \delta^4(x - z) \delta^4(x - w) \left[ f_{ABC} f_{DEC} (\eta^{\rho\nu} \eta^{\sigma\mu} - \eta^{\rho\mu} \eta^{\sigma\nu}) + f_{BDC} f_{AEC} (\eta^{\sigma\rho} \eta^{\mu\nu} - \eta^{\mu\rho} \eta^{\sigma\nu})
\]

\[
+ f_{DAC} f_{BEC} (\eta^{\rho\sigma} \eta^{\mu\nu} - \eta^{\mu\sigma} \eta^{\rho\nu}) \right] .
\]

We can now compute the first one-loop diagram as

\[
\frac{1}{2} G_0^{A_mB_n} S_{A_0A_kA_mA_n} = -\frac{g^2}{2} \delta^{DE} G_0^{\rho\sigma}(x, x) \left[ f_{ABC} f_{DEC} (\eta^{\rho\mu} \eta^{\sigma\nu} - \eta^{\rho\nu} \eta^{\sigma\mu}) + f_{BDC} f_{AEC} (\eta^{\sigma\rho} \eta^{\mu\nu} - \eta^{\rho\nu} \eta^{\sigma\mu})
\]

\[
+ f_{DAC} f_{BEC} (\eta^{\mu\rho} \eta^{\nu\sigma} - \eta^{\mu\sigma} \eta^{\rho\nu}) \right] \delta^4(x - y)
\]

\[
= -\frac{g^2}{2} \left[ G_0^{\rho\sigma}(x, x) \delta^4(x - y) \right] \left[ f_{BDC} f_{AEC} - f_{DAC} f_{BEC} \right] \delta^4(x - y)
\]

\[
= -g^2 \left[ G_0^{\rho\sigma}(x, x) \delta^4(x - y) \right] f_{BDC} f_{AEC} \delta^4(x - y).
\]
The quantity $f_{BDC} f_{ADC}$ is actually the Casimir operator in the adjoint representation of the group. The adjoint representation of $SU(N)$ is $(N^2 - 1)$-dimensional. The generators in the adjoint representation can be given by $(t^A_B)_C = i f_{ABC}$. Indeed we can easily check that these matrices satisfy the fundamental commutation relations $[t^A_B, t^C_D] = i f_{ABC} t^C_D$.

We compute then $f_{BDC} f_{ADC} = (t^C_D t^E_F)_B A = C_2(G) \delta_{BA}$. These generators must also satisfy $tr G t^A_B t^C_D = C(G) \delta^{AB}$. For $SU(N)$ we have

$$f_{BDC} f_{ADC} = C_2(G) \delta_{BA} = C(G) \delta_{BA} = N \delta_{BA}. \quad (3.263)$$

Hence

$$\frac{1}{2} G^{A_m A_n} S_{A_{i_0} A_{k_0} A_m A_n} = -g^2 C_2(G) \delta_{AB} \left[ G_{\eta^\mu}^{\eta^\nu} (x, x) \eta^{\mu\nu} - G_0^{\eta^\mu} (x, x) \right] \delta^4 (x - y). \quad (3.264)$$

In order to maintain gauge invariance we will use the powerful method of dimensional regularization. The above diagram takes now the form

$$\frac{1}{2} G^{A_m A_n} S_{A_{i_0} A_{k_0} A_m A_n} = -g^2 C_2(G) \delta_{AB} \int \frac{d^dp}{(2\pi)^d} \frac{1}{p^2} \left[ (d - 1) \eta^{\mu\nu} - (\xi - 1) \frac{p^\mu p^\nu}{k^2} \right] \delta^4 (x - y). \quad (3.265)$$

This simplifies further in the Feynman gauge. Indeed for $\xi = 1$ we get

$$\frac{1}{2} G^{A_m A_n} S_{A_{i_0} A_{k_0} A_m A_n} = -g^2 C_2(G) \delta_{AB} \int \frac{d^dp}{(2\pi)^d} \frac{1}{p^2} \left[ (d - 1) \eta^{\mu\nu} - (\xi - 1) \frac{p^\mu p^\nu}{k^2} \right] \delta^4 (x - y)$$

$$= -g^2 C_2(G) \delta_{AB} \int \frac{d^dp}{(2\pi)^d} \int \frac{d^dk}{p^2} \frac{(p + k)^2}{p(p + k)^2} \left[ (d - 1) \eta^{\mu\nu} \right] \epsilon^{ik(x - y)}. \quad (3.266)$$

We use now Feynman parameters, viz

$$\frac{1}{(p + k)^2 p^2} = \int_0^1 dx \int_0^1 dy \delta(x + y - 1) \frac{1}{x(p + k)^2 + yp^2} \frac{1}{2}$$

$$= \int_0^1 dx \frac{1}{(l - \Delta)^2}, \quad l = p + xk, \quad \Delta = -x(1 - x)k^2. \quad (3.267)$$

We have then (using also rotational invariance)

$$\frac{1}{2} G^{A_m A_n} S_{A_{i_0} A_{k_0} A_m A_n} = -g^2 C_2(G) \delta_{AB} \int \frac{d^dk}{(2\pi)^d} \left[ (d - 1) \eta^{\mu\nu} \right] \epsilon^{ik(x - y)} \int_0^1 dx \int \frac{d^dl}{(2\pi)^d} \frac{(l + (1 - x)k)^2}{(l^2 - \Delta)^2}$$

$$= -g^2 C_2(G) \delta_{AB} \int \frac{d^dk}{(2\pi)^d} \left[ (d - 1) \eta^{\mu\nu} \right] \epsilon^{ik(x - y)} \int_0^1 dx \left\{ \int \frac{d^dl}{(2\pi)^d} \frac{l^2}{(l^2 - \Delta)^2} + (1 - x)^2 k^2 \right\}. \quad (3.268)$$

---

*Exercise: Derive this result.*
The above two integrals over $l$ are given by (after dimensional regularization and Wick rotation)

$$
\int \frac{d^4l}{(2\pi)^d} \frac{l^2}{(l^2 - \Delta)^2} = -i \int \frac{d^4l_E}{(2\pi)^d} \frac{1}{(l_E^2 + \Delta)^2} \\
= -i \frac{1}{(4\pi)^\frac{d}{2}} \frac{1}{\Delta^{1-\frac{d}{2}}} \Gamma(2 - \frac{d}{2}).
$$

(3.269)

$$
\int \frac{d^4l}{(2\pi)^d} \frac{1}{(l^2 - \Delta)^2} = i \int \frac{d^4l_E}{(2\pi)^d} \frac{l_E^2}{(l_E^2 + \Delta)^2} \\
= i \frac{1}{(4\pi)^\frac{d}{2}} \frac{1}{\Delta^{1-\frac{d}{2}}} \Gamma(2 - \frac{d}{2}).
$$

(3.270)

We get the final result

$$
\frac{1}{2} G_{\alpha} A_{\alpha} S_{A_1 A_2 A_3 A_4} = i \frac{g^2}{(4\pi)^\frac{d}{2}} C_2(G) \delta_{AB} \int \frac{d^4k}{(2\pi)^d} \left[ \eta^{\mu\nu} k^2 \right] e^{ik(x-y)} \int_0^1 dx \int \frac{dx}{(x(1-x)k^2)^{\frac{d}{2}}} \left( -\frac{1}{2} d(d-1)x \right) \\
\times \Gamma(1 - \frac{d}{2}) - (d-1)(1-x)^2 \Gamma(2 - \frac{d}{2})).
$$

We compute now the second diagram. First we write the pure gauge field cubic interaction in the totally symmetric form

$$
g f_{ABC} \int d^4 x \partial^\mu A^{\mu} A^{\nu} A^{\rho} = \frac{g f_{ABC}}{3!} \int d^4 x \int d^4 y \int d^4 z A^{(A}_\mu (x) A^{B)_\nu (y) A^{C)}_\rho (z) \left[ \eta^{\mu\nu} \left( \partial^\rho \delta^4(x-z) \right) - \eta^{\mu\rho} \left( \partial^\nu \delta^4(x-y) \right) - \eta^{\nu\rho} \left( \partial^\mu \delta^4(y-x) \right) \right] \\
- \partial^\nu \delta^4(x-y) \partial^\mu \delta^4(x-z) \right) \right) - \partial^\mu \delta^4(y-x) \partial^\nu \delta^4(y-z) \right) \right) \\
- \partial^\rho \delta^4(x-z) \partial^\nu \delta^4(y-z) \right) \right) \right) \right).
$$

(3.272)

Thus we compute (with $j_0 = (x, \mu, A)$, $k_0 = (y, \nu, B)$ and $m = (z, \rho, C)$)

$$
S_{A_1 A_2 A_3 A_4} = ig f_{ABC} S^{\mu \nu \rho}(x, y, z).
$$

(3.273)

$$
\begin{align*}
\eta^{\mu\rho} & \left( \partial^\nu \delta^4(x-z) \right) - \eta^{\mu\nu} \left( \partial^\rho \delta^4(x-y) \right) - \eta^{\nu\rho} \left( \partial^\mu \delta^4(y-x) \right) \\
- \partial^\nu \delta^4(x-y) \partial^\mu \delta^4(x-z) & \right) \right) - \partial^\rho \delta^4(x-z) \partial^\nu \delta^4(y-z) \right) \right) \right) \right) \right) \right) \\
- \partial^\mu \delta^4(y-x) \partial^\nu \delta^4(y-z) \right) & \right) \right) \right) \right) \right) \right) \\
- \partial^\nu \delta^4(x-y) \partial^\mu \delta^4(x-z) & \right) \right) \right) \right) \right) \right) \right) \\
= i \int \frac{d^4k}{(2\pi)^d} \int \frac{d^4p}{(2\pi)^d} \int \frac{d^4l}{(2\pi)^d} S^{\mu \nu \rho}(k, p) (2\pi)^d \delta^4(p+k+l) \exp(ikx + ipy + ilz).
\end{align*}
$$

(3.274)
\[ S^\mu\nu (k, p) = (2p + k)\eta^\mu\nu - (p + 2k)^\nu \eta^\mu + (k - p)^\rho \eta^\mu\nu. \] (3.275)

The second diagram is therefore given by
\[
\frac{1}{2} G_0^{A_1 A_2 A_3 A_4} G_0^{A_5 A_6 A_7 A_8} S_{A_9 A_{10} A_1 A_2 A_3 A_4} S_{A_9 A_{10} A_1 A_2 A_3 A_4} = -\frac{g^2 C_2(G)\delta_{AB}}{2} \int d^4 z d^4 w d^4 w_0 G_0^{\rho\sigma}(z, z_0) G_0^{\rho\sigma}(w, w_0) S^\mu\rho\sigma(k, p) \times S^\nu_{\mu\rho\sigma}(y, z_0, w_0) \\
= -\frac{g^2 C_2(G)\delta_{AB}}{2} \int \frac{d^4 k}{(2\pi)^4} \int \frac{d^4 p}{(2\pi)^4} G_0^{\rho\sigma}(p) G_0^{\rho\sigma}(k + p) S^\mu\rho\sigma(k, p) \times S^\nu_{\mu\rho\sigma}(-k, -p) \exp i k(x - y).
\]

In the Feynman gauge this becomes
\[
\frac{1}{2} G_0^{A_1 A_2 A_3 A_4} G_0^{A_5 A_6 A_7 A_8} S_{A_9 A_{10} A_1 A_2 A_3 A_4} S_{A_9 A_{10} A_1 A_2 A_3 A_4} = -\frac{g^2 C_2(G)\delta_{AB}}{2} \int \frac{d^4 k}{(2\pi)^4} \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2(k + p)^2} S^\mu\rho\sigma(k, p) \times S^\nu_{\rho\sigma}(-k, -p) \exp i k(x - y).
\]

We use now Feynman parameters as before. We get
\[
\frac{1}{2} G_0^{A_1 A_2 A_3 A_4} G_0^{A_5 A_6 A_7 A_8} S_{A_9 A_{10} A_1 A_2 A_3 A_4} S_{A_9 A_{10} A_1 A_2 A_3 A_4} = -\frac{g^2 C_2(G)\delta_{AB}}{2} \int \frac{d^4 k}{(2\pi)^4} \exp i k(x - y) \int_0^1 dx \int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2 - \Delta)^2} S^\mu\rho\sigma(k, l - x k) S^\nu_{\rho\sigma}(-k, -l + x k).
\]

Clearly by rotational symmetry only quadratic and constant terms in \( l^\mu \) in the product \( S^\mu\rho\sigma(k, l - x k) S^\nu_{\rho\sigma}(-k, -l + x k) \) give non-zero contribution to the integral over \( l \). These are
\[
\frac{1}{2} G_0^{A_1 A_2 A_3 A_4} G_0^{A_5 A_6 A_7 A_8} S_{A_9 A_{10} A_1 A_2 A_3 A_4} S_{A_9 A_{10} A_1 A_2 A_3 A_4} = -\frac{g^2 C_2(G)\delta_{AB}}{2} \int \frac{d^4 k}{(2\pi)^4} \exp i k(x - y) \int_0^1 dx \left\{ \frac{6(1 - d)}{d - 1} \eta^\mu\nu \right. \times \left. \int \frac{d^4 l}{(2\pi)^4} \frac{l^2}{(l^2 - \Delta)^2} \right. + \left. \left( (2 - d)(1 - 2x)^2 k^\mu k^\nu + 2(1 + x)(2 - x)k^\mu\nu \right) \right. \\
- \left. \eta^\mu\nu k^2(2 - x)^2 - \eta^\mu\nu k^2(1 + x)^2 \right) \int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2 - \Delta)^2} \right\}.
\]

We now employ dimensional regularization and use the integrals (3.269) and (3.270). We obtain
\[
\frac{1}{2} G_0^{A_1 A_2 A_3 A_4} G_0^{A_5 A_6 A_7 A_8} S_{A_9 A_{10} A_1 A_2 A_3 A_4} S_{A_9 A_{10} A_1 A_2 A_3 A_4} = -\frac{ig^2 C_2(G)\delta_{AB}}{2(4\pi)^\frac{d}{2}} \int \frac{d^4 k}{(2\pi)^d} \exp i k(x - y) \int_0^1 \frac{dx}{(-x(1 - x)k^2)^2 - \frac{d}{2}} \left\{ \right. \\
- 3(d - 1)\eta^\mu\nu \Gamma(1 - d) x(1 - x)k^2 + \Gamma(2 - \frac{d}{2}) \right. \left. \left( (2 - d)(1 - 2x)^2 k^\mu k^\nu + 2(1 + x)(2 - x)k^\mu\nu \right) \right. \\
+ \left. 2(1 + x)(2 - x)k^\mu k^\nu - \eta^\mu\nu k^2(2 - x)^2 - \eta^\mu\nu k^2(1 + x)^2 \right) \right\}.
\]

Exercise: Derive explicitly these terms.
We go now to the third diagram which involves a ghost loop. We recall first the ghost field propagator

\[ < e^A(x) e^B(y) > = \int \frac{d^4k}{(2\pi)^4} \frac{i \delta^{AB}}{k^2 + i\epsilon} e^{ik(x-y)}. \]  

(3.281)

However we will need

\[ G_0^{e^A(x) e^B(y)} = \int \frac{d^4k}{(2\pi)^4} \frac{\delta^{AB}}{k^2 + i\epsilon} e^{ik(x-y)}. \]  

(3.282)

The interaction between the ghost and vector fields is given by

\[ -gf_{ABC} \int d^4x (e^A \partial^\mu A_\mu^C + \bar{e}^A \partial^\mu e^B A_\mu^C) = -gf_{ABC} \int d^4x \int d^4y \int d^4z e^A(x) e^B(y) A_\mu^C(z) \partial^\nu_x (\delta^4(x-y)\delta^4(x-z)) \]  

(3.283)

In other words (with \( j_0 = (z, C, \mu) \), \( m = (x, A) \) and \( n = (y, B) \))

\[ S_{A j_0, e_m e_n} = g f_{ABC} \partial^\nu_x (\delta^4(x-y)\delta^4(x-z)) \]

\[ = -ig f_{ABC} \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4l}{(2\pi)^4} k^\mu (2\pi)^4 \delta^4(p+k-l) \exp(-ikx - ipy + ilz). \]  

(3.284)

We compute the third diagram as follows. We have (with \( j_0 = (z, C, \mu), k_0 = (w, D, \nu) \), \( m = (x, A) \), \( n = (y, B) \) and \( m_0 = (x_0, A_0), n_0 = (y_0, B_0) \))

\[ G_0^{e_m e_n} G_0^{e_{m_0} e_{n_0}} S_{A j_0, e_m e_n} S_{A_0 j_0, e_{m_0} e_{n_0}} = \sum_{A, A_0, B, B_0} \int d^4x \int d^4x_0 \int d^4y \int d^4y_0 \int \frac{d^4k}{(2\pi)^4} \frac{\delta^{A_0 A}}{k^2} e^{ik(x_0-x)} \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4l}{(2\pi)^4} \delta^{A_0 A} k^\mu (2\pi)^4 \delta^4(p+k-l) \exp(-ikx - ipy + ilz) \]

\[ \times e^{ip(z-y)} (g f_{ABC} \partial^\nu_y (\delta^4(x-y)\delta^4(x-z))) - g f_{B_0 A_0 B} \partial^\nu_{y_0} (\delta^4(y_0 - x_0)\delta^4(y)) \]

\[ = g^2 f_{ABC} f_{ABD} \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4p}{(2\pi)^4} \frac{(p+k)^\mu p^\nu}{(p+k)^2 p^2} e^{ik(z-w)} \]

We use Feynman parameters as before. Also we use rotational invariance to bring the above loop integral to the form

\[ G_0^{e_m e_n} G_0^{e_{m_0} e_{n_0}} S_{A j_0, e_m e_n} S_{A_0 j_0, e_{m_0} e_{n_0}} = g^2 C_2(G) \delta_{CD} \int \frac{d^4k}{(2\pi)^4} e^{ik(z-w)} \int_0^1 dx \int \frac{d^4l}{(2\pi)^4} \frac{l^\mu l^\nu + x(x-1)k^\mu k^\nu}{l^2} \int \frac{d^4l}{(2\pi)^4} \frac{l^2}{(l^2 - \Delta)^2} \]

\[ \times \left[ \frac{1}{(2\pi)^4} \int \frac{d^4l}{(2\pi)^4} \frac{l^2}{(l^2 - \Delta)^2} \right]. \]  

(3.285)
Once more we employ dimensional regularization and use the integrals (3.269) and (3.270). Hence we get the loop integral (with $C \rightarrow A, D \rightarrow B, z \rightarrow x, w \rightarrow y$)

\[
G_0^{c_m c_0} G_0^{c_n c_n} S_{A_0 \bar{c}_m c_n} S_{A_0 \bar{c}_m c_n} = -g^2 C_2(G) \frac{i}{(4\pi)^{\frac{d}{2}}} \delta_{AB} \int \frac{d^d k}{(2\pi)^d} e^{ik(x-y)} \left( -\frac{1}{2} \eta^{\mu\nu} k^2 \Gamma(1 - \frac{d}{2}) + k^\mu k^\nu \Gamma(2 - \frac{d}{2}) \frac{\partial}{\partial x(x-1)} \right) \times \int_0^1 dx \frac{x(1-x)}{-(x-1)^2}.
\]

By putting equations (3.271), (3.280) and (3.287) together we get

\[
(3.271) + (3.280) - (3.287) = g^2 C_2(G) \frac{i}{(4\pi)^{\frac{d}{2}}} \delta_{AB} \int \frac{d^d k}{(2\pi)^d} e^{ik(x-y)} \int_0^1 dx \left( \frac{\eta^{\mu\nu} k^2 (d-2) \Gamma(2 - \frac{d}{2})}{(x-1)^2} \right) \times x(1-x) + \frac{\eta^{\mu\nu} k^2 (2 - \frac{d}{2})}{2} \left( -(d-1)(1-x)^2 + \frac{1}{2}(2-x)^2 + \frac{1}{2}(1+x)^2 \right) - k^\mu k^\nu \Gamma(2 - \frac{d}{2}) \left( 1 - \frac{d}{2} \right) (1 - 2x)^2 + 2 \right) \}.
\]

The pole at $d = 2$ cancels exactly since the gamma function $\Gamma(1 - \frac{d}{2})$ is completely gone. There remains of course the pole at $d = 4$. By using the symmetry of the integral over $x$ under $x \rightarrow 1 - x$ we can rewrite the above integral as

\[
(3.271) + (3.280) - (3.287) = g^2 C_2(G) \frac{i}{(4\pi)^{\frac{d}{2}}} \delta_{AB} \int \frac{d^d k}{(2\pi)^d} e^{ik(x-y)} \int_0^1 dx \left( \frac{\eta^{\mu\nu} k^2 (1 - \frac{d}{2}) \Gamma(2 - \frac{d}{2})}{(x-1)^2} \right) \times \left( (1-2x)^2 + (1-2x) \right) + \frac{\eta^{\mu\nu} k^2 (2 - \frac{d}{2})}{2} \cdot 4x - k^\mu k^\nu \Gamma(2 - \frac{d}{2}) \left( 1 - \frac{d}{2} \right) (1 - 2x)^2 \right) \}.
\]

Again by the symmetry $x \rightarrow 1 - x$ we can replace $x$ in every linear term in $x$ by $1/2^{10}$. We obtain the final result

\[
(3.271) + (3.280) - (3.287) = g^2 C_2(G) \frac{i\Gamma(2 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \delta_{AB} \int \frac{d^d k}{(2\pi)^d} e^{ik(x-y)} \left( \frac{\eta^{\mu\nu} k^2 - k^\mu k^\nu}{(k^2)^{\frac{d}{2}}-2} \right) \int_0^1 dx \left( \frac{1}{(x-1)^2} \right) \times \left( 1 - \frac{d}{2} \right) (1 - 2x)^2 + 2 \right) \) \]

\[
= g^2 C_2(G) \frac{i\Gamma(2 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \delta_{AB} \int \frac{d^d k}{(2\pi)^d} e^{ik(x-y)} \left( \frac{\eta^{\mu\nu} k^2 - k^\mu k^\nu}{(k^2)^{\frac{d}{2}}-2} \right) \left( \frac{5}{3} \right) + \text{regular terms} \)
\]

The gluon field is therefore transverse as it should be for any vector field with an underlying gauge symmetry. Indeed the exhibited the tensor structure $\eta^{\mu\nu} k^2 - k^\mu k^\nu$ is consistent.

\[\text{Exercise: Why.}\]
with Ward identity. This result does not depend on the gauge fixing parameter although
the proportionality factor actually does\(^\text{11}\).

There remains the fourth and final diagram which as it turns out is the only diagram
which is independent of the gauge fixing parameter. We recall the Dirac field propagator

\[
\langle \psi^a_\alpha(x) \bar{\psi}^b_\beta(y) \rangle = i \delta^{ab} \int \frac{d^4p}{(2\pi)^4} \frac{(\gamma^\mu p_\mu + m)_{\alpha\beta}}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}. \tag{3.291}
\]

However we will need something a little different. We have

\[
S_{\psi^a_\alpha(x) \bar{\psi}^b_\beta(y)} = \frac{\delta^2 S}{\delta \psi^a_\alpha(x) \delta \bar{\psi}^b_\beta(y)}|_{A, \psi, c=0} = (i\gamma^\mu \partial_\mu - m)_{\beta\alpha} \delta^4(y-x) \delta^{ab} = \int \frac{d^4k}{(2\pi)^4} \frac{(\gamma^\mu k_\mu - m)_{\beta\alpha} e^{ik(x-y)} \delta^{ab}}{k^2 - m^2 + i\epsilon}. \tag{3.292}
\]

Thus we must have

\[
G^0_{\psi^a_\alpha(x) \bar{\psi}^b_\beta(y)} = \delta^{ab} \int \frac{d^4k}{(2\pi)^4} \frac{(\gamma^\mu k_\mu + m)_{\alpha\beta} e^{-ik(x-y)}}{k^2 - m^2 + i\epsilon}. \tag{3.293}
\]

Indeed we can check

\[
\int d^4y \sum_{\beta, \alpha} S_{\psi^a_\alpha(x) \bar{\psi}^b_\beta(y)} G^0_{\psi^a_\alpha(x) \bar{\psi}^b_\beta(y)} = \delta^{\alpha\alpha_0} \delta^{\beta\beta_0} \delta^4(x - x_0). \tag{3.294}
\]

The interaction between the Dirac and vector fields is given by

\[
-gA^A \int d^4x \sum_{a,b} \bar{\psi}^a \gamma^\mu \psi^b A^A_{\mu} = -g A^A (\gamma^\mu)_{\alpha\beta} \int d^4x \int d^4y \int d^4z \bar{\psi}^a(x) \psi^b(y) A^A_{\mu}(z) \delta^4(x-y) \delta^4(x-z). \tag{3.295}
\]

In other words (with \(j_0 = (z, A, \mu)\), \(m = (x, a, \alpha)\) and \(n = (y, b, \beta)\))

\[
S_{A_{j_0} \psi_{m} \bar{\psi}_{n}} = g t^A A^A (\gamma^\mu)_{\alpha\beta} \delta^4(x-y) \delta^4(x-z). \tag{3.296}
\]

By using these results we compute the fourth diagram is given by (with \(j_0 = (z, A, \mu)\),
\(k_0 = (w, B, \nu)\), \(m = (x, a, \alpha)\), \(n = (y, b, \beta)\), \(m_0 = (x_0, a_0, \alpha_0)\) and \(n_0 = (y_0, b_0, \beta_0)\) and
\(\mathrm{tr}\gamma^\mu = 0, \mathrm{tr}\gamma^\mu\gamma^\nu = 4\gamma^\mu, \mathrm{tr}\gamma^\mu\gamma^\nu\gamma^\rho = 0, \mathrm{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = 4(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho})\))

\[
G^0_{\psi^a_\alpha(x) \bar{\psi}^b_\beta(y)} = 4g^2 A^A B \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{tr}\gamma^\rho p_\rho + m + \gamma^\rho(p + k)_\rho + m + \gamma^\nu(p + k)_\nu + m - \eta^{\mu\nu}(p^2 + pk - m^2)}{(p^2 - m^2)((p + k)^2 - m^2)} e^{-ik(z-w)}.
\]

\(^{11}\) Exercise: Determine the corresponding factor for an arbitrary value of the gauge fixing parameter \(\xi\).
We use now Feynman parameters in the form

\[
\frac{1}{(p^2 - m^2)((p + k)^2 - m^2)} = \int_0^1 dx \int_0^1 dy \delta(x + y - 1) \frac{1}{[x(p^2 - m^2) + y((p + k)^2 - m^2)]^2}
\]

\[
= \int_0^1 dx \frac{1}{(l^2 - \Delta)^2}, \quad l = p + (1 - x)k, \quad \Delta = m^2 - x(1 - x)k^2.
\]

By using also rotational invariance we can bring the integral to the form

\[
G_0^\psi_0^\psi_0 S_{A_0}^\psi_0^\psi_0 S_{A_0}^{\bar{\psi}_0^\bar{\psi}_0} = 4g^2 \text{tr} A^4 B \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} e^{-ik(z-w)}
\]

\[
\times \left[ \frac{1}{l^2} \eta^{\mu\nu} - 2x(1 - x)k^\mu k^\nu - \eta^{\mu\nu}(l^2 - x(1 - x)k^2 - m^2) \right] \frac{1}{(l^2 - \Delta)^2}
\]

\[
= 4g^2 \text{tr} A^4 B \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} e^{-ik(z-w)}
\]

\[
\times \left[ \frac{1}{l^2} \eta^{\mu\nu} - 2x(1 - x)k^\mu k^\nu - \eta^{\mu\nu}(l^2 - x(1 - x)k^2 - m^2) \right] \frac{1}{(l^2 - \Delta)^2}
\]

After using the integrals (3.269) and (3.270), the fourth diagram becomes (with \( z \to x \), \( w \to y \))

\[
G_0^\psi_0^\psi_0 S_{A_0}^\psi_0^\psi_0 S_{A_0}^{\bar{\psi}_0^\bar{\psi}_0} = 8ig^2 \frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \text{tr} A^4 B \int \frac{d^4k}{(2\pi)^d} (k^2 \eta^{\mu\nu} - k^\mu k^\nu) e^{-ik(x-y)}
\]

\[
\times \left[ \frac{(k^2)^{\frac{d}{2}}}{2} \right] \int_0^1 dx \frac{x(1 - x)}{(m^2 - x(1 - x))^{2 - \frac{d}{2}}}
\]

\[
= 4g^2 \frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}} C(N) \delta_{AB} \int \frac{d^4k}{(2\pi)^d} i(k^2 \eta^{\mu\nu} - k^\mu k^\nu) e^{-ik(x-y)} (k^2)^{\frac{d}{2}} - 2 \left( 1 + \text{regular terms} \right).
\]

For \( n_f \) flavors (instead of a single flavor) of fermions in the representation \( t^a_f \) (instead of
the fundamental representation \( t_a \) we obtain (with also a change \( k \rightarrow -k \))

\[
G_0 \bar{\psi}_m \gamma_\mu S \gamma_5 A_0 \bar{\psi}_n S A_0 \bar{\psi}_m \psi_n = \frac{4}{3} n_f g^2 \Gamma(2 - \frac{d}{2}) \frac{4}{(4\pi)^\frac{d}{2}} C(r) \delta_{AB} \int \frac{d^d k}{(2\pi)^d} i(k^2 \eta^{\mu \nu} - k^\mu k^\nu) e^{ik(x-y)} (k^2)^{\frac{d}{2}-2} \left( 1 + \text{regular terms} \right).
\]

(3.301)

By putting (3.290) and (3.301) together we get the final result

\[
\Gamma_{\mu \nu}^{AB}(x, y) = (3.290) - (3.301)
- \int \frac{d^d k}{(2\pi)^d} \left( k^2 \eta^{\mu \nu} + \left( \frac{1}{\xi} - 1 \right) k^\mu k^\nu \right) \delta_{AB} e^{ik(x-y)}
+ g^2 \Gamma(2 - \frac{d}{2}) \frac{5}{3} C_2(G) - \frac{4}{3} n_f C(r) \int \frac{d^d k}{(2\pi)^d} (k^2 \eta^{\mu \nu} - k^\mu k^\nu) \delta_{AB} e^{ik(x-y)} (k^2)^{\frac{d}{2}-2} \left( 1 + \text{regular terms} \right).
\]

(3.302)

The final step is to add the contribution of the counter terms. This leads to the one-loop result in the Feynman-t’Hooft gauge given by

\[
\Gamma_{\mu \nu}^{AB}(x, y) = - \int \frac{d^d k}{(2\pi)^d} \left( k^2 \eta^{\mu \nu} + \left( \frac{1}{\xi} - 1 \right) k^\mu k^\nu \right) \delta_{AB} e^{ik(x-y)}
+ g^2 \Gamma(2 - \frac{d}{2}) \frac{5}{3} C_2(G) - \frac{4}{3} n_f C(r) \int \frac{d^d k}{(2\pi)^d} (k^2 \eta^{\mu \nu} - k^\mu k^\nu) \delta_{AB} e^{ik(x-y)} (k^2)^{\frac{d}{2}-2} \left( 1 + \text{regular terms} \right) - \delta_3 \int \frac{d^d k}{(2\pi)^d} \left( k^2 \eta^{\mu \nu} + \left( \frac{1}{\xi} - 1 \right) k^\mu k^\nu \right) \delta_{AB} e^{ik(x-y)}.
\]

(3.303)

Equivalently

\[
\Gamma_{\mu \nu}^{AB}(k) = - \left( k^2 \eta^{\mu \nu} + \left( \frac{1}{\xi} - 1 \right) k^\mu k^\nu \right) \delta_{AB}
+ g^2 \Gamma(2 - \frac{d}{2}) \frac{5}{3} C_2(G) - \frac{4}{3} n_f C(r) (k^2 \eta^{\mu \nu} - k^\mu k^\nu) \delta_{AB} (k^2)^{\frac{d}{2}-2} \left( 1 + \text{regular terms} \right) - \delta_3 \left( k^2 \eta^{\mu \nu} + \left( \frac{1}{\xi} - 1 \right) k^\mu k^\nu \right) \delta_{AB}.
\]

(3.304)

Remark that the \( 1/\xi \) term in the classical contribution (the first term) can be removed by undoing the gauge fixing procedure. In 4 dimensions the coupling constant \( g^2 \) is dimensionless.
In dimension \( d = 4 - \epsilon \) the coupling constant \( g \) is in fact not dimensionless but has dimension of \( 1/\text{mass}^{(d/2-2)} \). The dimensionless coupling constant \( \hat{g} \) can therefore be given in terms of an arbitrary mass scale \( \mu \) by the formula

\[
\hat{g} = g \mu^{\frac{d}{2} - 2} \Leftrightarrow g^2 = \hat{g}^2 \mu^\epsilon.
\] (3.305)

We get then

\[
\Gamma_{AB}^{\mu\nu}(k) = -\left( k^2 \eta^{\mu\nu} + \left( \frac{1}{\xi} - 1 \right) k^\mu k^\nu \right) \delta_{AB} + \frac{\hat{g}^2}{16\pi^2} \Gamma\left( \frac{2}{\epsilon} \right) \left( \frac{5}{3} C_2(G) - \frac{4}{3} n_f C(r) \right) (k^2 \eta^{\mu\nu} - k^\mu k^\nu) \delta_{AB} \left( 1 + \text{regular terms} \right)
\]

\[
+ \delta_3 \left( k^2 \eta^{\mu\nu} - k^\mu k^\nu \right) \delta_{AB}
\]

\[
= -\left( k^2 \eta^{\mu\nu} + \left( \frac{1}{\xi} - 1 \right) k^\mu k^\nu \right) \delta_{AB} + \frac{\hat{g}^2}{16\pi^2} \left( \frac{2}{\epsilon} + \ln 4\pi - \gamma - \ln \frac{k^2}{\mu^2} \right) \left( \frac{5}{3} C_2(G) + \frac{4}{3} n_f C(r) \right) (k^2 \eta^{\mu\nu} - k^\mu k^\nu)
\]

\[
\times \left( 1 + \text{regular terms} \right) - \delta_3 \left( k^2 \eta^{\mu\nu} - k^\mu k^\nu \right) \delta_{AB}
\]

It is now clear that in order to eliminate the divergent term we need, in the spirit of minimal subtraction, only subtract the logarithmic divergence exhibited here by the term \( 2/\epsilon \) which has a pole at \( \epsilon = 0 \). In other words the counter term \( \delta_3 \) is chosen such that

\[
\delta_3 = \frac{\hat{g}^2}{16\pi^2} \left( \frac{2}{\epsilon} \right) \left( \frac{5}{3} C_2(G) - \frac{4}{3} n_f C(r) \right).
\] (3.307)

### 3.4.4 The Quark Field Self-Energy at 1-Loop

This is defined

\[
\Gamma_{\alpha\beta}(x, y) = \left. \frac{\delta^2 \Gamma}{\delta \bar{\psi}^\alpha_{\alpha}(x) \delta \psi^\beta_{\beta}(y)} \right|_{A, \psi, c=0}
\]

\[
= \left. \frac{\delta^2 S}{\delta \bar{\psi}^\alpha_{\alpha}(x) \delta \psi^\beta_{\beta}(y)} \right|_{A, \psi, c=0} + \frac{1}{i} \left. \frac{\delta^2 \Gamma}{\delta \bar{\psi}^\alpha_{\alpha}(x) \delta \psi^\beta_{\beta}(y)} \right|_{A, \psi, c=0}.
\] (3.308)

The first term is given by

\[
\left. \frac{\delta^2 S}{\delta \bar{\psi}^\alpha_{\alpha}(x) \delta \psi^\beta_{\beta}(y)} \right|_{A, \psi, c=0} = (i\gamma^\mu \partial^\mu - m)_{\beta\alpha} \delta^4(y - x) \delta^{ab}
\]

\[
= \int \frac{d^4 k}{(2\pi)^4} (\gamma^\mu k^\mu - m)_{\beta\alpha} e^{ik(x-y)} \delta^{ab}.
\] (3.309)

Again by using the elegant formula (3.257) we obtain (with \( j_0 = (x, \alpha, a) \) and \( k_0 = (y, \beta, b) \))

\[
\Gamma_{j_0 k_0} = -G_{0} \bar{\psi}_m \psi_n G_{0} A^a_{m} A^a_{n} S_{\bar{\psi}_m \bar{\psi}_n} S_{\psi_n \psi_m} A^a_{n}.
\] (3.310)
We recall the results
\[ G_0^{A\mu A}(x)A^\nu B(y) = \delta_{AB} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + i\epsilon} \left[ \eta^{\mu\nu} + (\xi - 1) \frac{k^\mu k^\nu}{k^2} \right] e^{ik(x-y)}. \]
(3.311)
\[ G_0^{\gamma\gamma}(x)\gamma^\beta(y) = \delta_{ab} \int \frac{d^4p}{(2\pi)^4} \frac{\gamma^{\mu}p_\mu + m}{p^2 - m^2} e^{-ip(x-y)}. \]
(3.312)
\[ S_{A\mu A}(x)\gamma^\beta(y) = g_{A\mu}^A (\gamma^\nu)_{\alpha\beta} \delta^A(x-y)\delta^A(x-z). \]
(3.313)

We compute then
\[ \Gamma_{1,\delta\Delta} = -g^2 (t^A t^A)_{ab} \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \left( \gamma^\mu (\gamma^\rho p_\rho + m) \gamma^\nu \right) \frac{1}{k^2((p+k)^2 - m^2)} \left( \eta^{\mu\nu} + (\xi - 1) \frac{k_\mu k_\nu}{k^2} \right) e^{i(k+p)(x-y)}. \]
(3.314)

This is given by the second diagram on figure 10. In the Feynman- t’Hooft gauge this reduces to (also using \( \gamma^\mu \gamma^\rho \gamma_\mu = -(2 - \epsilon) \gamma^\rho \), \( \gamma^\mu \gamma_\mu = d \) and \( (t^A t^A)_{ab} = C_2(r)\delta_{ab} \) where \( C_2(r) \) is the Casimir in the representation \( r \))
\[ \Gamma_{1,\delta\Delta} = -g^2 C_2(r) \delta_{ab} \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} (-2 - \epsilon) \gamma^\rho (p+k)_\rho + md \frac{1}{k^2((p+k)^2 - m^2)} e^{i(p-x-y)}. \]
(3.315)

We employ Feynman parameters in the form
\[ \frac{1}{k^2((p+k)^2 - m^2)} = \int_0^1 dx \frac{1}{(l^2 - \Delta)^2} , \ l = k + (1 - x)p , \ \Delta = -x(1-x)p^2 + (1-x)m^2. \]
(3.316)

We obtain
\[ \Gamma_{1,\delta\Delta} = -g^2 C_2(r) \delta_{ab} \int \frac{d^4p}{(2\pi)^4} e^{i(p-x-y)} \int_0^1 dx \int \frac{d^4l}{(2\pi)^4} \left( -2 - \epsilon \right) \gamma^\rho (l + xp)_\rho + md \frac{1}{(l^2 - \Delta)^2} \left( -2 - \epsilon \right) x \gamma^\rho p_\rho + md \frac{1}{(l^2 - \Delta)^2} \left( -2 - \epsilon \right) x \gamma^\rho p_\rho + md \frac{1}{(l^2 - \Delta)^2}. \]
(3.317)

After Wick rotation and dimensional regularization we can use the integral (3.269). We get
\[ \Gamma_{1,\delta\Delta} = -g^2 C_2(r) \frac{i}{(4\pi)^{\frac{d}{2}}} \delta_{ab} \int \frac{d^4p}{(2\pi)^4} \left( -\frac{1}{2} (2 - \epsilon) \gamma^\rho p_\rho + md \right) \frac{1}{(p^2 - \Delta)^{-\frac{d}{2}}} \left( -x(1-x) + (1-x) \right) \]
\[ = -g^2 C_2(r) \frac{i}{(4\pi)^{\frac{d}{2}}} \delta_{ab} \int \frac{d^4p}{(2\pi)^4} \left( -\frac{1}{2} (2 - \epsilon) \gamma^\rho p_\rho + md \right) \frac{1}{(p^2 - \Delta)^{-\frac{d}{2}}} \left( 1 + \text{regular terms} \right). \]
The quark field self-energy at 1-loop is therefore given by

\[
\Gamma_{\alpha\beta}^{ab}(x, y) = \int \frac{d^4p}{(2\pi)^4} (\gamma^\mu p_\mu - m)_{\beta\alpha} e^{ip(x-y)} \delta^{ab} \\
- g^2 C_2(r) \frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \delta_{ba} \int \frac{d^4p}{(2\pi)^4} \left( -\frac{1}{2}(2 - \epsilon)\gamma^\rho_p + md \right)_{\beta\alpha} e^{ip(x-y)} (p^2)^{-\frac{\epsilon}{2}} \left( 1 + \text{regular terms} \right). 
\]

(3.319)

We add the contribution of the counter terms. We obtain

\[
\Gamma_{\alpha\beta}^{ab}(x, y) = \int \frac{d^4p}{(2\pi)^4} (\gamma^\mu p_\mu - m)_{\beta\alpha} e^{ip(x-y)} \delta^{ab} \\
- \frac{g^2}{16\pi^2} C_2(r) \delta_{ba} \int \frac{d^4p}{(2\pi)^4} \left( \frac{2}{\epsilon} + \ln 4\pi - \gamma - \ln \frac{p^2}{\mu^2} \right) \left( -\gamma^\rho_p + md \right)_{\beta\alpha} e^{ip(x-y)} \left( 1 + \text{regular terms} \right) \\
+ \int \frac{d^4p}{(2\pi)^4} (\delta_2 \gamma^\mu p_\mu - \delta_m)_{\beta\alpha} e^{ip(x-y)} \delta^{ab}. 
\]

(3.320)

In order to cancel the divergence we must choose the counter terms \(\delta_2\) and \(\delta_m\) to be

\[
\delta_2 = - \frac{g^2}{16\pi^2} C_2(r) \left( \frac{2}{\epsilon} \right). 
\]

(3.321)

\[
\delta_m = - \frac{g^2}{16\pi^2} C_2(r) \left( \frac{8m}{\epsilon} \right). 
\]

(3.322)

These two counter terms allow us to determine the renormalized mass \(m\) in terms of the bare mass up to the one-loop order.

### 3.4.5 The Vertex at 1-Loop

The quark-gluon vertex at one-loop is given by

\[
\Gamma_{\alpha\beta}^{ab}(y, z) = \frac{\delta^3 \Gamma}{\delta \psi^A_\alpha(x) \delta \psi^A_\beta(y) \delta A^A_\mu(z)} \bigg|_{A, \psi, c=0} \\
+ \frac{\delta^3 S}{\delta \psi^A_\alpha(x) \delta \psi^A_\beta(y) \delta A^A_\mu(z)} \bigg|_{A, \psi, c=0} + \frac{1}{\mu} \frac{\delta^3 \Gamma_1}{\delta \psi^A_\alpha(x) \delta \psi^A_\beta(y) \delta A^A_\mu(z)} \bigg|_{A, \psi, c=0} \\
g(t^A)_{\alpha\beta} \delta^4(x - y) \delta^4(y - z) + \frac{1}{\mu} \frac{\delta^3 \Gamma_1}{\delta \psi^A_\alpha(x) \delta \psi^A_\beta(y) \delta A^A_\mu(z)} \bigg|_{A, \psi, c=0}. 
\]

(3.323)
In this section we compute the one-loop correction using Feynman rules directly. We write

\[ \int d^4x \int d^4y \int d^4z e^{-ikx-ipy-ilz} \Gamma^A_{a\beta\mu}(x, y, z) = g(t^A)_{ab}(\gamma^\mu)_{a\beta}(2\pi)^4\delta^4(k+p+l) + \frac{1}{i} \int d^4x \int d^4y \int d^4z e^{-ikx-ipy-ilz} \delta^A_{\alpha\beta}(y) \delta^B_{\gamma\beta}(z) |A, \psi, c\rangle = \left[ g(t^A)_{ab}(\gamma^\mu)_{a\beta} + \frac{1}{i} \right] (\text{Feynman diagrams}) (2\pi)^4\delta^4(k+p+l). \]

It is not difficult to convince ourselves that there are only two possible Feynman diagrams contributing to this 3-point proper vertex which we will only evaluate their leading divergent part in the Feynman-’t Hooft gauge. The first diagram on figure 12 is given explicitly by

\[ 12a = -ig^3 f_{CD\Lambda}(t^D, t^C)_{ab} \int \frac{d^d k}{(2\pi)^d} \left[ (-k + p_1 - 2p_2)^\alpha \gamma^\lambda \eta^\mu - (k + 2p_1 - p_2)^\alpha \eta^\rho + (2k + p_1 + p_2)^\mu \eta^\lambda \right] \]

\[ \times \frac{(\gamma^\lambda\gamma^\mu + m) \gamma^\rho}{(k^2 - m^2)(k + p_1)^2(k + p_2)^2} \]

\[ = -g^3 C_2(G) (t^A)_{ab} \int \frac{d^d k}{(2\pi)^d} \left[ (-k + p_1 - 2p_2)^\alpha \gamma^\lambda \eta^\mu - (k + 2p_1 - p_2)^\alpha \eta^\rho + (2k + p_1 + p_2)^\mu \eta^\lambda \right] \]

\[ \times \frac{(\gamma^\lambda\gamma^\mu + m) \gamma^\rho}{(k^2 - m^2)(k + p_1)^2(k + p_2)^2}. \] (3.325)

In the second line we have used the fact that \( f_{CD\Lambda t^D t^C} = f_{CD\Lambda t^D, t^C}/2 = if_{CD\Lambda f_D C E}\sqrt{E}/2. \)

We make now the approximation of neglecting the quark mass and all external momenta since the divergence is actually independent of both \( \gamma^\mu \). The result reduces to

\[ 12a = -g^3 C_2(G) (t^A)_{ab} \int \frac{d^d k}{(2\pi)^d} \left[ -k^\rho \eta^\lambda - k^\lambda \eta^\rho + 2k^\mu \eta^\lambda \right] \]

\[ \times \frac{(\gamma^\lambda\gamma^\mu + m) \gamma^\rho}{(k^2)^3} \]

\[ = -g^3 C_2(G) (t^A)_{ab} \int \frac{d^d k}{(2\pi)^d} \left[ -2(\gamma^\mu)_{a\beta} k^2 - 2(2 - \epsilon) k^d \eta^\mu (\gamma^\nu)_{a\beta} \right] \frac{1}{(k^2)^3} \]

\[ = -g^3 C_2(G) (t^A)_{ab} \int \frac{d^d k}{(2\pi)^d} \left[ -2(\gamma^\mu)_{a\beta} k^2 - 2(2 - \epsilon) k^d \eta^\mu (\gamma^\nu)_{a\beta} \right] \frac{1}{(k^2)^3} \]

\[ = \frac{g^3 C_2(G)}{2} \frac{4(d - 1)}{d} (t^A)_{ab} \left( \gamma^\mu \right)_{a\beta} \int \frac{d^d k}{(2\pi)^d (k^2)^2} \frac{1}{d^2}. \] (3.326)

\( \text{Exercise: Compute this integral without making these approximations and show that the divergence is indeed independent of the quark mass and all external momenta.} \)
Again if the quarks are in the representation $t^C$ we can make the approximation of dropping the quark mass and all external momenta.

$$12b = g^3(t^C t^A t^C)_{ab} \int \frac{d^d k}{(2\pi)^d} \frac{\left(\lambda_\alpha(\gamma_\beta(k + p_2) + m_\gamma^\mu(\gamma_\beta(k + p_1) + m_\gamma^\lambda)\right)}{k^2((k + p_1)^2 - m^2)((k + p_2)^2 - m^2)}. \quad (3.327)$$

We compute

$$t^C t^A t^C = t^C t^C t^A + t^C[t^A, t^C] = C_2(N)t^A + i f_{ABC} t^C t^B$$

and

$$12b = g^3(C_2(N) - \frac{1}{2}C_2(G))(t^A)_{ab} \int \frac{d^d k}{(2\pi)^d} \frac{\left(\gamma_\lambda(\gamma_\beta(k + p_2) + m_\gamma^\mu(\gamma_\beta(k + p_1) + m_\gamma^\lambda)\right)}{k^2((k + p_1)^2 - m^2)((k + p_2)^2 - m^2)}. \quad (3.329)$$

Again as before we are only interested at this stage in the leading divergent part and thus we can make the approximation of dropping the quark mass and all external momenta.\(^\text{13}\)

We obtain thus

$$12b = g^3(C_2(N) - \frac{1}{2}C_2(G))(t^A)_{ab} \int \frac{d^d k}{(2\pi)^d} \frac{\left(\gamma_\lambda(\gamma_\beta(k + p_2) + m_\gamma^\mu(\gamma_\beta(k + p_1) + m_\gamma^\lambda)\right)}{(k^2)^3}\frac{k^\rho k^\sigma}{k^2}} {k^2} \quad (3.330)$$

By putting the two results $12a$ and $12b$ together we obtain

$$12a + 12b = \frac{ig^3}{(4\pi)^2}(C_2(N) + C_2(G))(t^A)_{ab}(\gamma^\mu)_{\alpha\beta}\Gamma(2 - \frac{d}{2}). \quad (3.331)$$

Again if the quarks are in the representation $t^6$ instead of the fundamental representation $t^a$ we would have obtained

$$12a + 12b = \frac{ig^3}{(4\pi)^2}(C_2(r) + C_2(G))(t^A)_{ab}(\gamma^\mu)_{\alpha\beta}\Gamma(2 - \frac{d}{2}). \quad (3.332)$$

\(^\text{13}\)Exercise: Compute this integral without making these approximations.
The dressed quark-gluon vertex at one-loop is therefore given by
\[
\int d^4x \int d^4y \int d^4z e^{-ikx-ipy-ilz} \Gamma^{abA}_{\alpha\beta\mu}(x,y,z) = \left[ g(t^A)_{ab}(\gamma^\mu)_{\alpha\beta} + \frac{g^3}{(4\pi)^2} (C_2(r) + C_2(G))(t^A)_{ab}(\gamma^\mu)_{\alpha\beta} \Gamma(2 - \frac{d}{2}) \right] (2\pi)^4 \delta^4(k + p + l)
\]
\[
\times (2\pi)^4 \delta^4(k + p + l)
\]
\[
\times \left[ g(t^A)_{ab}(\gamma^\mu)_{\alpha\beta} + \frac{g^3}{(4\pi)^2} (C_2(r) + C_2(G))(t^A)_{ab}(\gamma^\mu)_{\alpha\beta} \right] (\frac{2}{\epsilon} + \ldots)
\]

Adding the contribution of the counter terms is trivial since the relevant counter term is of the same form as the bare vertex. We get
\[
\int d^4x \int d^4y \int d^4z e^{-ikx-ipy-ilz} \Gamma^{abA}_{\alpha\beta\mu}(x,y,z) = \left[ g(t^A)_{ab}(\gamma^\mu)_{\alpha\beta} + \frac{g^3}{(4\pi)^2} (C_2(r) + C_2(G))(t^A)_{ab}(\gamma^\mu)_{\alpha\beta} \right] (2\pi)^4 \delta^4(k + p + l).
\]
We conclude that, in order to subtract the logarithmic divergence in the vertex, the counter term \( \delta_1 \) must be chosen such that
\[
\delta_1 = - \frac{g^2}{(4\pi)^2} (C_2(r) + C_2(G)) \left( \frac{2}{\epsilon} \right). \tag{3.335}
\]

In a more careful treatment we should get
\[
\delta_1 = - \frac{g^2}{(4\pi)^2} (C_2(r) + C_2(G)) \left( \frac{2}{\epsilon} \right). \tag{3.336}
\]

We recall that the renormalized coupling \( g \) is related to the bare coupling \( g_0 \) by the relation
\[
\frac{g}{g_0} = \frac{Z_2 \sqrt{Z_3}}{1 + \delta_1}
\]
\[
= 1 - \delta_1 + \delta_2 + \frac{1}{2} \delta_3
\]
\[
= 1 + \frac{g^2}{16\pi^2} \frac{1}{\epsilon} \left[ \frac{11}{3} C_2(G) - \frac{4}{3} n_f C(r) \right]
\]
\[
= 1 + \mu^{-\epsilon} \frac{g^2}{16\pi^2} \frac{1}{\epsilon} \left[ \frac{11}{3} C_2(G) - \frac{4}{3} n_f C(r) \right]. \tag{3.337}
\]
This is equivalent to
\[
g = g_0 + \mu^{-\epsilon} \frac{g_0^3}{16\pi^2} \frac{1}{\epsilon} \left[ \frac{11}{3} C_2(G) - \frac{4}{3} n_f C(r) \right]. \tag{3.338}
\]

\[\text{This should become apparent if you solve the previous two exercises.}\]
We compute then
\[
\mu \frac{\partial g}{\partial \mu} = -\mu^{-\epsilon} \frac{g_0^3}{16\pi^2} \left[ \frac{11}{3} C_2(G) - \frac{4}{3} n_f C(r) \right] \\
= -\frac{g^3}{16\pi^2} \left[ \frac{11}{3} C_2(G) - \frac{4}{3} n_f C(r) \right]. 
\] (3.339)
\[ A, \mu \rightarrow \bar{\psi}_\alpha^a \rightarrow \psi^b_\beta \rightarrow B, \nu \]

\[ = \delta^{AB} \left[ \eta_{\mu\nu} + (\xi - 1) \frac{k_\mu k_\nu}{k^2} \right]. \]

\[ \bar{\psi}^a_\alpha \rightarrow \psi^b_\beta \]

\[ = \delta^{ab} (\gamma.p + m)_{\beta\alpha} \]

\[ \frac{p^2 - m^2}{p^2}. \]

\[ e^A \rightarrow e^B \]

\[ = \frac{\delta^{AB}}{p^2}. \]

\[ \bar{\psi}^a_\alpha \rightarrow \mu, A \rightarrow \psi^b_\beta \]

\[ = g(t^A)_{ab} (\gamma^\mu)_{a\beta}. \]

\[ \nu, B, p \rightarrow \rho, C, q \rightarrow \mu, A, k \]

\[ = ig f_{ABC} \left[ (k + 2p)^\mu \eta^{\rho\nu} - (p + 2k)^\nu \eta^{\rho\mu} + (k - p)^\rho \eta^{\mu\nu} \right]. \]
\[\begin{align*}
\rho, D & \quad \sigma, E \\
\mu, A & \quad \nu, B \\
\frac{\partial}{\partial x^{\alpha}} & = -g^{2} \left[ f_{ABC} f_{DEC} (\eta^{\alpha \mu} \eta^{\rho \nu} - \eta^{\alpha \rho} \eta^{\mu \nu}) + f_{BDC} f_{AEC} (\eta^{\sigma \mu} \eta^{\rho \nu} - \eta^{\rho \mu} \eta^{\sigma \nu}) \\
& \quad + f_{DAC} f_{BEC} (\eta^{\sigma \mu} \eta^{\rho \nu} - \eta^{\rho \mu} \eta^{\sigma \nu}) \right].
\end{align*}\]
4.1 Critical Phenomena and The $\phi^4$ Theory

4.1.1 Critical Line and Continuum Limit

We are interested in the critical properties of systems which are ergodic at finite volume, i.e. they can access all regions of their phase space with non zero probability. In the infinite volume limit these systems may become non ergodic and as a consequence the phase space decomposes into disjoint sets corresponding to different phases. The thermodynamical limit is related to the largest eigenvalue of the so-called transfer matrix. If the system remains ergodic then the largest eigenvalue of the transfer matrix is non degenerate while it becomes degenerate if the system becomes non ergodic.

The boundary between the different phases is demarcated by a critical line or a second order phase transition which is defined by the requirement that the correlation length, which is the inverse of the smallest decay rate of correlation functions or equivalent the smallest physical mass, diverges at the transition point.

The properties of these systems near the transition line are universal and are described by the renormalization group equations of Euclidean scalar field theory. The requirement of locality in field theory is equivalent to short range forces in second order phase transitions. The property of universality is intimately related to the property of renormalizability of the field theory. More precisely universality in second order phase transitions emerges in the regime in which the correlation length is much larger than the macroscopic scale which corresponds, on the field theory side, to the fact that renormalizable local field theory is insensitive to short distance physics in the sense that we obtain a unique renormalized Lagrangian in the limit in which all masses and momenta are much smaller than the UV cutoff $\Lambda$.

The Euclidean $O(N)\phi^4$ action is given by (with some change of notation compared
to previous chapters and sections)

\[
S[\phi] = - \int d^d x \left( \frac{1}{2} (\partial_\mu \phi^i)^2 + \frac{1}{2} m^2 \phi^i \phi^i + \frac{\lambda}{4} (\phi^i \phi^i)^2 \right). \tag{4.1}
\]

We will employ lattice regularization in which \(x = an\), \(\int d^d x = a^d \sum_n\), \(\phi^i(x) = \phi^i_n\) and \(\partial_\mu \phi^i = (\phi^i_{n+\hat{\mu}} - \phi^i_n)/a\). The lattice action reads

\[
S[\phi] = \sum_n \left( a^{d-2} \sum_\mu \phi^i_n \phi^i_{n+\hat{\mu}} - \frac{a^{d-2}}{2} (m^2 a^2 + 2d) \phi^i_n \phi^i_n - \frac{a^d \lambda}{4} (\phi^i_n \phi^i_n)^2 \right)
= \sum_n \left( 2\kappa \sum_\mu \Phi^i_n \Phi^i_{n+\hat{\mu}} - \Phi^i_n \Phi^i_n - g(\Phi^i_n \Phi^i_n - 1)^2 \right). \tag{4.2}
\]

The mass parameter \(m^2\) is replaced by the so-called hopping parameter \(\kappa\) and the coupling constant \(\lambda\) is replaced by the coupling constant \(g\) where

\[
m^2 a^2 = \frac{1 - 2g}{\kappa} - 2d , \quad \frac{\lambda}{a^{d-4}} = \frac{g}{\kappa^2}. \tag{4.3}
\]

The fields \(\phi^i_n\) and \(\Phi^i_n\) are related by

\[
\phi^i_n = \sqrt{\frac{2\kappa}{a^{d-2}}} \Phi^i_n. \tag{4.4}
\]

\[
m^2 a^2 = \frac{1 - 2g}{\kappa} - 2d , \quad \frac{\lambda}{a^{d-4}} = \frac{g}{\kappa^2}. \tag{4.5}
\]

The partition function is given by

\[
Z = \int \prod_n d\Phi^i_n e^{S[\phi]}
= \int d\mu(\Phi) e^{2\kappa \sum_n \Phi^i_n \Phi^i_n - g(\Phi^i_n \Phi^i_n - 1)^2}. \tag{4.6}
\]

The measure \(d\mu(\phi)\) is given by

\[
d\mu(\Phi) = \prod_n d\Phi^i_n e^{-\sum_n (\Phi^i_n \Phi^i_n + g(\Phi^i_n \Phi^i_n - 1)^2)}
= \prod_n (d^N \tilde{\Phi}_n e^{-\tilde{\Phi}_n^2 - g(\tilde{\Phi}_n^2 - 1)^2})
= \prod_n d\mu(\Phi_n). \tag{4.7}
\]

This is a generalized Ising model. Indeed in the limit \(g \to \infty\) the dominant configurations are such that \(\Phi^1_n + ... + \Phi^N_n = 1\), i.e. points on the sphere \(S^{N-1}\). Hence
\[
\int \frac{d\mu(\Phi_n) f(\Phi_n)}{d\mu(\Phi_n)} = \int \frac{d\Omega_{N-1} f(\Phi_n)}{d\Omega_{N-1}}, \ g \to \infty. \tag{4.8}
\]

For \( N = 1 \) we obtain
\[
\int \frac{d\mu(\Phi_n) f(\Phi_n)}{d\mu(\Phi_n)} = \frac{1}{2} (f(+1) + f(-1)) , \ g \to \infty. \tag{4.9}
\]

Thus the limit \( g \to \infty \) of the \( O(1) \) model is precisely the Ising model in \( d \) dimensions. The limit \( g \to \infty \) of the \( O(3) \) model corresponds to the Heisenberg model in \( d \) dimensions. The \( O(N) \) models on the lattice are thus intimately related to spin models.

There are two phases in this model. A disordered (paramagnetic) phase characterized by \( <\Phi_i^n> = 0 \) and an ordered (ferromagnetic) phase characterized by \( <\Phi_i^n> = v_i \neq 0 \). This can be seen in various ways. The easiest way is to look for the minima of the classical potential
\[
V[\phi] = -\int d^d x \left( \frac{1}{2} m^2 \phi^i \phi^i + \frac{\lambda}{4} (\phi^i \phi^i)^2 \right). \tag{4.10}
\]

The equation of motion reads
\[
[m^2 + \frac{\lambda}{2} \phi^j \phi^j] \phi^i = 0. \tag{4.11}
\]

For \( m^2 > 0 \) there is a unique solution \( \phi^i = 0 \) whereas for \( m^2 < 0 \) there is a second solution given by \( \phi^i \phi^j = -2m^2/\lambda \).

A more precise calculation is as follows. Let us compute the expectation value \( <\Phi_i^n> \) on the lattice which is defined by
\[
<\phi_i^n> = \int \frac{d\mu(\Phi_n) \Phi_i^n e^{2\kappa \sum_n \sum_{\mu} \Phi_i^n \Phi_{n+\mu}^i}}{d\mu(\Phi_n) e^{2\kappa \sum_n \sum_{\mu} \Phi_i^n \Phi_{n+\mu}^i}} = \int \frac{d\mu(\Phi_n) \Phi_i^n e^{\kappa \sum_n \Phi_i^n \sum_{\mu} (\Phi_{n+\mu}^i + \Phi_{n-\mu}^i)}}{d\mu(\Phi_n) e^{\kappa \sum_n \Phi_i^n \sum_{\mu} (\Phi_{n+\mu}^i + \Phi_{n-\mu}^i)}}. \tag{4.12}
\]

Now we approximate the spins \( \Phi_i^n \) at the \( 2d \) nearest neighbors of each spin \( \Phi_i^n \) by the average \( v^i = <\Phi_i^n> \), viz
\[
\frac{\sum_{\mu} (\Phi_{n+\mu}^i + \Phi_{n-\mu}^i)}{2d} = v^i. \tag{4.13}
\]

This is a crude form of the mean field approximation. Equation (4.12) becomes
\[
v^i = \int \frac{d\mu(\Phi_n) \Phi_i^n e^{4\kappa d \sum_n \Phi_i^n v^i}}{d\mu(\Phi_n) e^{4\kappa d \sum_n \Phi_i^n v^i}} \tag{4.14}
\]
The extra factor of 2 in the exponents comes from the fact the coupling between any two nearest neighbor spins on the lattice occurs twice. We write the above equation as

\[ v^i = \frac{\partial}{\partial J^i} \ln Z[J] |_{J^i=4\kappa_c d v^i}, \]  
(4.15)

\[ Z[J] = \int d\mu(\Phi_n) e^{\Phi^i_n J^i} \]

\[ = \int d^N \Phi^i_n e^{-\Phi^i_n \Phi^i_n - g(\Phi^i_n \Phi^i_n - 1)^2 + \Phi^i_n J^i}. \]  
(4.16)

**The limit** \( g \rightarrow 0 \): In this case we have

\[ Z[J] = \int d^N \Phi^i_n e^{-\Phi^i_n \Phi^i_n + \Phi^i_n J^i} = Z[0] e^{\frac{J^i J^i}{4}}. \]  
(4.17)

In other words

\[ v^i = 2\kappa_c d v^i \Rightarrow \kappa_c = \frac{1}{2d}. \]  
(4.18)

**The limit** \( g \rightarrow \infty \): In this case we have

\[ Z[J] = \mathcal{N} \int d^N \Phi^i_n \delta(\Phi^i_n \Phi^i_n - 1) e^{\Phi^i_n J^i} \]

\[ = \mathcal{N} \int d^N \Phi^i_n \delta(\Phi^i_n \Phi^i_n - 1) \left[ 1 + \Phi^i_n J^i + \frac{1}{2} \Phi^i_n \Phi^j_n J^i J^j + ... \right]. \]  
(4.19)

By using rotational invariance in \( N \) dimensions we obtain

\[ \int d^N \Phi^i_n \delta(\Phi^i_n \Phi^i_n - 1) \Phi^i_n = 0. \]  
(4.20)

\[ \int d^N \Phi^i_n \delta(\Phi^i_n \Phi^i_n - 1) \Phi^i_n \Phi^i_n = \frac{\delta^{ij}}{N} \int d^N \Phi^i_n \delta(\Phi^i_n \Phi^i_n - 1) \Phi^k_n \Phi^k_n = \frac{\delta^{ij} Z[0]}{N^2 \mathcal{N}}. \]  
(4.21)

Hence

\[ Z[J] = Z[0] \left[ 1 + \frac{J^i J^i}{2N} + ... \right]. \]  
(4.22)

Thus

\[ v^i = \frac{J^i}{N} = \frac{4\kappa_c d v^i}{N} \Rightarrow \kappa_c = \frac{N}{4d}. \]  
(4.23)
The limit of The Ising Model: In this case we have
\[ N = 1, \ g \to \infty. \]  
(4.24)

We compute then
\[ Z[J] = N \int d\Phi_n \delta(\Phi_n^2 - 1) e^{\Phi_n J} \]
\[ = Z[0] \cosh J. \]  
(4.25)
Thus
\[ v = \tanh 4\kappa dv. \]  
(4.26)

A graphical sketch of the solutions of this equation is shown on figure 17. Clearly for \( \kappa \) near \( \kappa_c \) the solution \( v \) is near 0 and thus we can expand the above equation as
\[ v = 4\kappa dv - \frac{1}{3}(4\kappa d)^3 v^2 + \ldots \]  
(4.27)
The solution is
\[ \frac{1}{3}(4d)^2 \kappa^3 v^2 = \kappa - \kappa_c. \]  
(4.28)
Thus only for \( \kappa > \kappa_c \) there is a non zero solution.

In summary we have the two phases
\[ \kappa > \kappa_c : \text{broken, ordered, ferromagnetic} \]  
(4.29)
\[ \kappa < \kappa_c : \text{symmetric, disordered, paramagnetic}. \]  
(4.30)
The critical line \( \kappa_c = \kappa_c(g) \) interpolates in the \( \kappa - g \) plane between the two lines given by
\[ \kappa_c = \frac{N}{4d}, \ g \to \infty. \]  
(4.31)
\[ \kappa_c = \frac{1}{2d}, \ g \to 0. \]  
(4.32)
See figure 18.

For \( d = 4 \) the critical value at \( g = 0 \) is \( \kappa_c = 1/8 \) for all \( N \). This critical value can be derived in a different way as follows. From equation (2.172) we know that the renormalized mass at one-loop order in the continuum \( \phi^4 \) with \( O(N) \) symmetry is given by the equation (with \( \lambda \to 6\lambda \))
\[ m_R^2 = m^2 + (N + 2)\lambda I(m^2, \Lambda) \]
\[ = m^2 + \frac{(N + 2)\lambda}{16\pi^2} \Lambda^2 + \frac{(N + 2)\lambda}{16\pi^2} m^2 \ln \frac{m^2}{\Lambda^2} + \frac{(N + 2)\lambda}{16\pi^2} m^2 C + \text{finite terms}. \]  
(4.33)
This equation reads in terms of dimensionless quantities as follows
\[
a^2 m_R^2 = a m^2 + \frac{(N + 2)\lambda}{16\pi^2} + \frac{(N + 2)\lambda}{16\pi^2} a^2 m^2 \ln a^2 m^2 + \frac{(N + 2)\lambda}{16\pi^2} a^2 m^2 C + a^2 \times \text{finite terms.}
\] (4.34)

The lattice space \( a \) is formally identified with the inverse cut off \( 1/\Lambda \), viz
\[
a = \frac{1}{\Lambda}.
\] (4.35)

Thus we obtain in the continuum limit \( a \rightarrow 0 \) the result
\[
a^2 m^2 \rightarrow -\frac{(N + 2)\lambda}{16\pi^2} + \frac{(N + 2)\lambda}{16\pi^2} a^2 m^2 \ln a^2 m^2 + \frac{(N + 2)\lambda}{16\pi^2} a^2 m^2 C + a^2 \times \text{finite terms.}
\] (4.36)

In other words (with \( r_0 = (N + 2)/8\pi^2 \))
\[
a^2 m^2 \rightarrow a^2 m_c^2 = -\frac{r_0}{2} \lambda + O(\lambda^2).
\] (4.37)

This is the critical line for small values of the coupling constant as we will now show. Expressing this equation in terms of \( \kappa \) and \( g \) we obtain
\[
\frac{1 - 2g}{\kappa} - 8 \rightarrow -\frac{r_0}{2} \frac{g}{\kappa^2} + O(\lambda^2).
\] (4.38)

This can be brought to the form
\[
\left[ \kappa - \frac{1}{16} (1 - 2g) \right]^2 \rightarrow \frac{1}{256} \left[ 1 + 16r_0g - 4g \right] + O(g^2/\kappa^2).
\] (4.39)

We get the result
\[
\kappa \rightarrow \kappa_c = \frac{1}{8} + (\frac{r_0}{2} - \frac{1}{4})g + O(g^2).
\] (4.40)

This result is of fundamental importance. The continuum limit \( a \rightarrow 0 \) corresponds precisely to the limit in which the mass approaches its critical value. This happens for every value of the coupling constant and hence the continuum limit \( a \rightarrow 0 \) is the limit in which we approach the critical line. The continuum limit is therefore a second order phase transition.

### 4.1.2 Mean Field Theory

We start from the partition function of an \( O(1) \) model given by
\[
Z(J) = \int d\mu(\Phi_n) e^{\sum_{n,m} \Phi_n V_{nm} \Phi_m + \sum_n J_n \Phi_n}.
\] (4.41)
The positive matrix $V_{nm}$ (for the case of ferromagnetic interactions with $\kappa > 0$) is defined by

$$V_{nm} = \kappa \sum_\mu (\delta_{m,n+\mu} + \delta_{m,n-\mu}). \tag{4.42}$$

The measure is defined by

$$d\mu(\Phi_n) = d\Phi_n e^{-\Phi_n^2 - g(\Phi_n^2 - 1)^2}. \tag{4.43}$$

We introduce the Hubbard transformation

$$\int \prod_n dX_n e^{-\frac{1}{4} \sum_{m,m} X_n V_{nm}^{-1} X_m + \sum_n \Phi_n X_n} = Ke^{\sum_{n,m} \Phi_n V_{nm} \Phi_m}. \tag{4.44}$$

We obtain

$$Z(J) = \frac{1}{K} \int \prod_n dX_n e^{-\frac{1}{4} \sum_{m,m} X_n V_{nm}^{-1} X_m \sum_n d\mu(\Phi_n) e^{\sum_n (X_n + J_n) \Phi_n}}.$$

$$= \frac{1}{K} \int \prod_n dX_n e^{-\frac{1}{4} \sum_{m,m} X_n V_{nm}^{-1} X_m - \sum_n A(X_n + J_n)}. \tag{4.45}$$

The function $A$ is defined by

$$A(X_n + J_n) = -\ln z(X_n + J_n), \quad z(X_n + J_n) = \int d\mu(\Phi_n) e^{(X_n + J_n) \Phi_n}. \tag{4.46}$$

In the case of the Ising model we have explicitly

$$z(X_n + J_n) = \int d\mu(\Phi_n) e^{(X_n + J_n) \Phi_n} = K^{\frac{1}{2}} \left(e^{(X_n + J_n)} + e^{-(X_n + J_n)} \right) = K c \cosh(X_n + J_n). \tag{4.47}$$

We introduce a new variable $\phi_n$ as follows

$$\phi_n = X_n + J_n. \tag{4.48}$$

The partition function becomes (using also the fact that $V$ and $V^{-1}$ are symmetric matrices)

$$Z(J) = \frac{1}{K} \int \prod_n d\phi_n e^{-\frac{1}{4} \sum_{m,m} \phi_n V_{nm}^{-1} \phi_m + \frac{i}{2} \sum_{n,m} \phi_n V_{nm}^{-1} J_m - \frac{1}{4} \sum_{n,m} J_n V_{nm}^{-1} J_m - \sum_n A(\phi_n)}. \tag{4.49}$$

We replace $V_{ij}$ by $W_{ij} = V_{ij}/L$ and we replace every spin $\Phi_n$ by $\hat{\Phi}_n = \sum_{l=1}^L \Phi_n^l$, i.e. by the sum of $L$ spins $\Phi_n^l$ which are assumed to be distributed with the same probability $d\mu(\Phi_n^l)$. We get the partition function
\[ Z(J) = \int \prod_{n,l} d\mu(\Phi_n) e^{\sum_{n,m} \hat{\Phi}_n W_{nm} \hat{\Phi}_m + \sum_n J_n \hat{\Phi}_n} \]
\[ = \frac{1}{K} \int \prod_n dX_n e^{-\frac{1}{4} \sum_{n,m} X_n W_{nm}^{-1} X_m \left( \int \prod_n d\mu(\Phi_n) e^{\sum_n (X_n + J_n) \Phi_n} \right)^L} \]
\[ = \frac{1}{K} \int \prod_n d\phi_n e^{-L \left[ \frac{1}{4} \sum_{n,m} \phi_n V_{nm}^{-1} \phi_m - \frac{1}{2} \sum_{n,m} \phi_n V_{nm}^{-1} J_m + \frac{1}{4} \sum_{n,m} J_n V_{nm}^{-1} J_m + \sum_n A(\phi_n) \right]} \]
\[ \equiv \frac{1}{K} \int \prod_n d\phi_n e^{-L V(\phi_n)}. \] (4.50)

In the limit \( L \rightarrow \infty \) we can apply the saddle point method. The partition function is dominated by the configuration which solves the equation of motion
\[ \frac{dV}{d\phi_n} = 0 \iff \phi_n - J_n + 2 \sum_m V_{nm} \frac{dA}{d\phi_m} = 0. \] (4.51)

In other words we replace the field at each site by the best equivalent magnetic field. This approximation performs better at higher dimensions. Clearly steepest descent allows an expansion in powers of \( 1/L \). We see that mean field is the tree level approximation of the field theory obtained from (4.50) by neglecting the quadratic term in \( J_n \) and redefining the current \( J_n \) as \( J_n^{\text{redefined}} = \sum_m V_{nm}^{-1} J_m/2 \).

The partition function becomes (up to a multiplicative constant factor)
\[ Z(J) = e^{-L \left[ \frac{1}{4} \sum_{n,m} \phi_n V_{nm}^{-1} \phi_m - \frac{1}{2} \sum_{n,m} \phi_n V_{nm}^{-1} J_m + \frac{1}{4} \sum_{n,m} J_n V_{nm}^{-1} J_m + \sum_n A(\phi_n) \right]}|_{\text{saddle point}}. \] (4.52)

The vacuum energy (which plays the role of the thermodynamic free energy) is then given by
\[ W(J) = \frac{1}{L} \ln Z[J] \]
\[ = -\left[ \frac{1}{4} \sum_{n,m} \phi_n V_{nm}^{-1} \phi_m - \frac{1}{2} \sum_{n,m} \phi_n V_{nm}^{-1} J_m + \frac{1}{4} \sum_{n,m} J_n V_{nm}^{-1} J_m + \sum_n A(\phi_n) \right]|_{\text{saddle point}}. \] (4.53)

The order parameter is the magnetization which is conjugate to the magnetic field \( J_n \). It is defined by
\[ M_m = \frac{\partial W}{\partial J_m} \]
\[ = \frac{1}{2} \sum_n (\phi_n - J_n) V_{nm}^{-1} \]
\[ = -\frac{dA}{d\phi_m}. \] (4.54)
The effective action (which plays the role of the thermodynamic energy) is the Legendre transform of $W(J)$ defined by
\[
\Gamma(M) = \sum_n M_n J_n - W(J) \\
= \sum_n M_n J_n + \sum_{n,m} M_n V_{nm} M_m + \sum_n A(\phi_n) \\
= -\sum_{n,m} M_n V_{nm} M_m + \sum_n B(M_n).
\] (4.55)

The function $B(M_n)$ is the Legendre transform of $A(\phi_n)$ given by
\[
B(M_n) = M_n \phi_n + A(\phi_n). \tag{4.56}
\]

For the Ising model we compute (up to an additive constant)
\[
A(\phi_n) = -\ln \cosh \phi_n \\
= -\phi_n - \ln \frac{1 + e^{-2\phi_n}}{2}.
\] (4.57)

The magnetization in the Ising model is given by
\[
M_n = \frac{1 - e^{-2\phi_n}}{1 + e^{-2\phi_n}} \iff \phi_n = \frac{1}{2} \ln(1 + M_n) - \frac{1}{2} \ln(1 - M_n).
\] (4.58)

Thus
\[
A(\phi_n) = \frac{1}{2} \ln(1 + M_n) + \frac{1}{2} \ln(1 - M_n). \tag{4.59}
\]
\[
B(M_n) = \frac{1}{2} (1 + M_n) \ln(1 + M_n) + \frac{1}{2} (1 - M_n) \ln(1 - M_n). \tag{4.60}
\]

From the definition of the effective potential we get the equation of motion
\[
\frac{\partial \Gamma}{\partial M_n} = -2 \sum_m V_{nm} M_m + \frac{\partial B}{\partial M_n} = (J_n - \phi_n) + \phi_n = J_n.
\] (4.61)

Thus for zero magnetic field the magnetization is given by an extremum of the effective potential. On the other hand the partition function for zero magnetic field is given by $Z = \exp(-L\Gamma)$ and hence the saddle point configurations which dominate the partition function correspond to extrema of the effective potential.

In systems where translation is a symmetry of the physics we can assume that the magnetization is uniform, i.e. $M_n = M = \text{constant}$ and as a consequence the effective potential per degree of freedom is given by
\[
\frac{\Gamma(M)}{N} = -vM^2 + B(M).
\] (4.62)
The number $N$ is the total number of degrees of freedom, viz $N = \sum_n 1$. The positive parameter $v$ is finite for short range forces and plays the role of the inverse temperature $\beta = 1/T$. It is given explicitly by

$$v = \frac{\sum_{n,m} V_{nm}}{N}. \quad (4.63)$$

It is a famous exact result of statistical mechanics that the effective potential $\Gamma(M)$ is a convex function of $M$, i.e. for $M, M_1$ and $M_2$ such that $M = xM_1 + (1 - x)M_2$ with $0 < x < 1$ we must have

$$\Gamma(M) \leq x\Gamma(M_1) + (1 - x)\Gamma(M_2). \quad (4.64)$$

This means that a linear interpolation is always greater than the potential which means that $\Gamma(M)$ is an increasing function of $M$ for $|M| \to \infty$. This can be made more precise as follows. First we compute

$$\frac{d^2A}{d\phi^2} = -\langle (\Phi - <\Phi>)^2 \rangle. \quad (4.65)$$

Thus $-d^2A/d\phi^2 > 0$ and as a consequence $A$ is a convex function of $\phi$. From the definition of the partition function $z(\phi)$ and the explicit form of the measure $d\mu(\Phi)$ we can see that $\Phi \to 0$ for $\phi \to \pm\infty$ and hence we obtain the condition

$$\frac{d^2A}{d\phi^2} \to 0, \quad \phi \to \infty. \quad (4.66)$$

Since $M = <\Phi>$ this condition also means that $M^2 - <\Phi^2> \to 0$ for $\phi \to \pm\infty$. Now by differentiating $M_n = \partial W/\partial J_n$ with respect to $M_n$ and using the result $\partial J_n/\partial M_n = \partial^2\Gamma/\partial M_n^2$ we obtain

$$1 = \frac{\partial^2W}{\partial J_n^2} \frac{\partial^2\Gamma}{\partial M_n^2}. \quad (4.67)$$

We compute (using $V_{nn} = 0$) the result $\partial^2\Gamma/\partial M_n^2 = d^2B/dM_n^2$. By recalling that $\phi_n = X_n + J_n$ we also compute (using $V_{nn}^{-1} = 0$) the result $\partial^2W/\partial J_n^2 = -d^2A/d\phi_n^2$. Hence we obtain

$$-1 = \frac{d^2B}{dM_n^2} \frac{d^2A}{d\phi_n^2}. \quad (4.68)$$

Thus the function $B$ is also convex in the variable $M$. Furthermore the condition (4.65) leads to the condition that the function $B$ goes to infinity faster than $M^2$ for $M \to \pm\infty$ (or else that $|M|$ is bounded as in the case of the Ising model).

The last important remark is to note that the functions $A(\phi)$ and $B(M)$ are both even in their respective variables.

There are two possible scenario we now consider:

---

1Exercise: Verify this explicitly.

2Exercise: Verify this explicitly.
• **First Order Phase Transition:** For high temperature (small value of \( v \)) the effective action is dominated by the second term \( B(M) \) which is a convex function. The minimum of \( \Gamma(M) \) is \( M = 0 \). We start decreasing the temperature by increasing \( v \). At some \( T = T_c \) (equivalently \( v = v_c \)) new minima of \( \Gamma(M) \) appear which are degenerate with \( M = 0 \). For \( T < T_c \) the new minima become absolute minima and as a consequence the magnetization jumps discontinuously from 0 to a finite value corresponding to these new minima. See figure 14. In this case the second derivative of the effective potential at the minimum \( \Gamma''(0) \) is always strictly positive and as a consequence the correlation length, which is inversely proportional to the square root of \( \Gamma''(0) \), is always finite.

• **Second Order Phase Transition:** The more interesting possibility occurs when the minimum at the origin \( M = 0 \) becomes at some critical temperature \( T = T_c \) a maximum and simultaneously new minima appear which start moving away from the origin as we decreasing the temperature. The critical temperature \( T_c \) is defined by the condition \( \Gamma''(0) = 0 \) or equivalently

\[
2v_c = B''(0).
\]  

(4.69)

Above \( T_c \) we have only the solution \( M = 0 \) whereas below \( T_c \) we have two minima moving continuously away from the origin. In this case the magnetization remains continuous at \( v = v_c \) and as a consequence the transition is also termed continuous. Clearly the correlation length diverges at \( T = T_c \). See figure 14.

### 4.1.3 Critical Exponents in Mean Field

In the following we will only consider the second scenario. Thus we assume that we have a second order phase transition at some temperature \( T = T_c \) (equivalently \( v = v_c \)). We are interested in the thermodynamic of the system for temperatures \( T \) near \( T_c \). The transition is continuous and thus we can assume that the magnetization \( M \) is small near \( T = T_c \) and as a consequence we can expand the effective action (thermodynamic energy) in powers of \( M \). We write then

\[
\Gamma(M) = -\sum_{n,m} M_n V_{nm} M_m + \sum_n B(M_n)
\]

\[
= -\sum_{n,m} M_n V_{nm} M_m + \sum_n \left[ \frac{a}{2!} M_n^2 + \frac{b}{4!} M_n^4 + \ldots \right].
\]  

(4.70)

The function \( B(M_n) \) is the Legendre transform of \( A(\phi_n) \), i.e.

\[
B(M_n) = M_n \phi_n + A(\phi_n).
\]  

(4.71)

We expand \( A(\phi_n) \) in powers of \( \phi_n \) as

\[
A(\phi_n) = \frac{a}{2!} \phi_n^2 + \frac{b}{4!} \phi_n^4 + \ldots
\]  

(4.72)
Thus

\[ M_n = -\frac{dA}{d\phi_n} = -a' \phi_n - \frac{b'}{6} \phi_n^3 + \ldots \]  \hspace{1cm} (4.73)

We compute

\[ \frac{d^2A}{d\phi_n^2} = -\left( \frac{d^2B}{dM_n^2} \right)^{-1} \]

\[ = -\frac{1}{a} + \frac{b}{2a^2} M_n^2 + \ldots \]

\[ = -\frac{1}{a} + \frac{b}{2a^2} \phi_n^2 + \ldots \]  \hspace{1cm} (4.74)

By integration this equation we obtain

\[ A = -\frac{1}{2a} \phi_n^2 + \frac{b}{4!a^2} \phi_n^4 + \ldots \]  \hspace{1cm} (4.75)

Hence

\[ a' = -\frac{1}{a}, \quad b' = \frac{b}{a^3}. \]  \hspace{1cm} (4.76)

The critical temperature is given by the condition \( \Gamma''(0) = 0 \) (where \( \Gamma \) here denotes the effective potential \( \Gamma(M) = N(-vM^2 + B(M)) \)). This is equivalent to the condition \( B''(0) = 2v_c \) which gives the value (recall that the coefficient \( a \) is positive since \( B \) is convex)

\[ v_c = \frac{a}{2}. \]  \hspace{1cm} (4.77)

The equation of motion \( \Gamma'(0) = 0 \) gives the condition \( B'(M) = 2vM \). For \( v < v_c \) we have no spontaneous magnetization whereas for \( v > v_c \) we have a non zero spontaneous magnetization given by

\[ M = \sqrt{\frac{12}{b}} (v - v_c)^{1/2}. \]  \hspace{1cm} (4.78)

The magnetization is associated with the critical exponent \( \beta \) defined for \( T \) near \( T_c \) from below by

\[ M \sim (T_c - T)^\beta. \]  \hspace{1cm} (4.79)

We have clearly

\[ \beta = \frac{1}{2}. \]  \hspace{1cm} (4.80)
The inverse of the (magnetic) susceptibility is defined by (with $J$ being the magnetic field)

$$\chi^{-1} = \frac{\partial M}{\partial J} = \frac{\delta^2 \Gamma}{\delta M^2} = \mathcal{N}(-2v + a + \frac{b}{2}M^2). \quad (4.81)$$

We have the 2-cases

$$v < v_c \quad M = 0 \Rightarrow \chi^{-1} = 2(v_c - v)$$

$$v > v_c \quad M = \sqrt{\frac{12}{b}}(v - v_c)^{1/2} \Rightarrow \chi^{-1} = 4(v - v_c). \quad (4.82)$$

The susceptibility is associated with the critical exponent $\gamma$ defined by

$$\chi \sim |T - T_c|^{-\gamma}. \quad (4.83)$$

Clearly we have

$$\gamma = 1. \quad (4.84)$$

The quantum equation of motion (equation of state) relates the source (external magnetic field), the temperature and the spontaneous magnetization. It is given by

$$J = \frac{\partial \Gamma}{\partial M} = \mathcal{N}(2(v_c - v)M + \frac{b}{6}M^3) = \frac{\mathcal{N}b}{3}M^3. \quad (4.85)$$

The equation of state is associated with the critical exponent $\delta$ defined by

$$J \sim M^\delta. \quad (4.86)$$

Clearly we have

$$\delta = 3. \quad (4.87)$$

Let us derive the 2-point correlation function given by

$$G_{nm}^{(2)} = \left[ \frac{\delta^2 \Gamma}{\delta M_n \delta M_m} \right]^{-1} = \left[ -2V_{nm} + a\delta_{nm} + \frac{b}{2}M_n^2\delta_{nm} \right]^{-1}. \quad (4.88)$$
Define
\[ \Gamma^{(2)}_{nm} = -2V_{nm} + a\delta_{nm} + \frac{b}{2}M_n^2\delta_{nm}. \] (4.89)

The two functions \( g^{(2)}_{nm} \) and \( \Gamma^{(2)}_{nm} \) can only depend on the difference \( n - m \) due to invariance under translation. Thus Fourier transform is and its inverse are defined by
\[ K_{nm} = \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \tilde{K}(k) e^{i k (n - m)}, \quad \tilde{K}(k) = \sum_n K_{nm} e^{-i k (n - m)}. \] (4.90)

For simplicity we assume a uniform magnetization, viz \( M = M_n \). Thus
\[ \tilde{\Gamma}^{(2)}(k) = \sum_n \Gamma^{(2)}_{nm} e^{-i k (n - m)} = -2\tilde{V}(k) + a + \frac{b}{2}M^2. \] (4.91)

Hence
\[ G^{(2)}_{nm} = \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \frac{1}{-2\tilde{V}(k) + a + \frac{b}{2}M^2} e^{i k (n - m)}. \] (4.92)

The function \( \tilde{V}(k) \) is given explicitly by
\[ \tilde{V}(k) = \sum_n V_{nm} e^{-i k (n - m)}. \] (4.93)

We assume a short range interaction which means that the potential \( V_{nm} \) decays exponentially with the distance \( |n - m| \). In other words we must have
\[ V_{nm} < Me^{-\kappa|n-m|}, \quad \kappa > 0. \] (4.94)

This condition implies that the Fourier transform \( \tilde{V}(k) \) is analytic for \( |\text{Im } k| < \kappa \). Furthermore positivity of the potential \( V_{nm} \) and its invariance under translation gives the requirement
\[ |\tilde{V}(k)| \leq \sum_n V_{nm} = \tilde{V}(0) = v. \] (4.95)

For small momenta \( k \) we can then expand \( \tilde{V}(k) \) as
\[ \tilde{V}(k) = v(1 - \rho^2 k^2 + O(k^4)). \] (4.96)

The 2–point function admits therefore the expansion
\[ G^{(2)}_{nm} = \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \frac{\tilde{G}^{(2)}(0)}{1 + \xi^2 k^2 + O(k^4)} e^{i k (n - m)}. \] (4.97)

Exercise: Construct an explicit argument.
\begin{align*}
\tilde{G}^{(2)}(0) &= \frac{1}{2(v_c - v) + \frac{b}{2}M^2}. \quad (4.98) \\
\xi^2 &= \frac{2v\rho^2}{2(v_c - v) + \frac{b}{2}M^2}. \quad (4.99)
\end{align*}

The length scale $\xi$ is precisely the so-called correlation length which measures the exponential decay of the 2–point function. Indeed we can write the 2–point function as

$$G^{(2)}_{\text{nm}} = \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \tilde{G}^{(2)}(0)e^{-\xi^2 k^2} e^{ik(n-m)}. \quad (4.100)$$

More generally it is not difficult to show that the denominator $-2\tilde{\hat{V}}(k) + a + bM^2/2$ is strictly positive for $v > v_c$ and hence the 2–point function decays exponentially which indicates that the correlation length is finite.

We have the two cases

\begin{align*}
v < v_c : & \quad M = 0 \Rightarrow \xi^2 = \frac{v\rho^2}{v_c - v} \\
v > v_c : & \quad M = \sqrt{\frac{12}{b}(v - v_c)^{1/2}} \Rightarrow \xi^2 = \frac{v\rho^2}{2(v - v_c)}. \quad (4.101)
\end{align*}

The correlation length $\xi$ is associated with the critical exponent $\nu$ defined by

$$\xi \sim |T - T_c|^{-\nu}. \quad (4.102)$$

Clearly we have

$$\nu = \frac{1}{2}. \quad (4.103)$$

The correlation length thus diverges at the critical temperature $T = T_c$.

A more robust calculation which shows this fundamental result is easily done in the continuum. In the continuum limit the 2–point function (4.97) becomes

$$G^{(2)}(x,y) = \frac{1}{(2\pi)^d} \int \frac{d^d k}{m^2 + k^2} e^{ik(x-y)}. \quad (4.104)$$

The squared mass parameter is given by

$$m^2 = \frac{1}{\xi^2} = \frac{2(v_c - v) + \frac{b}{2}M^2}{2v\rho^2} \sim |v - v_c| \sim |T - T_c|. \quad (4.105)$$

We compute

$$G^{(2)}(x,y) = \frac{2}{(4\pi)^{d/2}} \frac{2m}{r^{d/2-1}} K_{1-d/2}(mr). \quad (4.106)$$

\text{Exercise: Do this important integral.}
For large distances we obtain
\[ G^{(2)}(x, y) = \frac{1}{2m} \left( \frac{m}{2\pi} \right)^{(d-1)/2} \frac{e^{-mr}}{r^{d-1/2}}, \quad r \to \infty. \] (4.107)

The last crucial critical exponent is the anomalous dimension \( \eta \). This is related to the behavior of the 2-point function at \( T = T_c \). At \( T = T_c \) we have \( v = v_c \) and \( M = 0 \) and hence the 2-point function becomes
\[ G^{(2)}(x, y) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{2v_c(\rho^2 k^2 - O(k^4))} e^{ik(x-y)}. \] (4.108)

Thus the denominator vanishes only at \( k = 0 \) which is consistent with the fact that the correlation length is infinite at \( T = T_c \). This also leads to algebraic decay. This can be checked more easily in the continuum limit where the 2-point function becomes
\[ G^{(2)}(x, y) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2} e^{ik(x-y)}. \] (4.109)

We compute
\[ G^{(2)}(x, y) = \frac{2^{d-2}}{(4\pi)^{d/2}} \Gamma(d/2 - 1) \frac{1}{r^{d-2}}. \] (4.110)

The critical exponent \( \eta \) is defined by the behavior
\[ G^{(2)}(x, y) \sim \frac{1}{r^{d-2+\eta}}. \] (4.111)

The mean field prediction is therefore given by
\[ \eta = 0. \] (4.112)

In this section we have not used any particular form for the potential \( V_{mn} \). It will be an interesting exercise to compute directly all the critical exponents \( \beta, \gamma, \delta, \nu \) and \( \eta \) for the case of the \( O(1) \) model corresponding to the nearest-neighbor interaction (4.42). This of course includes the Ising model as a special case.

### 4.2 The Callan-Symanzik Renormalization Group Equation

#### 4.2.1 Power Counting Theorems

We consider a \( \phi^4 \) theory in \( d \) dimensions given by the action
\[ S[\phi] = \int d^d x \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{\mu^2}{2} \phi^2 - \frac{g}{4} \phi^4 \right]. \] (4.113)

\(^5\)Exercise: Check this limit
\(^6\)Exercise: Do this important integral.
\(^7\)Exercise: Compute the exponents \( \beta, \gamma, \delta, \nu \) and \( \eta \) for the potential (4.42).
The case of interest is of course $d = 4$ and $r = 4$. In natural units where $\hbar = c = 1$ the action is dimensionless, viz $[S] = 1$. In these units time and length has the same dimension whereas mass, energy and momentum has the same dimension. We take the fundamental dimension to be that of length or equivalently that of mass. We have clearly (for example from Heisenberg uncertainty principle)

$$L = \frac{1}{M}. \quad (4.14)$$

$$[t] = [x] = L = M^{-1}, \ [m] = [E] = [p] = M. \quad (4.15)$$

It is clear that the Lagrangian density is of mass dimension $M_d$ and as a consequence the field is of mass dimension $M^{(d-2)/2}$ and the coupling constant $g$ is of mass dimension $M^{d-rd/2+r}$ (use the fact that $[\partial] = M$). We write

$$[\phi] = M^{d-2}. \quad (4.16)$$

$$[g] = M^{d-rd/2} \equiv M^{\delta_r}, \ \delta_r = d - r \frac{d-2}{2}. \quad (4.17)$$

The main result of power counting states that $\phi^r$ theory is renormalizable only in $d_c$ dimension where $d_c$ is given by the condition

$$\delta_r = 0 \Leftrightarrow d_c = \frac{2r}{r-2}. \quad (4.18)$$

The effective action is given by

$$\Gamma[\phi_c] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^d x_1 ... \int d^d x_n \Gamma^{(n)}(x_1,...,x_n) \phi_c(x_1) ... \phi_c(x_n). \quad (4.19)$$

Since the effective action is dimensionless the $n$-point proper vertices $\Gamma^{(n)}(x_1,...,x_n)$ have mass dimension such that

$$1 = \frac{1}{M^{nd}}[\Gamma^{(n)}(x_1,...,x_n)]M^{n \frac{d-2}{2}} \Leftrightarrow [\Gamma^{(n)}(x_1,...,x_n)] = M^{\frac{nd}{2}+n}. \quad (4.20)$$

The Fourier transform is defined as usual by

$$\int d^d x_1 ... \int d^d x_n \Gamma^{(n)}(x_1,...,x_n) e^{ip_1 x_1 + \ldots + ip_n x_n} = (2\pi)^d \delta^d(p_1 + \ldots + p_n) \tilde{\Gamma}^{(n)}(p_1,...,p_n). \quad (4.21)$$

From the fact that $\int d^d p \delta^d(p) = 1$ we conclude that $[\delta^d(p)] = M^{-d}$ and hence

$$[\tilde{\Gamma}^{(n)}(p_1,...,p_n)] = M^{d-n(\frac{d}{2}-1)}. \quad (4.22)$$
The \( n \)-point function \( G^{(n)}(x_1, ..., x_n) \) is the expectation value of the product of \( n \) fields and hence it has mass dimension
\[
[G^{(n)}(x_1, ..., x_n)] = M^{n \frac{d-2}{2}}.
\] (4.123)

The Fourier transform is defined by
\[
\int d^d x_1 \cdots d^d x_n G^{(n)}(x_1, ..., x_n) e^{ip_1 x_1 + \cdots + ip_n x_n} = (2\pi)^d \delta^d(p_1 + \cdots + p_n) \tilde{G}^{(n)}(p_1, ..., p_n). \] (4.124)

Hence
\[
[\tilde{G}^{(n)}(p_1, ..., p_n)] = M^{d-n\left(\frac{d}{2}+1\right)}. \] (4.125)

We consider now an arbitrary Feynman diagram in a \( \phi^r \) theory in \( d \) dimensions. This diagram is contributing to some \( n \)-point proper vertex \( \tilde{\Gamma}^{(n)}(p_1, ..., p_n) \) and it can be characterized by the following:

- \( L \)=number of loops.
- \( V \)=number of vertices.
- \( P \)=number of propagators (internal lines).
- \( n \)=number of external lines (not to be considered propagators).

We remark that each propagator is associated with a momentum variable. In other words we have \( P \) momenta which must be constrained by the \( V \) delta functions associated with the \( V \) vertices and hence there can only be \( P - V \) momentum integrals in this diagram. However, only one delta function (which enforces energy-momentum conservation) survives after integration and thus only \( V - 1 \) delta functions are actually used. The number of loops \( L \) must be therefore given by
\[
L = P - (V - 1). \] (4.126)

Since we have \( r \) lines coming into a vertex the total number of lines coming to \( V \) vertices is \( rV \). Some of these lines are propagators and some are external lines. Clearly among the \( rV \) lines we have precisely \( n \) external lines. Since each propagator connects two vertices it must be counted twice. We have then
\[
rV = n + 2P. \] (4.127)

It is clear that \( \tilde{\Gamma}^{(n)}(p_1, ..., p_n) \) must be proportional to \( g^V \), viz
\[
\tilde{\Gamma}^{(n)}(p_1, ..., p_n) = g^V f(p_1, ..., p_n). \] (4.128)

We have clearly
\[
[f(p_1, ..., p_n)] = M^\delta, \quad \delta = -V\delta_r + d - n\left(\frac{d}{2} - 1\right). \] (4.129)
The index $\delta$ is called the superficial degree of divergence of the Feynman graph. The physical significance of $\delta$ can be unraveled as follows. Schematically the function $f$ is of the form

$$f(p_1, ..., p_n) \sim \int_0^\Lambda d^d k_1 \ldots \int_0^\Lambda d^d k_P \frac{1}{k_1^2 - \mu^2} \ldots \frac{1}{k_P^2 - \mu^2} \left[ \delta^d \left( \sum p - \sum k \right) \right] V^{-1}. \quad (4.130)$$

If we neglect, in a first step, the delta functions than we can see immediately that the asymptotic behavior of the integral $f(p_1, ..., p_n)$ is $\Lambda^{P(d-2)}$. This can be found by factoring out the dependence of $f$ on $\Lambda$ via the rescaling $k \rightarrow \Lambda k$. By taking the delta functions into considerations we see immediately that the number of independent variables reduces and hence the asymptotic behavior of $f(p_1, ..., p_n)$ becomes

$$f(p_1, ..., p_n) \sim \Lambda^{P(d-2)-d(V-1)}. \quad (4.131)$$

By using $P = (rV - n)/2$ we arrive at the result

$$f(p_1, ..., p_n) \sim \Lambda^{-V\delta_r + d-n(\frac{d}{2}-1)} \sim \Lambda^\delta. \quad (4.132)$$

The index $\delta$ controls therefore the ultraviolet behavior of the graph. From the last two equations it is obvious that $\delta$ is the difference between the power of $k$ in the numerator and the power of $k$ in the denominator, viz

$$\delta = \text{(power of } k \text{ in numerator)} - \text{(power of } k \text{ in denominator)} \quad (4.133)$$

Clearly a negative index $\delta$ corresponds to convergence whereas a positive index $\delta$ corresponds to divergence. Since $\delta$ is only a superficial degree of divergence there are exceptions to this simple rule. More precisely we have the following first power counting theorem:

- For $\delta > 0$ the diagram diverges as $\Lambda^\delta$. However symmetries (if present) can reduce/eliminate divergences in this case.
- For $\delta = 0$ the diagram diverges as $\ln \Lambda$. An exception is the trivial diagram ($P = L = 0$).
- For $\delta < 0$ the diagram converges absolutely if it contains no divergent subdiagrams. In other words a diagram with $\delta < 0$ which contains divergent subdiagrams is generically divergent.

As an example let us consider $\phi^4$ in 4 dimensions. In this case

$$\delta = 4 - n. \quad (4.134)$$

Clearly only the 2-point and the 4-point proper vertices are superficially divergent, i.e. they have $\delta \geq 0$. In particular for $n = 4$ we have $\delta = 0$ indicating possible logarithmic
divergence which is what we had already observed in actual calculations. For \( n = 6 \) we observe that \( \delta = -2 < 0 \) which indicates that the 6–point proper vertex is superficially convergent. In other words the diagrams contributing to the 6–point proper vertex may or may not be convergent depending on whether or not they contain divergent subdiagrams. For example the one-loop diagram on figure 13 is convergent whereas the two-loop diagrams are divergent.

The third rule of the first power counting theorem can be restated as follows:

- A Feynman diagram is absolutely convergent if and only if it has a negative superficial degree of divergence and all its subdiagrams have negative superficial degree of divergence.

The \( \phi^4 \) theory in \( d = 4 \) is an example of a renormalizable field theory. In a renormalizable field theory only a finite number of amplitudes are superficially divergent. As we have already seen, the divergent amplitudes in the case of the \( \phi^4 \) theory in \( d = 4 \) theory, are the 2–point and the 4–point amplitudes. All other amplitudes may diverge only if they contain divergent subdiagrams corresponding to the 2–point and the 4–point amplitudes.

Another class of field theories is non-renormalizable field theories. An example is \( \phi^4 \) in \( D = 6 \). In this case

\[
\delta_r = -2, \; \delta = 2V + 6 - 2n. \tag{4.135}
\]

The formula for \( \delta \) depends now on the order of perturbation theory as opposed to what happens in the case of \( D = 4 \). Thus for a fixed \( n \) the superficial degree of divergence increases by increasing the order of perturbation theory, i.e. by increasing \( V \). In other words at a sufficiently high order of perturbation theory all amplitudes are divergent.

In a renormalizable field theory divergences occur generally at each order in perturbation theory. For \( \phi^4 \) theory in \( d = 4 \) all divergences can be removed order by order in perturbation theory by redefining the mass, the coupling constant and the wave function renormalization. This can be achieved by imposing three renormalization conditions on \( \tilde{\Gamma}^{(2)}(p), \; d\tilde{\Gamma}^{(2)}(p)/dp^2 \) and \( \tilde{\Gamma}^{(4)}(p_1, ..., p_4) \) at 0 external momenta corresponding to three distinct experiments.

In contrast we will require an infinite number of renormalization conditions in order to remove the divergences occurring at a sufficiently high order in a non-renormalizable field theory since all amplitudes are divergent in this case. This corresponds to an infinite number of distinct experiments and as a consequence the theory has no predictive power.

From the formula for the superficial degree of divergence \( \delta = -\delta_r V + d - n(d/2 - 1) \) we see that \( \delta_r \), the mass dimension of the coupling constant, plays a central role. For \( \delta_r = 0 \) (such as \( \phi^4 \) in \( d = 4 \) and \( \phi^3 \) in \( d = 6 \) we see that the index \( \delta \) is independent of the order of perturbation theory which is a special behavior of renormalizable theory.

For \( \delta_r < 0 \) (such as \( \phi^4 \) in \( d > 4 \) we see that \( \delta \) depends on \( V \) in such a way that it increases as \( V \) increases and hence we obtain more divergencies at each higher order of perturbation theory. Thus \( \delta_r < 0 \) defines the class of non-renormalizable field theories as \( \delta_r = 0 \) defines the class of renormalizable field theories.
Another class of field theories is super-renormalizable field theories for which $\delta_r > 0$ (such as $\phi^3$ in $D = 4$). In this case the superficial degree of divergence $\delta$ decreases with increasing order of perturbation theory and as a consequence only a finite number of Feynman diagrams are superficially divergent. In this case no amplitude diverges.

The second (main) power counting theorem can be summarized as follows:

- **Super-Renormalizable Theories**: The coupling constant $g$ has positive mass dimension. There are no divergent amplitudes and only a finite number of Feynman diagrams superficially diverge.

- **Renormalizable Theories**: The coupling constant $g$ is dimensionless. There is a finite number of superficially divergent amplitudes. However since divergences occur at each order in perturbation theory there is an infinite number of Feynman diagrams which are superficially divergent.

- **Non-Renormalizable Theories**: The coupling constant $g$ has negative mass dimension. All amplitudes are superficially divergent at a sufficiently high order in perturbation theory.

### 4.2.2 Renormalization Constants and Renormalization Conditions

We write the $\phi^4$ action in $d = 4$ as

$$S[\phi] = \int d^4x \left[ \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} (\phi^2)^2 \right].$$

The bare field $\phi$, the bare coupling constant $\lambda$ and the bare mass $m^2$ are given in terms of the renormalized field $\phi_R$, the renormalized coupling constant $\lambda_R$ and the renormalized mass $m^2_R$ respectively by the relations

$$\phi = \sqrt{Z} \phi_R. \quad \quad (4.137)$$

$$\lambda = Z_g/Z^2 \lambda_R. \quad \quad (4.138)$$

$$m^2 = (m^2_R + \delta_m)/Z. \quad \quad (4.139)$$

The renormalization constant $Z$ is called wave function renormalization constant (or equivalently field amplitude renormalization constant) whereas $Z_g/Z^2$ is the coupling constant renormalization constant.

The action $S$ given by equation (4.136) can be split as follows

$$S = S_R + \delta S. \quad \quad (4.140)$$

The renormalized action $S_R$ is given by

$$S_R[\phi_R] = \int d^4x \left[ \frac{1}{2} \partial_{\mu} \phi_R \partial^{\mu} \phi_R - \frac{1}{2} m^2_R \phi_R^2 - \frac{\lambda_R}{4!} (\phi_R^2)^2 \right].$$

(4.141)
The counter-term action $\delta S$ is given by
\[
\delta S[\phi_R] = \int d^4x \left[ \frac{\delta Z}{2} \partial_\mu \phi_R \partial^\mu \phi_R - \frac{1}{2} \delta m \phi_R^2 - \frac{\delta \lambda}{4!} (\phi_R^2)^2 \right].
\] (4.142)

The counterterms $\delta Z$, $\delta m$ and $\delta \lambda$ are given by
\[
\delta Z = Z - 1, \quad \delta m = Zm^2 - m_R^2, \quad \delta \lambda = \lambda Z^2 - \lambda_R = (Z_g - 1)\lambda_R.
\] (4.143)

The renormalized $n$–point proper vertex $\Gamma_R^{(n)}$ is given in terms of the bare $n$–point proper vertex $\Gamma^{(n)}$ by
\[
\Gamma_R^{(n)}(x_1, ..., x_n) = Z^{\frac{n}{2}} \Gamma^{(n)}(x_1, ..., x_n).
\] (4.144)

The effective action is given by (where $\phi$ denotes here the classical field)
\[
\Gamma_R[\phi_R] = \sum_{n=0}^{\infty} \frac{1}{n!} \Gamma_R^{(n)}(x_1, ..., x_n) \phi_R(x_1) ... \phi_R(x_n).
\] (4.145)

We assume a momentum cutoff regularization. The renormalization constants $Z$ and $Z_g$ and the counterterm $\delta m$ are expected to be of the form
\[
\begin{align*}
\delta m &= a_1(\Lambda)\lambda_R + a_2(\Lambda)\lambda_R^2 + ... \\
Z &= 1 + b_1(\Lambda)\lambda_R + b_2(\Lambda)\lambda_R^2 + ... \\
Z_g &= 1 + c_1(\Lambda)\lambda_R + c_2(\Lambda)\lambda_R^2 + ...
\end{align*}
\] (4.146)

All other quantities can be determined in terms of $Z$ and $Z_g$ and the counterterm $\delta m$. We can state our third theorem as follows:

- Renormalizability of the $\phi^4$ theory in $d = 4$ means precisely that we can choose the constants $a_i$, $b_i$ and $c_i$ such that all correlation functions have a finite limit order by order in $\lambda_R$ when $\Lambda \to \infty$.

We can eliminate the divergences by imposing appropriate renormalization conditions at zero external momentum. For example we can choose to impose conditions consistent with the tree level action, i.e.
\[
\begin{align*}
\hat{\Gamma}^{(2)}_R(p)|_{p^2=0} &= m_R^2 \\
\frac{d}{dp^2}\hat{\Gamma}^{(2)}_R(p)|_{p^2=0} &= 1 \\
\hat{\Gamma}^{(4)}(p_1, ..., p_4)|_{p_i^2=0} &= \lambda_R.
\end{align*}
\] (4.147)

This will determine the superficially divergent amplitudes completely and removes divergences at all orders in perturbation theory.\(^8\)

\(^8\)Exercise:
It is well established that a far superior regularization method, than the simple cutoff used above, is dimensional regularization in which case we use, instead of renormalization conditions, the so-called minimal subtraction (MS) and modified minimal subtraction (MMS) schemes to renormalize the theory. In minimal subtraction scheme we subtract only the pole term and nothing else.

In dimension $d \neq 4$ the coupling constant $\lambda$ is not dimensionless. The dimensionless coupling constant in this case is given by $g$ defined by

$$g = \mu^{-\epsilon} \lambda, \quad \epsilon = 4 - d. \quad (4.148)$$

The bare action can then be put in the form

$$S = \int dx \left[ \frac{Z}{2} \partial_{\mu} \phi_{R} \partial^{\mu} \phi_{R} - \frac{Z_{m}m_{R}^{2}}{2} \phi_{R}^{2} - \frac{\mu^{\epsilon} g_{R} Z_{g}(\phi_{R}^{2})^{2}}{4!} \right]. \quad (4.149)$$

The new renormalization condition $Z_{m}$ is defined through the equation

$$m^{2} = m_{R}^{2} \frac{Z_{m}}{Z}. \quad (4.150)$$

The mass $\mu^{2}$ is an arbitrary mass scale parameter which plays a central role in dimensional regularization and minimal subtraction. The mass $\mu^{2}$ will define the subtraction point. In other words the mass scale at which we impose renormalization conditions in the form

$$\tilde{\Gamma}_{R}^{(2)}(p)\big|_{p^{2}=0} = m_{R}^{2}$$

$$\frac{d}{dp^{2}} \tilde{\Gamma}_{R}^{(2)}(p)\big|_{p^{2}=\mu^{2}} = 1$$

$$\tilde{\Gamma}_{R}^{(4)}(p_{1}, ..., p_{4})\big|_{SP} = \mu^{\epsilon} g_{R}. \quad (4.151)$$

The symmetric point SP is defined by $p_{i} \cdot p_{j} = \mu^{2}(4\delta_{ij} - 1)/3$. For massive theories we can simply choose $\mu = m_{R}$. According to Weinberg’s theorem (and other considerations) the only correlation functions of massless $\phi^{4}$ which admit a zero momentum limit is the $2-$point function. This means in particular that the second and third renormalization conditions (4.147) do not make sense in the massless limit $m_{R}^{2} \rightarrow 0$ and should be replaced by the second and third renormalization conditions (4.151). This is also the reason why we have kept the first renormalization condition unchanged. The renormalization conditions (4.151) are therefore better behaved.

As pointed above the renormalization prescription known as minimal subtraction is far superior than the above prescription of imposing renormalization conditions since it

- Show that the loopwise expansion is equivalent to an expansion in powers of $\lambda$.
- Write down the one-loop effective action of the $\phi^{4}$ theory in $d = 4$. Use a Gaussian cutoff.
- Compute $a_{1}, b_{1}$ and $c_{1}$ at the one-loop order of perturbation theory.
- Consider one-loop renormalization of $\phi^{3}$ in $d = 6$. 
is intimately tied to dimensional regularization. In this prescription the mass scale $\mu^2$ appears only via (4.148). We will keep calling $\mu^2$ the subtraction point since minimal subtraction must be physically equivalent to imposing the renormalization conditions (4.151) although the technical detail is generically different in the two cases.

The renormalized proper vertices $\tilde{\Gamma}_R^{(n)}$ depend on the momenta $p_1, \ldots, p_n$ but also on the renormalized mass $m_R^2$, the renormalized coupling constant $g_R$ and the cutoff $\Lambda$. In the case of dimensional regularization the cutoff is $\epsilon = 4 - d$ whereas in the case of lattice regularization the cutoff is the inverse lattice spacing. The proper vertices $\tilde{\Gamma}_R^{(n)}$ will also depend on the mass scale $\mu^2$ explicitly and implicitly through $m_R^2$ and $g_R$. The renormalized proper vertices are related to the bare proper vertices as

$$\tilde{\Gamma}_R^{(n)}(p_i; \mu^2; m_R^2, g_R, \Lambda) = Z_R^{(n)}(p_i; m^2, g, \Lambda) \tilde{\Gamma}_R^{(n)}(p_i; m^2, g, \Lambda).$$

(4.152)

The renormalization constant $Z$ (and also other renormalization constants $Z_g$, $Z_m$ and counterterms $\delta Z$, $\delta m$ and $\delta \lambda$) will only depend on the dimensionless parameters $g_R$, $m_R^2/\mu^2$, $\Lambda^2/\mu^2$, $m_R^2/\Lambda^2$ as well as on $\Lambda$, viz

$$Z = Z(g_R, m_R^2/\mu^2, \Lambda^2/\mu^2, \Lambda).$$

(4.153)

In dimensional regularization we have

$$\tilde{\Gamma}_R^{(n)}(p_i; \mu^2; m_R^2, g_R, \mu^2) = Z_R^{(n)}(g_R, m_R^2/\mu^2, \epsilon) \tilde{\Gamma}_R^{(n)}(p_i; m^2, g, \epsilon).$$

(4.154)

Renormalizability of the $\phi^4$ theory in $d = 4$ via renormalization conditions (the fourth theorem) can be stated as follows:

- The renormalized proper vertices $\tilde{\Gamma}_R^{(n)}(p_i; \mu^2; m_R^2, g_R, \Lambda)$ at fixed $p_i$, $\mu^2$, $g_R$, $m_R^2$ have a large cut-off limit $\tilde{\Gamma}_R^{(n)}(p_i; \mu^2; m_R^2, g_R)$ which are precisely the physical proper vertices, viz

$$\tilde{\Gamma}_R^{(n)}(p_i; \mu^2; m_R^2, g_R, \Lambda) = \tilde{\Gamma}_R^{(n)}(p_i; \mu^2; m_R^2, g_R) + O\left(\frac{(\ln \Lambda)^L}{\Lambda^2}\right).$$

(4.155)

The renormalized physical proper vertices $\tilde{\Gamma}_R^{(n)}(p_i; \mu^2; m_R^2, g_R)$ are universal in the sense that they do not depend on the specific cut-off procedure as long as the renormalization conditions (4.151) are kept unchanged. In the above equation $L$ is the number of loops.

### 4.2.3 Renormalization Group Functions and Minimal Subtraction

The bare mass $m^2$ and the bare coupling constant $\lambda$ are related to the renormalized mass $m_R^2$ and renormalized coupling constant $\lambda_R$ by the relations

$$m^2 = m_R^2 \frac{Z_m}{Z}.$$
\[ \lambda = \frac{Z_g}{Z^2} \lambda_R = \frac{Z_g}{Z^2} \mu^2 g_R. \]  

(4.157)

In dimensional regularization the renormalization constants will only depend on the dimensionless parameters \( g_R, m_R^2/\mu^2 \) as well as on \( \epsilon \). We may choose the subtraction mass scale \( \mu^2 = m_R^2 \). Clearly the bare quantities \( m^2 \) and \( \lambda \) are independent of the mass scale \( \mu \).

Thus by differentiating both sides of the above second equation with respect to \( \mu^2 \) keeping \( m^2 \) and \( \lambda \) fixed we obtain

\[ 0 = \left( \mu \frac{\partial \lambda}{\partial \mu} \right)_{\lambda, m^2} \Rightarrow \beta = -\epsilon g_R - \left( \mu \frac{\partial}{\partial \mu} \ln \frac{Z_g}{Z^2} \right)_{\lambda, m^2} g_R. \]  

(4.158)

The so-called renormalization group beta function \( \beta \) (also called the Gell-Mann Law function) is defined by

\[ \beta = \beta(g_R, m^2_R) = \left( \mu \frac{\partial g_R}{\partial \mu} \right)_{\lambda, m^2}. \]  

(4.159)

Let us define the new dimensionless coupling constant

\[ G = \frac{Z_g}{Z^2} g_R. \]  

(4.160)

Alternatively by differentiating both sides of equation (4.157) with respect to \( \mu \) keeping \( m^2 \) and \( \lambda \) fixed we obtain

\[ 0 = \left( \mu \frac{\partial m^2}{\partial \mu} \right)_{\lambda, m^2} \Rightarrow 0 = \epsilon G + \beta \frac{\partial}{\partial g_R} G + \left( \mu \frac{\partial}{\partial \mu} m_R \right)_{\lambda, m^2} \frac{\partial}{\partial m^2} G. \]  

(4.161)

The last term is absent when \( \mu = m_R \).

Next by differentiating both sides of equation (4.156) with respect to \( \mu \) keeping \( m^2 \) and \( \lambda \) fixed we obtain

\[ 0 = \left( \mu \frac{\partial m^2}{\partial \mu} \right)_{\lambda, m^2} \Rightarrow 0 = \left( \mu \frac{\partial m^2}{\partial \mu} \right)_{\lambda, m^2} + m_R^2 \left( \mu \frac{\partial}{\partial \mu} \ln \frac{Z_m}{Z} \right)_{\lambda, m^2}. \]  

(4.162)

We define the renormalization group function \( \gamma \) by

\[ \gamma_m = \gamma_m(g_R, m^2_R) = \left( \mu \frac{\partial}{\partial \mu} \ln m^2_R \right)_{\lambda, m^2} = -\left( \mu \frac{\partial}{\partial \mu} \ln \frac{Z_m}{Z} \right)_{\lambda, m^2}. \]  

(4.163)

In the minimal subtraction scheme the renormalization constants will only depend on the dimensionless parameters \( g_R \) and as a consequence the renormalization group functions will only depend on \( g_R \). In this case we find

\[ \beta(g_R) = -\epsilon g_R \left[ 1 + g_R \frac{d}{dg_R} \ln \frac{Z_g}{Z} \right]^{-1} \]  

\[ = -\epsilon \left[ \frac{d}{dg_R} \ln G(g_R) \right]^{-1}. \]  

(4.164)
\[ \gamma_m(g_R) = -\beta(g_R) \frac{d}{dg_R} \ln \frac{Z_m}{Z}. \] (4.165)

We go back to the renormalized proper vertices (in minimal subtraction) given by
\[ \hat{\Gamma}^{(n)}_R(p_i, \mu^2; m^2_R, g_R, \epsilon) = Z^{n/2}(g_R, \epsilon) \tilde{\Gamma}^{(n)}(p_i; m^2, g, \epsilon). \] (4.166)

Again the bare proper vertices must be independent of the subtraction mass scale, viz
\[ 0 = \left( \mu \frac{\partial}{\partial \mu} \hat{\Gamma}^{(n)}_R \right)_{\lambda, m^2}. \] (4.167)

By differentiating both sides of equation (4.166) with respect to \( \mu \) keeping \( m^2 \) and \( \lambda \) fixed we obtain
\[ \left( \mu \frac{\partial}{\partial \mu} \hat{\Gamma}^{(n)}_R \right)_{\lambda, m^2} = \frac{n}{2} \left( \mu \frac{\partial}{\partial \mu} \ln Z \right)_{\lambda, m^2} \hat{\Gamma}^{(n)}_R. \] (4.168)

Equivalently we have
\[ \left( \mu \frac{\partial}{\partial \mu} + \mu \frac{m^2_R}{d m^2_R} \frac{\partial}{\partial m^2_R} + \mu \frac{g_R}{d g_R} \frac{\partial}{\partial g_R} \right) \hat{\Gamma}^{(n)}_R = \frac{n}{2} \left( \mu \frac{\partial}{\partial \mu} \ln Z \right)_{\lambda, m^2} \hat{\Gamma}^{(n)}_R. \] (4.169)

We get finally
\[ \left( \mu \frac{\partial}{\partial \mu} + m^2_R \gamma_m \frac{\partial}{\partial m^2_R} + \beta \frac{\partial}{\partial g_R} - \frac{n}{2} \eta \right) \hat{\Gamma}^{(n)}_R = 0. \] (4.170)

This is our first renormalization group equation. The new renormalization group function \( \eta \) (also called the anomalous dimension of the field operator) is defined by
\[ \eta(g_R) = \left( \mu \frac{\partial}{\partial \mu} \ln Z \right)_{\lambda, m^2} = \beta(g_R) \frac{d}{dg_R} \ln Z. \] (4.171)

Renormalizability of the \( \phi^4 \) theory in \( d = 4 \) via minimal subtraction (the fifth theorem) can be stated as follows:

- The renormalized proper vertices \( \hat{\Gamma}^{(n)}_R(p_i, \mu^2; m^2_R, g_R, \epsilon) \) and the renormalization group functions \( \beta(g_R), \gamma(g_R) \) and \( \eta(g_R) \) have a finite limit when \( \epsilon \rightarrow 0 \).

By using the above the theorem and the fact that \( G(g_R) = g_R + \ldots \) we conclude that the beta function must be of the form
\[ \beta(g_R) = -\epsilon g_R + \beta_2(\epsilon) g_R^2 + \beta_3(\epsilon) g_R^3 + \ldots \] (4.172)
The functions \( \beta_i(\epsilon) \) are regular in the limit \( \epsilon \to 0 \). By using the result (4.164) we find

\[
g_R \frac{G'}{G} = -\frac{\epsilon g_R}{\beta} = 1 + \frac{\beta_2(\epsilon)}{\epsilon} g_R + \left( \frac{\beta_3(\epsilon)}{\epsilon^2} + \frac{\beta_3(\epsilon)}{\epsilon} \right) g_R^2 + \ldots
\]

(4.173)

The most singular term in \( \epsilon \) is captured by the function \( \beta_2(\epsilon) \). By integrating this equation we obtain

\[
G(g_R) = g_R \left[ 1 - \frac{\beta_2(0)}{\epsilon} g_R \right]^{-1} + \text{less singular terms.}
\]

(4.174)

The function \( G(g_R) \) can then be expanded as

\[
G(g_R) = g_R + \sum_{n=2}^{n} g_R^n \tilde{G}_n(\epsilon).
\]

(4.175)

The functions \( \tilde{G}_n(\epsilon) \) behave as

\[
\tilde{G}_n(\epsilon) = \frac{\beta_n^{-1}(0)}{\epsilon^{n-1}} + \text{less singular terms.}
\]

(4.176)

Alternatively we can expand \( G \) as

\[
G(g_R) = g_R + \sum_{n=1}^{n} \frac{G_n(g_R)}{\epsilon^n} + \text{regular terms} \ , \ G_n(g_R) = O(g_R^{n+1}).
\]

(4.177)

This is equivalent to

\[
\frac{Z_g}{Z^2} = 1 + \sum_{n=1}^{n} \frac{H_n(g_R)}{\epsilon^n} + \text{regular terms} \ , \ H_n(g_R) = O(g_R^n).
\]

(4.178)

We compute the beta function

\[
\beta(g_R) = -\epsilon \left[ g_R + \sum_{n=1}^{n} \frac{G_n(g_R)}{\epsilon^n} \right] \left[ 1 + \sum_{n=1}^{n} \frac{G_n'(g_R)}{\epsilon^n} \right]^{-1} = -\epsilon \left[ g_R + \frac{G_1}{\epsilon} + \ldots \right] \left[ 1 - \frac{G_1'}{\epsilon} + \frac{(G_1')^2}{\epsilon^2} - \frac{G_2'}{\epsilon^2} + \ldots \right] = -\epsilon g_R - G_1(g_R) + g_R G_1'(g_R) + \sum_{n=1}^{n} \frac{b_n(g_R)}{\epsilon^n}.
\]

(4.179)

The beta function is finite in the limit \( \epsilon \to 0 \) and as a consequence we must have \( b_n(g_R) = 0 \) for all \( n \). The beta function must therefore be of the form

\[
\beta(g_R) = -\epsilon g_R - G_1(g_R) + g_R G_1'(g_R).
\]

(4.180)
The beta function $\beta$ is completely determined by the residue of the simple pole of $G$, i.e. by $G_1$. In fact all the functions $G_n$ with $n \geq 2$ are determined uniquely by $G_1$ (from the condition $b_n = 0$).

Similarly from the finiteness of $\eta$ in the limit $\epsilon \rightarrow 0$ we conclude that the renormalization constant $Z$ is of the form

$$Z(g_R) = 1 + \sum_{n=1}^{\infty} \frac{\alpha_n(g_R)}{\epsilon^n} + \text{regular terms} , \quad \alpha_n(g_R) = O(g_R^{n+1}). \quad (4.181)$$

We compute the anomalous dimension

$$\eta = \beta(g_R)\frac{d}{dg_R} \ln Z(g_R)$$

$$= \left[ -\epsilon g_R - G_1(g_R) + g_R G'_1(g_R) \right] \left[ \frac{1}{\epsilon} \alpha'_1 + \ldots \right]. \quad (4.182)$$

Since $\eta$ is finite in the limit $\epsilon \rightarrow 0$ we must have

$$\eta = -g_R \alpha'_1. \quad (4.183)$$

### 4.2.4 CS Renormalization Group Equation in $\phi^4$ Theory

We will assume $d = 4$ in this section although much of what we will say is also valid in other dimensions. We will also use a cutoff regularization throughout.

**Inhomogeneous CS RG Equation:** Let us consider now $\phi^4$ theory with $\phi^2$ insertions. We add to the action (4.136) a source term of the form $\int d^d x K(x) \phi^2(x)/2$, i.e.

$$S[\phi, K] = \int d^d x \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m_R^2 \phi^2 - \frac{\lambda}{4!} (\phi^2)^2 + \frac{Z_2}{2} K \phi^2 \right]. \quad (4.184)$$

Then we consider the path integral

$$Z[J, K] = \int \mathcal{D}\phi \exp(iS[\phi, K] + i \int d^d x J \phi). \quad (4.185)$$

It is clear that differentiation with respect to $K(x)$ generates insertions of the operator $-\phi^2/2$. The corresponding renormalized field theory will be given by the path integral

$$Z_R[J, K] = \int \mathcal{D}\phi_R \exp(iS[\phi_R, K] + i \int d^d x J \phi_R). \quad (4.186)$$

$$S[\phi_R, K] = \int d^d x \left[ \frac{1}{2} \partial_\mu \phi_R \partial^\mu \phi_R - \frac{1}{2} m_R^2 \phi_R^2 - \frac{\lambda_R}{4!} (\phi_R^2)^2 + \frac{Z_2}{2} K \phi_R^2 \right] + \delta S. \quad (4.187)$$
Z_2 is a new renormalization constant associated with the operator \( \int d^d x K(x) \phi^2(x)/2 \).
We have clearly the relations

\[
W_R[J,K] = W[\sqrt{\frac{J}{Z}}, \frac{Z_2}{Z} K].
\]

\[
\Gamma_R[\phi_c, K] = \Gamma[\sqrt{Z} \phi_c, \frac{Z_2}{Z} K].
\]

The renormalized \((l,n)\)−point proper vertex \(\Gamma_R^{(l,n)}\) is given in terms of the bare \((l,n)\)−point proper vertex \(\Gamma^{(l,n)}\) by

\[
\Gamma_R^{(l,n)}(y_1, \ldots, y_l; x_1, \ldots, x_n) = Z_2^{l-n} Z^{l} \Gamma_R^{(l,n)}(y_1, \ldots, y_l; x_1, \ldots, x_n).
\]

The proper vertex \(\Gamma_R^{(1,2)}(y_1, x_1, x_2)\) is a new superficially divergent proper vertex which requires a new counterterm and a new renormalization condition. For consistency with the tree level action we choose the renormalization condition

\[
\tilde{\Gamma}_R^{(1,2)}(q; p_1, p_2)|_{q=p_i=0} = 1.
\]

Let us remark that correlation functions with one operator insertion \(i \phi^2(y)/2\) are defined by

\[
< i \frac{\phi^2(y)\phi(x_1) \ldots \phi(x_n)}{2} > = \frac{1}{i^n} \frac{1}{Z[J,K]} \frac{\delta}{\delta K(y_1)} \ldots \frac{\delta}{\delta J(x_1)} Z[J,K]|_{J=K=0}.
\]

This can be generalized easily to

\[
< i \frac{\phi^2(y_1) \ldots \phi^2(y_l)\phi(x_1) \ldots \phi(x_n)}{2^l} > = \frac{1}{i^n} \frac{1}{Z[J,K]} \frac{\delta}{\delta K(y_1)} \ldots \frac{\delta}{\delta K(y_l)} \frac{\delta}{\delta J(x_1)} \ldots \frac{\delta}{\delta J(x_n)} Z[J,K]|_{J=K=0}.
\]

From this formula we see that the generating functional of correlation functions with \(l\) operator insertions \(i \phi^2(y)/2\) is defined by

\[
Z[y_1, \ldots, y_l; J] = \frac{\delta}{\delta K(y_1)} \ldots \frac{\delta}{\delta K(y_l)} Z[J,K]|_{K=0}.
\]

The generating functional of the connected correlation functions with \(l\) operator insertions \(i \phi^2(y)/2\) is then defined by

\[
W[y_1, \ldots, y_l; J] = \frac{\delta}{\delta K(y_1)} \ldots \frac{\delta}{\delta K(y_l)} W[J,K]|_{K=0}.
\]

We write the effective action as

\[
\Gamma[\phi_c, K] = \sum_{l,n=0}^{\infty} \frac{1}{l! n!} \int d^d y_1 \ldots \int d^d x_n \Gamma_R^{(l,n)}(y_1, \ldots, y_l; x_1, \ldots, x_n) K(y_1) \ldots K(y_l) \phi_c(x_1) \ldots \phi_c(x_n).
\]
The generating functional of 1PI correlation functions with \( l \) operator insertions \( i \phi_c^2(y)/2 \) is defined by

\[
\frac{\delta \Gamma[\phi_c, K]}{\delta K(y_1)\ldots\delta K(y_l)} = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^d x_1 \ldots \int d^d x_n \Gamma^{(l,n)}(y_1, \ldots, y_l; x_1, \ldots, x_n) \phi_c(x_1) \ldots \phi_c(x_n). 
\]

(4.197)

Clearly

\[
\frac{\delta^{l+n} \Gamma[\phi_c, K]}{\delta K(y_1)\ldots\delta K(y_l)\delta \phi_c(x_1)\ldots\delta \phi_c(x_n)} = \Gamma^{(l,n)}(y_1, \ldots, y_l; x_1, \ldots, x_n). 
\]

(4.198)

We also write

\[
\Gamma[\phi_c, K] = \sum_{l,n=0}^{\infty} \frac{1}{l! n!} \int \frac{d^d q_1}{(2\pi)^d} \ldots \int \frac{d^d p_n}{(2\pi)^d} \tilde{\Gamma}^{(l,n)}(q_1, \ldots, q_l; p_1, \ldots, p_n) K(q_1) \ldots K(q_l) \tilde{\phi}_c(p_1) \ldots \tilde{\phi}_c(p_n). 
\]

(4.199)

We have defined

\[
\int d^d y_1 \ldots \int d^d x_n \Gamma^{(l,n)}(y_1, \ldots, y_l; x_1, \ldots, x_n) e^{iq_1 y_1} \ldots e^{iq_l y_l} e^{ip_1 x_1} \ldots e^{ip_n x_n} = \tilde{\Gamma}^{(l,n)}(q_1, \ldots, q_l; p_1, \ldots, p_n). 
\]

(4.200)

The definition of the proper vertex \( \tilde{\Gamma}^{(l,n)}(q_1, \ldots, q_l; p_1, \ldots, p_n) \) in this equation includes a delta function. We recall that

\[
\Gamma[\phi_c, K] = W[J, K] - \int d^d x J(x) \phi_c(x), \ \phi_c(x) = \frac{\delta W[J, K]}{\delta J(x)}. 
\]

(4.201)

We calculate immediately

\[
\frac{\partial W}{\partial m^2}_{\lambda, \Lambda} = - \int d^d z \frac{\delta W}{\delta K(z)} \Rightarrow \frac{\partial \Gamma}{\partial m^2}_{\lambda, \Lambda} = - \int d^d z \frac{\delta \Gamma}{\delta K(z)}. 
\]

(4.202)

As a consequence

\[
\frac{\partial \Gamma^{(l,n)}(y_1, \ldots, y_l; x_1, \ldots, x_n)}{\partial m^2}_{\lambda, \Lambda} = - \int d^d z \frac{\delta \Gamma^{(l,n)}(y_1, \ldots, y_l; x_1, \ldots, x_n)}{\delta K(z)} = - \int d^d z \Gamma^{(l+1,n)}(z, y_1, \ldots, y_l; x_1, \ldots, x_n) 
\]

(4.203)

Fourier transform then gives

\[
\frac{\partial \tilde{\Gamma}^{(l,n)}(q_1, \ldots, q_l; p_1, \ldots, p_n)}{\partial m^2}_{\lambda, \Lambda} = - \tilde{\Gamma}^{(l+1,n)}(0, q_1, \ldots, q_l; p_1, \ldots, p_n). 
\]

(4.204)
By using equation (4.190) to convert bare proper vertices into renormalized proper vertices we obtain

\[
\left( \frac{\partial}{\partial m^2} - \frac{n}{2} \frac{\partial \ln Z}{\partial m^2} \right) \Gamma_{R}^{(l,n)} = -ZZ_2^{-1}\tilde{\Gamma}_{R}^{(l+1,n)}. \tag{4.205}
\]

The factor of $-1$ multiplying the right hand side of this equation will be absent in the Euclidean rotation of the theory. The renormalized proper vertices $\tilde{\Gamma}_{R}^{(l,n)}$ depend on the momenta $q_1, \ldots, q_l, p_1, \ldots, p_n$ but also on the renormalized mass $m_R^2$, the renormalized coupling constant $\lambda_R$ and the cutoff $\Lambda$. They also depend on the subtraction mass scale $\mu^2$. We will either assume that $\mu^2 = 0$ or $\mu^2 = m_R^2$. We have then

\[
\left( \frac{\partial m_R}{\partial m^2} \frac{\partial}{\partial m_R} + \frac{\partial \lambda_R}{\partial m^2} \frac{\partial}{\partial \lambda_R} - \frac{n}{2} \frac{\partial \ln Z_2/Z}{\partial m^2} - l \frac{\partial \ln Z_2/Z}{\partial m^2} \right) \Gamma_{R}^{(l,n)} = -ZZ_2^{-1}\tilde{\Gamma}_{R}^{(l+1,n)}. \tag{4.206}
\]

We write this as

\[
\frac{\partial m_R}{\partial m^2} \left( m_R \frac{\partial}{\partial m_R} + m_R \frac{\partial m^2}{\partial m^2} \frac{\partial}{\partial \lambda_R} - \frac{n}{2} m_R \frac{\partial m^2}{\partial m^2} \frac{\partial \ln Z_2/Z}{\partial m^2} - l m_R \frac{\partial m^2}{\partial m_R} \frac{\partial \ln Z_2/Z}{\partial m^2} \right) \Gamma_{R}^{(l,n)} = -m_R Z Z_2^{-1}\tilde{\Gamma}_{R}^{(l+1,n)}. \tag{4.207}
\]

We define

\[
\beta(\lambda_R, m_R) = m_R \frac{\partial^2 m^2}{\partial m_R \partial \lambda_R} = \left( m_R \frac{\partial \lambda_R}{\partial m_R} \right)_{\lambda, \Lambda}. \tag{4.208}
\]

\[
\eta(\lambda_R, m_R) = m_R \frac{\partial^2 m^2}{\partial m_R \partial m^2} \frac{\partial \ln Z}{\partial m^2} = \left( m_R \frac{\partial \ln Z}{\partial m_R} \right)_{\lambda, \Lambda}. \tag{4.209}
\]

\[
\eta_2(\lambda_R, m_R) = m_R \frac{\partial^2 m^2}{\partial m_R \partial m^2} \frac{\partial \ln Z_2/Z}{\partial m^2} = \left( m_R \frac{\partial \ln Z_2/Z}{\partial m_R} + \beta \frac{\partial \ln Z_2/Z}{\partial \lambda_R} \right)_{\lambda, \Lambda}. \tag{4.210}
\]

\[
m_R^2 \sigma(\lambda_R, m_R) = ZZ_2^{-1} \left( m_R \frac{\partial m^2}{\partial m_R} \right)_{\lambda, \Lambda}. \tag{4.211}
\]

\textsuperscript{9}Exercise: Check this.
The above differential equation becomes with these definitions
\[
\left( m_R \frac{\partial}{\partial m_R} + \beta \frac{\partial}{\partial \lambda_R} - \frac{n}{2} \eta - n_2 \right) \tilde{\Gamma}^{(l,n)}_{R} = -m_R^2 \sigma \tilde{\Gamma}^{(l+1,n)}_{R}. \tag{4.212}
\]

This is the original Callan-Symanzik equation. This equation represents only the response of the proper vertices to rescaling \((\phi \rightarrow \phi_R)\) and to reparametrization \((m^2 \rightarrow m^2_R, \lambda \rightarrow \lambda_R)\). We still need to impose on the Callan-Symanzik equation the renormalization conditions in order to determine the renormalization constants and show that the renormalized proper vertices have a finite limit when \(\Lambda \rightarrow \infty\). The functions \(\beta, \eta, \eta_2\) and \(\sigma\) can be expressed in terms of renormalized proper vertices and as such they have an infinite cutoff limit. The Callan-Symanzik equation (4.212) can be used to provide an inductive proof of renormalizability of \(\phi^4\) theory in 4 dimensions. We will not go through this involved exercise at this stage.

**Homogeneous CS RG Equation-Massless Theory:** The renormalization conditions for a massless \(\phi^4\) theory in \(d = 4\) are given by
\[
\begin{align*}
\tilde{\Gamma}^{(2)}_{R}(p)|_{p^2=0} & = 0 \\
\frac{d}{dp^2} \tilde{\Gamma}^{(2)}_{R}(p)|_{p^2=\mu^2} & = 1 \\
\tilde{\Gamma}^{(4)}(p_1,...,p_4)|_{SP} & = \lambda_R. \tag{4.213}
\end{align*}
\]

The renormalized proper vertices \(\tilde{\Gamma}^{(n)}_{R}\) depend on the momenta \(p_1,...,p_n\), the mass scale \(\mu^2\), the renormalized coupling constant \(\lambda_R\) and the cutoff \(\Lambda\). The bare proper vertices \(\bar{\Gamma}^{(n)}\) depend on the momenta \(p_1,...,p_n\), the bare coupling constant \(\lambda\) and the cutoff \(\Lambda\). The bare mass is fixed by the condition that the renormalized mass is 0. We have then
\[
\tilde{\Gamma}^{(n)}_{R}(p_i, \mu^2; \lambda_R, \Lambda) = Z^2(\lambda, \frac{\Lambda^2}{\mu^2}, \Lambda) \bar{\Gamma}^{(n)}(p_i; \lambda, \Lambda). \tag{4.214}
\]

The bare theory is obviously independent of the mass scale \(\mu^2\). This is expressed by the condition
\[
\left( \mu \frac{\partial}{\partial \mu} \tilde{\Gamma}^{(n)}_{R}(p_i; \lambda, \Lambda) \right)_{\lambda, \Lambda} = 0. \tag{4.215}
\]

We differentiate equation (4.214) with respect to \(\mu^2\) keeping \(\lambda\) and \(\Lambda\) fixed. We get
\[
\frac{\partial}{\partial \mu} \tilde{\Gamma}^{(n)}_{R} + \left( \frac{\partial \lambda_R}{\partial \mu} \right) \frac{\partial}{\partial \lambda_R} \tilde{\Gamma}^{(n)}_{R} = \frac{n}{2} \left( \frac{\partial \ln Z}{\partial \mu} \right)_{\lambda, \Lambda} Z^2 \bar{\Gamma}^{(n)}. \tag{4.216}
\]

We obtain immediately the differential equation
\[
\left( \mu \frac{\partial}{\partial \mu} + \beta(\lambda_R) \frac{\partial}{\partial \lambda_R} - \frac{n}{2} \eta(\lambda_R) \right) \tilde{\Gamma}^{(n)}_{R} = 0. \tag{4.217}
\]
\[ \beta(\lambda_R) = \left( \frac{\partial \lambda_R}{\partial \mu} \right)_{\lambda,\Lambda}, \quad \eta(\lambda_R) = \left( \frac{\partial \ln Z}{\partial \mu} \right)_{\lambda,\Lambda}. \] (4.218)

This is the Callan-Symanzik equation for the massless theory. The functions \( \beta \) and \( \eta \) do not depend on \( \Lambda/\mu \) since they can be expressed in terms of renormalized proper vertices and as such they have an infinite cutoff limit.

For the massless theory with \( \phi^2 \) insertions we need, as in the massive case, an extra renormalization constant \( Z_2 \) and an extra renormalization condition to fix it given by

\[ \tilde{\Gamma}_R^{(1,2)}(q; p_1, p_2)|_{q^2 = p_1^2} = 1. \] (4.219)

We will also need an extra RG function given by

\[ \eta_2(\lambda_R) = \left( \frac{\partial}{\partial \mu} \ln Z_2/Z \right)_{\lambda,\Lambda}. \] (4.220)

The Callan-Symanzik equation for the massless theory with \( \phi^2 \) insertions is then given (by the almost obvious) equation

\[ \left( \mu \frac{\partial}{\partial \mu} + \beta(\lambda_R) \frac{\partial}{\partial \lambda_R} - \frac{n}{2} \eta(\lambda_R) - n \eta_2(\lambda_R) \right) \tilde{\Gamma}_R^{(l,n)} = 0. \] (4.221)

**Homogeneous CS RG Equation-Massive Theory:** We consider again a massless \( \phi^4 \) theory in \( d = 4 \) dimensions with \( \phi^2 \) insertions. The action is given by the massless limit of the action (4.187), namely

\[ S[\phi_R, K] = \int d^d x \left[ \frac{1}{2} Z \partial_\mu \phi_R \partial^\mu \phi_R - \frac{1}{2} \delta_m \phi_R^2 - \frac{\lambda_R Z g}{4!} (\phi_R^2)^2 + \frac{Z_2}{2} K \phi_R^2 \right]. \] (4.222)

The effective action is still given by

\[ \Gamma[\phi_c, K] = \sum_{l,n=0} \frac{1}{l! n!} \int d^d y_1 \ldots \int d^d x_n \Gamma^{(l,n)}(y_1, \ldots, y_l; x_1, \ldots, x_n) K(y_1) \ldots K(y_l) \phi_c(x_1) \ldots \phi_c(x_n). \] (4.223)

An arbitrary proper vertex \( \tilde{\Gamma}_R^{(n)}(p_1, \ldots, p_n; K) \) can be expanded in terms of the proper vertices \( \Gamma^{(l,n)}_R(q_1, \ldots, q_l; p_1, \ldots, p_n) \) as follows

\[ \tilde{\Gamma}_R^{(n)}(p_1, \ldots, p_n; K) = \sum_{l=0} \frac{1}{l!} \int \frac{d^d q_1}{(2\pi)^d} \ldots \int \frac{d^d q_l}{(2\pi)^d} \Gamma^{(l,n)}_R(q_1, \ldots, q_l; p_1, \ldots, p_n) K(q_1) \ldots K(q_l). \] (4.224)

We consider the differential operator

\[ D = \mu \frac{\partial}{\partial \mu} + \beta(\lambda_R) \frac{\partial}{\partial \lambda_R} - \frac{n}{2} \eta(\lambda_R) - \eta_2(\lambda_R) \int d^d q K(q) \frac{\delta}{\delta K(q)}. \] (4.225)
We compute

\[
\int d^d q \tilde{K}(q) \frac{\delta}{\delta \tilde{K}(q)} \tilde{\Gamma}^{(n)}_R(p_1, ..., p_n; K) = \sum_{l=0}^{1} \frac{1}{l!} \int d^d q_1 ... \int d^d q_l \tilde{\Gamma}^{(l,n)}_R(q_1, ..., q_l; p_1, ..., p_n) \times \int d^d q \tilde{K}(q) \frac{\delta}{\delta \tilde{K}(q)} \tilde{K}(q) \tilde{K}(q_1) ... \tilde{K}(q_l).
\] (4.226)

By using now the Callan-Symanzik equation (4.221) we get

\[
\left( \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial \lambda_R} - \frac{n}{2} \eta - \eta_2 \int d^d q \tilde{K}(q) \frac{\delta}{\delta \tilde{K}(q)} \right) \tilde{\Gamma}^{(n)}_R(p_1, ..., p_n; K) = 0.
\] (4.227)

A massive theory can be obtained by setting the source \(-K(x)\) equal to a constant which will play the role of the renormalized mass \(m_R^2\). We will then set \(K(x) = -m_R^2 \Leftrightarrow \tilde{K}(q) = -m_R^2(2\pi)^d \delta^d(q)\). (4.228)

We obtain therefore the Callan-Symanzik equation

\[
\left( \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial \lambda_R} - \frac{n}{2} \eta - \eta_2 m_R^2 \frac{\partial}{\partial m_R^2} \right) \tilde{\Gamma}^{(n)}_R(p_1, ..., p_n; m_R^2) = 0.
\] (4.229)

This needs to be compared with the renormalization group equation (4.170) and as a consequence the renormalization function \(-\eta_2\) must be compared with the renormalization constant \(\gamma_m\). The renormalized proper vertices \(\tilde{\Gamma}^{(n)}_R\) will also depend on the coupling constant \(\lambda_R\), the subtraction mass scale \(\mu\) and the cutoff \(\Lambda\).

4.2.5 Summary

We end this section by summarizing our main results so far. The bare action and with \(\phi^2\) insertion is

\[
S[\phi, K] = \int d^d x \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} (\phi^2)^2 + \frac{1}{2} K \phi^2 \right].
\] (4.230)

The renormalized action is

\[
S_R[\phi_R, K] = \int d^d x \left[ \frac{1}{2} \partial_\mu \phi_R \partial^\mu \phi_R - \frac{1}{2} m_R^2 \phi_R^2 - \frac{\lambda_R}{4!} (\phi_R^2)^2 + \frac{1}{2} Z_R^2 K \phi_R^2 \right].
\] (4.231)

The dimensionless coupling \(g_R\) and the renormalization constants \(Z, Z_g\) and \(Z_m\) are defined by the equations

\[
g_R = \mu^{-\epsilon} \lambda_R \ , \ \epsilon = 4 - d.
\] (4.232)
\[ \phi = \sqrt{Z} \phi_R \]
\[ \lambda = \lambda_R \frac{Z_g}{Z} \]
\[ m^2 = m^2_R \frac{Z_m}{Z}. \tag{4.233} \]

The arbitrary mass scale \( \mu \) defines the renormalization scale. For example renormalization conditions must be imposed at the scale \( \mu \) as follows

\[ \tilde{\Gamma}_R^{(2)}(p)|_{p^2=0} = m^2_R \]
\[ \frac{d}{dp^2} \tilde{\Gamma}_R^{(2)}(p)|_{p^2=\mu^2} = 1 \]
\[ \tilde{\Gamma}_R^{(4)}(p_1, \ldots, p_4)|_{SP} = \mu^4 R \]
\[ \tilde{\Gamma}_R^{(1,2)}(q; p_1, p_2)|_{q=p_1=p_2=\mu} = 1. \tag{4.234} \]

However we will use in the following minimal subtraction to renormalize the theory instead of renormalization conditions. In minimal subtraction, which is due to ’t Hooft, the renormalization functions \( \beta, \gamma_m, \eta \) and \( \eta_2 \) depend only on the coupling constant \( g_R \) and they are defined by

\[ \beta(g_R) = \left( \mu \frac{\partial g_R}{\partial \mu} \right)_{\lambda,m^2} = -\epsilon \left[ \frac{d}{dg_R} \ln G(g_R) \right]^{-1}, \quad G = \frac{Z_g}{Z R} \tag{4.235} \]
\[ \gamma_m(g_R) = \left( \mu \frac{\partial \ln m^2_R}{\partial \mu} \right)_{\lambda,m^2} = -\beta(g_R) \frac{d}{dg_R} \ln \frac{Z_m}{Z}. \tag{4.236} \]
\[ \eta(g_R) = \left( \mu \frac{\partial \ln Z}{\partial \mu} \right)_{\lambda,m^2} = \beta(g_R) \frac{d}{dg_R} \ln Z. \tag{4.237} \]
\[ \eta_2(g_R) = \left( \mu \frac{\partial \ln Z^2}{\partial \mu} \right)_{\lambda,m^2} = \beta(g_R) \frac{d}{dg_R} \ln Z^2. \tag{4.238} \]

We may also use the renormalization function \( \gamma \) defined simply by

\[ \gamma(g_R) = \frac{\eta(g_R)}{2}. \tag{4.239} \]

The renormalized proper vertices are given by

\[ \tilde{\Gamma}_R^{(n)}(p_i, \mu^2; m_R^2, g_R) = Z^{n/2}(g_R, \epsilon) \tilde{\Gamma}^{(n)}(p_i; m^2, \lambda, \epsilon). \tag{4.240} \]
\[ \tilde{\Gamma}_R^{(1,n)}(q_i; p_i; \mu^2; m_R^2, g_R) = Z^{n-1} Z_g^{(1,n)}(q_i; p_i; m^2, \lambda, \epsilon). \tag{4.241} \]
They satisfy the renormalization group equations

\[
\left( \mu \frac{\partial}{\partial \mu} + \gamma m^2 R \frac{\partial}{\partial m_R} + \beta \frac{\partial}{\partial g_R} - \frac{n}{2} \eta \right) \tilde{\Gamma}^{(n)}_R = 0. \tag{4.242}
\]

\[
\left( m_R \frac{\partial}{\partial m_R} + \beta \frac{\partial}{\partial \lambda_R} - \frac{n}{2} \eta - \eta_2 \right) \tilde{\Gamma}^{(l,n)}_R = -m^2_R \sigma \tilde{\Gamma}^{(l+1,n)}_R. \tag{4.243}
\]

In the first equation we have set \( K = 0 \) and in the second equation the renormalization scale is \( \mu = m_R \). The renormalization function \( \sigma \) is given by

\[
\sigma(g_R) = \frac{Z}{Z_2} \frac{1}{m^2_R} \left( m_R \frac{\partial m^2}{\partial m_R} \right)_\lambda \left( 2 + \beta(g_R) \frac{d}{dg_R} \ln \frac{Z_m}{Z} \right). \tag{4.244}
\]

An alternative renormalization group equation satisfied by the proper vertices \( \tilde{\Gamma}^{(n)}_R \) can be obtained by starting from a massless theory, i.e. \( m = m_R = 0 \) with \( K \neq 0 \) and then setting \( K = -m^2_R \) at the end. We obtain

\[
\left( \mu \frac{\partial}{\partial \mu} + \frac{\beta}{\partial \lambda_R} - \frac{n}{2} \eta - \eta_2 \frac{\partial}{\partial m^2_R} \right) \tilde{\Gamma}^{(n)}_R = 0. \tag{4.245}
\]

In this form the massless limit is accessible. As we can see from (4.242) and (4.245) the renormalization functions \( \gamma (g_R) \) and \( -\eta_2(g_R) \) are essentially the same object. Indeed since the two equations describe the same theory one must have

\[
\eta_2(g_R) = -\gamma (g_R). \tag{4.246}
\]

Alternatively we see from equation (4.244) that the renormalization constant \( Z_2 \) is not an independent renormalization constant since \( \sigma \) is finite. In accordance with (4.246) we choose

\[
Z_2 = Z_m. \tag{4.247}
\]

Because \( Z_2 = Z_m \) equation (4.244) becomes

\[
\sigma(g_R) = \left( m_R \frac{\partial}{\partial m_R} \ln m^2 \right)_\lambda = 2 - \gamma (g_R). \tag{4.248}
\]
4.3 Renormalization Constants and Renormalization Functions at Two-Loop

4.3.1 The Divergent Part of the Effective Action

The 2 and 4–Point Proper Vertices: Now we will renormalize the $O(N)$ sigma model at the two-loop order using dimensional regularization and (modified) minimal subtraction. The main divergences in this theory occur in the 2–point proper vertex (quadratic) and the 4–point proper vertex (logarithmic). Indeed all other divergences in this theory stem from these two functions. Furthermore only the divergence in the 2–point proper vertex is momentum dependent.

The 2–point and 4–point (at zero momentum) proper vertices of the $O(N)$ sigma model at the two-loop order in Euclidean signature are given by equations (2.201) and (2.227), viz

\[ \Gamma^{(2)}_{ij}(p) = \delta_{ij} \left[ p^2 + m^2 + \frac{1}{2} \frac{\lambda N + 2}{3} (a) - \frac{\lambda^2}{4} \left( \frac{N + 2}{3} \right)^2 (b) - \frac{\lambda^2}{6} \frac{N + 2}{3} (c) \right] \] (4.249)

\[ \Gamma^{(4)}_{i_1...i_4}(0,0,0) = \frac{\delta_{i_1i_2i_3i_4}}{3} \left[ \lambda - \frac{3}{2} \frac{N + 8}{9} \lambda^2 (d) + \frac{3}{2} \lambda^3 (N + 2)(N + 8) \frac{27}{27} (g) \right. \]
\[ \left. + \frac{3}{4} \lambda^3 (N + 2)(N + 4) + 12 \frac{27}{27} (e) + 3\lambda^3 \frac{5N + 22}{27} (f) \right]. \] (4.250)

The Feynman diagrams corresponding to (a), (b), (c), (d), (g), (e) and (f) are shown on figure 16. Explicitly we have

\[ (a) = I(m^2) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + m^2}. \] (4.251)

\[ (d) = J(0, m^2) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + m^2)^2}. \] (4.252)

\[ (b) = I(m^2)J(0, m^2) = (a)(d). \] (4.253)

\[ (c) = K(p^2, m) = \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2 + m^2)(k^2 + m^2)(((l + k + p)^2 + m^2)} \]. (4.254)

\[ (g) = I(m^2)L(0, m^2) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + m^2} \int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2 + m^2)^3}. \] (4.255)

\[ (e) = J(0, m^2)^2 = (d)^2. \] (4.256)
\[
(f) = M(0,0,m^2) = \int \frac{d^d l}{(2\pi)^d} \frac{d^d k}{(2\pi)^d} \frac{1}{(l^2 + m^2)(k^2 + m^2)((l + k)^2 + m^2)}. \quad (4.257)
\]

We remark that the two-loop graph \((g)\) is a superposition of the one-loop graphs \((a)\) and \((d)\) and thus it will be made finite once \((a)\) and \((d)\) are renormalized. At the two-loop order only the diagram \((c)\) is momentum dependent. We introduce the notation

\[
(c) = \Sigma^{(2)}(p) = \Sigma^{(2)}(0) + p^2 \frac{\partial}{\partial p^2} \Sigma^{(2)}(0) + \ldots = m^{2d-6}I_2 + p^2 m^{2d-8}I_3 + \ldots \quad (4.258)
\]

We will also introduce the notation

\[
(a) = m^{d-2}I_1. \quad (4.259)
\]

All other integrals can be expressed in terms of \(I_1\) and \(I_2\). Indeed we can show \(^{10}\)

\[
(d) = -\frac{\partial}{\partial m^2}(a) = (1 - \frac{d}{2})m^{d-4}I_1. \quad (4.260)
\]

\[
(b) = (1 - \frac{d}{2})m^{2d-6}I_1^2. \quad (4.261)
\]

\[
(e) = (1 - \frac{d}{2})^2m^{2d-8}I_1^2. \quad (4.262)
\]

\[
(f) = -\frac{1}{3}\frac{\partial}{\partial m^2}\Sigma^{(2)}(0) = -\frac{1}{3}(d - 3)m^{2d-8}I_2. \quad (4.263)
\]

\[
(g) = -\frac{1}{3}(a)\frac{\partial}{\partial m^2}(d) = \frac{1}{2}(1 - \frac{d}{2})(2 - \frac{d}{2})m^{2d-8}I_1^2. \quad (4.264)
\]

**Calculation of The Poles:** We have already met the integral \(I_1\) before. We compute

\[
I_1 = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + 1} = \frac{2}{(4\pi)^{d/2}\Gamma(d/2)} \int \frac{k^{d-1}dk}{k^2 + 1} = \frac{2}{(4\pi)^{d/2}\Gamma(d/2)} \frac{1}{2} \int \frac{x^{d/2-1}dx}{x + 1}. \quad (4.265)
\]

We use the formula

\[
\int \frac{u^\alpha du}{(u+a)\beta} = a^{\alpha+1-\beta} \frac{\Gamma(\alpha + 1)\Gamma(\beta - \alpha - 1)}{\Gamma(\beta)}. \quad (4.266)
\]

\(^{10}\)Exercise: Derive these results.
Thus (with $d = 4 - \epsilon$)

$$
(a) = \frac{m^2}{16\pi^2} \frac{(m^2)^{-\epsilon/2}}{(4\pi)^{-\epsilon/2}} \Gamma(-1 + \epsilon/2) \tag{4.267}
$$

We use the result

$$
\Gamma(-1 + \epsilon/2) = -\frac{2}{\epsilon} - 1 + \gamma + O(\epsilon). \tag{4.268}
$$

Hence we obtain

$$
(a) = \frac{m^2}{16\pi^2} \left[ -\frac{2}{\epsilon} - 1 + \gamma + \ln \frac{m^2}{4\pi} + O(\epsilon) \right]. \tag{4.269}
$$

The first Feynman graph is then given by

$$
\lambda(a) = g \frac{m^2}{16\pi^2} \left[ -\frac{2}{\epsilon} - 1 + \gamma - \ln 4\pi + \ln \frac{m^2}{\mu^2} + O(\epsilon) \right]. \tag{4.270}
$$

In minimal subtraction (MS) we subtract only the pole term $-2/\epsilon$ whereas in modified minimal subtraction (MMS) we subtract also any other extra constant such as the term $-1 + \gamma - \ln 4\pi$.

We introduce

$$
N_d = \frac{2}{(4\pi)^{d/2}\Gamma(d/2)}. \tag{4.271}
$$

We compute

$$
I_1 = \frac{N_d}{2} \Gamma(d/2) \Gamma(1 - d/2) = \frac{N_d}{2} \left( -\frac{2}{\epsilon} + O(\epsilon) \right). \tag{4.272}
$$

Then

$$
\lambda(a) = g \mu^\epsilon \frac{N_d}{2} \left( \frac{m^2}{\mu} \right)^{d/2 - 1} I_1 = \frac{N_d}{2} \left[ -\frac{2}{\epsilon} + \ln \frac{m^2}{\mu^2} + O(\epsilon) \right]. \tag{4.273}
$$

From this formula it is now obvious that subtracting $-N_d/\epsilon$ is precisely the above modified minimal subtraction.

We have also met the integral $\Sigma^{(2)}(p)$ before (see (2.202)). By following the same steps that led to equation (2.205) we obtain

$$
K(p^2; m^2) = \frac{1}{(4\pi)^d} \int dx_1 dx_2 dx_3 \frac{e^{-m^2(x_1 + x_2 + x_3) - \frac{x_1 x_2 x_3}{\Delta}}}{\Delta^{d/2}}, \Delta = x_1 x_2 + x_1 x_3 + x_2 x_3. \tag{4.274}
$$
Thus

\[
I_2 = K(0, 1) = \frac{1}{(4\pi)^d} \int dx_1 dx_2 dx_3 \frac{e^{-(x_1 + x_2 + x_3)}}{\Delta^{d/2}}.
\]

(4.275)

\[
I_3 = \frac{\partial}{\partial p^2} K(p^2, 1)|_{p^2 = 0} = -\frac{1}{(4\pi)^d} \int dx_1 dx_2 dx_3 \frac{x_1 x_2 x_3 e^{-(x_1 + x_2 + x_3)}}{\Delta^{1+d/2}}.
\]

(4.276)

We perform the change of variables \(x_1 = stu, x_2 = st(1-u)\) and \(x_3 = s(1-t)\). Thus \(x_1 + x_2 + x_3 = s, dx_1 dx_2 dx_3 = s^2 dudsdt\) and \(\Delta = s^2 t(1-t+ut(1-u))\). The above integrals become

\[
I_2 = \frac{1}{(4\pi)^d} \int_0^\infty ds e^{-s} s^{2-d} \int_0^1 du \int_0^1 dt \frac{t^{1-d/2}}{(1-t+ut(1-u))^{d/2}}
\]

\[
= \frac{\Gamma(3-d)}{(4\pi)^d} \int_0^1 du \int_0^1 dt \frac{t^{1-d/2}}{(1-t+ut(1-u))^{d/2}}.
\]

(4.277)

\[
I_3 = -\frac{1}{(4\pi)^d} \int_0^\infty ds e^{-s} s^{3-d} \int_0^1 duu(1-u) \int_0^1 dt \frac{t^{2-d/2}(1-t)}{(1-t+ut(1-u))^{1+d/2}}
\]

\[
= -\frac{\Gamma(4-d)}{(4\pi)^d} \int_0^1 duu(1-u) \int_0^1 dt \frac{t^{2-d/2}(1-t)}{(1-t+ut(1-u))^{1+d/2}}.
\]

(4.278)

We want to evaluate the integral

\[
J = \int_0^1 du \int_0^1 dt \frac{t^{1-d/2}(1-t+ut(1-u))^{-d/2}}{(1-t+ut(1-u))^{1+d/2}}
\]

\[
= \int_0^1 du \int_0^1 dt \left[ t^{1-d/2} + \left( (1-t+ut(1-u))^{-d/2} - 1 \right) \right]
\]

\[
+ (t^{1-d/2} - 1) \left( (1-t+ut(1-u))^{-d/2} - 1 \right)
\]

(4.279)

The first term gives the first contribution to the pole term. The last term is finite at \(d = 4\). Indeed with the change of variable \(x = t(u(1-u) - 1) + 1\) we calculate

\[
\int_0^1 du \int_0^1 dt (t^{-1} - 1) \left( (1-t+ut(1-u))^{-2} - 1 \right) = -\int_0^1 du \ln u(1-u)
\]

\[
= -2 \int_0^1 du \ln u = 2.
\]

(4.280)
We have then

\[ J = \frac{2}{\epsilon} - 1 + \int_0^1 du \int_0^1 dt (1 - t + ut(1 - u))^{d/2} + (2 + O_1(\epsilon)) \]

\[ = \frac{2}{\epsilon} - 1 + \frac{2}{\epsilon - 2} \int_0^1 du \frac{1}{u(1 - u) - 1} \left( (u(1 - u))^{1-d/2} - 1 \right) + (2 + O_1(\epsilon)) \]

\[ = \frac{2}{\epsilon} - 1 - \frac{2}{\epsilon - 2} \int_0^1 du \left( (u(1 - u))^{1-d/2} - 1 \right) + \frac{2}{\epsilon - 2} \int_0^1 du \frac{u(1 - u)}{u(1 - u) - 1} \]

\times \left( (u(1 - u))^{1-d/2} - 1 \right) + (2 + O_1(\epsilon)) \]

\[ = \frac{2}{\epsilon} - 1 + \frac{2}{\epsilon - 2} \int_0^1 du (u(1 - u))^{1-d/2} + \frac{2}{\epsilon - 2} (-1 + O_2(\epsilon)) + (2 + O_1(\epsilon)) \]

\[ = \frac{2}{\epsilon} - 1 + \frac{2}{\epsilon - 2} - \frac{2}{\epsilon - 2} \Gamma^2(2 - d/2) + \frac{2}{\epsilon - 2} \Gamma(4 - d) \]

\( \times \left( (u(1 - u))^{1-d/2} - 1 \right) + (2 + O_1(\epsilon)) \]

\[ = \frac{2}{\epsilon} - 1 + \frac{2}{\epsilon - 2} - \frac{2}{\epsilon - 2} \frac{\Gamma^2(2 - d/2)}{\Gamma(4 - d)} + \frac{2}{\epsilon - 2} (-1 + O_2(\epsilon)) + (2 + O_1(\epsilon)) \quad (4.281) \]

In the last line we have used the result (2.275) whereas in the third line we have used the fact that the second remaining integral is finite at \( d = 4 \). From this result we deduce that

\[ J = \frac{6}{\epsilon} + 3 + O(\epsilon). \quad (4.282) \]

\[ I_2 = \frac{\Gamma(3 - d)}{(4\pi)^d} J \]

\[ = \frac{\Gamma(-1 + \epsilon)}{(4\pi)^d} J \]

\[ = \frac{1}{(4\pi)^d} \left( -\frac{6}{\epsilon^2} - \frac{9}{\epsilon} + \frac{6\gamma}{\epsilon} + O(1) \right). \quad (4.283) \]

We compute \( \Gamma(d/2) = 1 + \gamma\epsilon/2 - \epsilon/2 + O(\epsilon^2) \) and hence \( \Gamma^2(d/2) = 1 + \gamma\epsilon - \epsilon + O(\epsilon^2) \). Thus the above result can be rewritten as

\[ I_2 = -N_d^2 \frac{3}{2\epsilon^2} (1 + \frac{\epsilon}{2}) + O(1). \quad (4.284) \]

Now we compute the integral \( I_3 \). The only divergence has already been exhibited by the term \( \Gamma(4 - d) \). Thus we have

\[ I_3 = -\frac{\Gamma(\epsilon)}{(4\pi)^d} \int_0^1 du (1 - u) \int_0^1 dt \frac{1 - t}{(1 - t + ut(1 - u))^2} + O(1) \]

\[ = -\frac{\Gamma(\epsilon)}{(4\pi)^d} \int_0^1 du \frac{1}{2} + O(1) \]

\[ = -N_d^2 \frac{1}{8} \Gamma^2(2 - d) + O(1) \]

\[ = -N_d^2 \frac{1}{8 \epsilon} + O(1). \quad (4.285) \]
4.3.2 Renormalization Constants

One-Loop Renormalization: We prefer to go back to the original expressions

\[ \Gamma_{ij}^{(2)}(p) = \delta_{ij} \left[ p^2 + m^2 + \frac{1}{2} \lambda \frac{N + 2}{3} I(m^2) \right] \]

\[ = \delta_{ij} \left[ p^2 + m^2 + \frac{1}{2} \lambda \frac{N + 2}{3} I(m^2) \right]. \quad (4.286) \]

\[ \Gamma_{i_1 \ldots i_4}^{(4)}(0, 0, 0, 0) = \frac{\delta_{i_1 i_2 i_3 i_4}}{3} \left[ \lambda - \frac{3}{2} \frac{N + 8}{9} \lambda^2 J(d) \right] \]

\[ = \frac{\delta_{i_1 i_2 i_3 i_4}}{3} \left[ \lambda - \frac{3}{2} \frac{N + 8}{9} \lambda^2 J(0, m^2) \right]. \quad (4.287) \]

The renormalized mass and the renormalized coupling constant are given by

\[ m^2 = m^2 R Z_m, \lambda = \lambda R Z_g. \quad (4.288) \]

We will expand the renormalization constants as

\[ Z = 1 + \lambda^2 Z^{(2)}. \quad (4.289) \]

\[ Z_m = 1 + \lambda R Z_m^{(1)} + \lambda^2 R Z_m^{(2)}. \quad (4.290) \]

\[ Z_g = 1 + \lambda R Z_g^{(1)} + \lambda^2 R Z_g^{(2)}. \quad (4.291) \]

At one-loop of course \( Z^{(2)} = Z_m^{(2)} = Z_g^{(2)} = 0. \) We will also define the massless coupling constant by

\[ g = \lambda m^{-\varepsilon}. \quad (4.292) \]

The renormalized 2–point and 4–point proper vertices are given by

\[ (\Gamma_R)_{ij}^{(2)}(p) = Z \Gamma_{ij}^{(2)}(p) \]

\[ = \delta_{ij} \left[ p^2 + m^2 R + \lambda R \left( m^2 R Z_m^{(1)} + \frac{1}{2} \frac{N + 2}{3} I(m^2 R) \right) \right]. \quad (4.293) \]

\[ (\Gamma_R)_{i_1 \ldots i_4}^{(4)}(0, 0, 0, 0) = Z^2 \Gamma_{i_1 \ldots i_4}^{(4)}(0, 0, 0, 0) \]

\[ = \frac{\delta_{i_1 i_2 i_3 i_4}}{3} \left[ \lambda R + \lambda^2 R \left( Z_g^{(1)} - \frac{3}{2} \frac{N + 8}{9} J(0, m^2 R) \right) \right]. \quad (4.294) \]
Minimal subtraction gives immediately

\[ Z_m^{(1)} = \frac{N + 2}{6} I(m_R^2) = \frac{N + 2}{6} m_R \frac{N_d}{\epsilon}. \]  

(4.295)

\[ Z_g^{(1)} = \frac{N + 8}{6} J(0, m_R^2) = \frac{N + 8}{6} m_R \frac{N_d}{\epsilon}. \]  

(4.296)

The renormalized mass and the renormalized coupling constant at one-loop order are given by

\[ m^2 = m_R^2 - \frac{N + 2}{6} \lambda R I(m_R^2), \quad \lambda = \lambda_R + \frac{N + 8}{6} \lambda_R^2 J(0, m_R^2). \]  

(4.297)

**Two-Loop Renormalization of The 2–Point Proper Vertex:** The original expression of the 2–point vertex reads

\[ \Gamma^{(2)}_{ij}(p) = \delta_{ij} \left[ p^2 + m^2 + \frac{1}{2} \lambda \left( \frac{N + 2}{3} \right)^2 (a) - \frac{\lambda^2}{4} \left( \frac{N + 2}{3} \right)^2 (b) - \frac{\lambda^2}{6} \left( \frac{N + 2}{3} \right)^2 (c) \right]. \]  

(4.298)

We use the result

\[ I(m^2) = I(m_R^2) + \frac{N + 2}{6} \lambda R I(m_R^2) J(0, m_R^2) + O(\lambda_R^2). \]  

(4.299)

By using the one-loop results we find the renormalized 2–point proper vertex to be given by

\[ (\Gamma_R)^{(2)}_{ij}(p) = Z \Gamma^{(2)}_{ij}(p) \\
= \delta_{ij} \left[ p^2 + m^2 + Z^{(2)} \lambda_R^2 p^2 + Z_m^{(2)} \lambda_R^2 m_R^2 + \frac{N + 2}{6} Z_g^{(1)} \lambda_R^2 I(m_R^2) + \frac{(N + 2)^2}{36} \lambda_R^2 I(m_R^2) J(0, m_R^2) \right. \]

\[ - \frac{\lambda_R^2}{4} \left( \frac{N + 2}{3} \right)^2 (b) - \frac{\lambda_R^2}{6} \left( \frac{N + 2}{3} \right) (c) \right] + \left. \frac{N + 2}{6} Z_g^{(1)} \lambda_R^2 I(m_R^2) - \frac{\lambda_R^2}{6} \left( \frac{N + 2}{3} \right) (m_R^{2d-6} I_2 + \right. \]

\[ + \left. p^2 m_R^{2d-8} I_3 \right). \]  

(4.300)

In the last equation we have used the results \((b)_R = I(m_R^2) J(0, m_R^2)\) and \((c)_R = m_R^{2d-6} I_2 + \right. \]

\[ p^2 m_R^{2d-8} I_3. \]  

By requiring finiteness of the kinetic term we obtain the result

\[ Z^{(2)} = \frac{N + 2}{18} m_R^{2d-8} I_3 = - \frac{N + 2}{144} m_R^{-2} \frac{N_R^2}{\epsilon}. \]  

(4.301)
Cancellation of the remaining divergences gives

\[
Z_m^{(2)} = \frac{N + 2}{6} \frac{Z_g^{(1)}}{m_R^2} + \frac{N + 2}{18} m_R^{2d-8} I_2
\]

\[
= \frac{(N + 2)(N + 8)}{36} m_R^{2e} \frac{N_d^2}{\epsilon^2} - \frac{N + 2}{12} m_R^{2e} \frac{N_d^2}{\epsilon^2} - \frac{N + 2}{24} m_R^{2e} \frac{N_d^2}{\epsilon^2}
\]

Two-Loop Renormalization of The 4-Point Proper Vertex: The original expression of the 4-point vertex reads

\[
\Gamma^{(4)}_{i_1...i_4}(0,0,0,0) = \frac{\delta_{i_1i_2i_3i_4}}{3} \left[ \lambda - \frac{3N + 8}{9} \lambda^2(d) + \frac{3}{2} \lambda^3(N + 2)(N + 8)(g) + \frac{3}{4} \lambda^3(N + 2)(N + 4) + \frac{12}{27} \lambda^3(5N + 22)(f) \right].
\]

We use the result

\[
J(0, m^2) = J(0, m_R^2) + \frac{N + 2}{3} \lambda_R I(m_R^2) L(0, m_R^2) + O(\lambda_R^2).
\]

By using the one-loop results we find the renormalized 4-point proper vertex to be given by

\[
(\Gamma^{(4)}_R)_{i_1...i_4}(0,0,0,0) = Z^2 \Gamma^{(4)}_{i_1...i_4}(0,0,0,0)
\]

\[
= \frac{\lambda_R + \lambda_R \lambda^3 Z_g^{(2)}(d)_R - \frac{N + 8}{3} \lambda^3 Z_g^{(1)}(d)_R - \frac{(N + 8)(N + 2)}{18} \lambda^3 I(m_R^2)_R L(0, m_R^2)}{3}
\]

\[
+ \frac{3}{2} \lambda_R^3 (N + 2)(N + 8)(g)_R + \frac{3}{4} \lambda_R^3 (N + 2)(N + 4) + \frac{12}{27} \lambda_R^3 (5N + 22)(f)_R
\]

\[
= \frac{\lambda_R + \lambda_R \lambda^3 Z_g^{(2)}(d)_R - \frac{N + 8}{3} \lambda_R^3 Z_g^{(1)}(d)_R + \frac{3}{4} \lambda_R^3 (N + 2)(N + 4) + \frac{12}{27} \lambda_R^3 (5N + 22)(f)_R}{3}
\]

Cancellation of the remaining divergences gives

\[
Z_g^{(2)} = \frac{N + 8}{3} \frac{Z_g^{(1)}}{m_R^2} - \frac{3}{4} \frac{(N + 2)(N + 4)}{27} \langle e \rangle_R + \frac{5N + 22}{27} \langle f \rangle_R - \frac{3}{2} m_R^{2e} I_1 - \frac{(N + 8)^2}{18} m_R^{2e} \frac{N_d^2}{\epsilon^2} - \frac{N_d^2}{2\epsilon^2} - \frac{N_d^2}{2\epsilon^2} - \frac{5N + 22}{27} m_R^{2e} \frac{N_d^2}{\epsilon^2} - \frac{N_d^2}{2\epsilon^2}
\]

\[
= \frac{(N + 8)^2}{36} m_R^{2e} \frac{N_d^2}{\epsilon^2} - \frac{5N + 22}{36} m_R^{2e} \frac{N_d^2}{\epsilon^2}.
\]
4.3.3 Renormalization Functions

The renormalization constants up to two-loop order are given by

\[ Z = 1 - g_R^2 \left( \frac{N + 2 N_d^2}{144 \epsilon} \right). \]  

(4.307)

\[ Z_g = 1 + \frac{N + 8}{6} g_R \frac{N_d}{\epsilon} + g_R^2 \left( \frac{(N + 8)^2 N_d^2}{36 \epsilon^2} - \frac{5N + 22 N_d^2}{36 \epsilon} \right). \]  

(4.308)

\[ Z_m = 1 + \frac{N + 2}{6} g_R \frac{N_d}{\epsilon} + g_R^2 \left( \frac{(N + 2)(N + 5) N_d^2}{36 \epsilon^2} - \frac{N + 2 N_d^2}{24 \epsilon} \right). \]  

(4.309)

The beta function is given by

\[ \beta(g_R) = -\frac{\epsilon g_R}{1 + g_R \frac{d}{dg_R} \ln Z - 2 g_R \frac{d}{dg_R} \ln Z}. \]  

(4.310)

We compute

\[ g_R \frac{d}{dg_R} \ln Z_g = \frac{N + 8}{6} g_R \frac{N_d}{\epsilon} + g_R^2 \left( \frac{(N + 8)^2 N_d^2}{36 \epsilon^2} - \frac{5N + 22 N_d^2}{18 \epsilon} \right). \]  

(4.311)

\[ \frac{d}{dg_R} \ln Z = -\frac{N + 2}{72} g_R \frac{N_d^3}{\epsilon} \Rightarrow -2 g_R \frac{d}{dg_R} \ln Z = \frac{N + 2}{36} g_R \frac{N_d^2}{\epsilon}. \]  

(4.312)

We get then the fundamental result

\[ \beta(g_R) = -\epsilon g_R + \frac{N + 8}{6} g_R N_d - \frac{3N + 14}{12} g_R^3 N_d^2. \]  

(4.313)

The second most important renormalization function is \( \eta \). It is defined by

\[ \eta(g_R) = \beta(g_R) \frac{d}{dg_R} \ln Z \]  

\[ = \left( -\epsilon g_R + \frac{N + 8}{6} g_R N_d - \frac{3N + 14}{12} g_R^3 N_d^2 \right) \left( -\frac{N + 2}{72} g_R \frac{N_d^2}{\epsilon} \right) \]  

\[ = \frac{N + 2}{72} g_R^2 N_d^2. \]  

(4.314)

The renormalization constant \( Z_m \) is associated with the renormalization function \( \gamma_m \) defined by

\[ \gamma_m(g_R) = -\beta(g_R) \frac{d}{dg_R} \ln \frac{Z_m}{Z}. \]  

(4.315)
We compute
\[
\frac{d}{dg_R} \ln Z_m = \frac{N + 2}{6} N_d \epsilon + g_R \left( \frac{(N + 2)(N + 8)}{36} \frac{N_d^2}{\epsilon^2} - \frac{N + 2}{12} \frac{N_d^2}{\epsilon} \right).
\] (4.316)

The renormalization function \( \gamma_m \) at the two-loop order is then found to be given by
\[
\gamma_m = \frac{N + 2}{6} N_d g_R - \frac{5(N + 2)}{72} g_R^2 N_d^2.
\] (4.317)

From this result we conclude immediately that
\[
\eta_2 = -\gamma_m = -\frac{N + 2}{6} N_d g_R + \frac{5(N + 2)}{72} g_R^2 N_d^2.
\] (4.318)

### 4.4 Critical Exponents

#### 4.4.1 Critical Theory and Fixed Points

We will postulate that quantum scalar field theory, in particular \( \phi^4 \), describes the critical domain of second order phase transitions which includes the critical line \( T = T_c \) where the correlation length \( \xi \) diverges and the scaling region with \( T \) near \( T_c \) where the correlation length \( \xi \) is large but finite. This is confirmed for example by mean field calculations.

From now on we will work in Euclidean signature. We write the action in the form
\[
S[\phi] = \beta \mathcal{H}(\phi) = \int d^d x \left[ \frac{1}{2} (\partial_{\mu} \phi)^2 + \frac{1}{2} m^2 \phi^2 - \frac{\Lambda^4}{4!} (\phi^2)^2 \right].
\] (4.319)

In the above action the cutoff \( \Lambda \) reflects the original lattice structure, i.e. \( \Lambda = 1/a \). The cutoff procedure is irrelevant to the physics and as a consequence we will switch back and forth between cutoff regularization and dimensional regularization as needed. The critical domain is defined by the conditions
\[
|m^2 - m_c^2| << \Lambda^2
\]
\[
\text{momenta} << \Lambda
\]
\[
< \phi(x) >> < \Lambda^2-1.
\] (4.320)

In above \( m_c^2 \) is the value of the mass parameter \( m^2 \) at the critical temperature \( T_c \) where \( m^2_R = 0 \) or the correlation length \( \xi \) diverges. Clearly \( m_c^2 \) is essentially mass renormalization. We will set
\[
m^2 = m_c^2 + t, \quad t \propto \frac{T - T_c}{T_c}.
\] (4.321)

The critical theory should be renormalized at a scale \( \mu \) in such a way that the renormalized mass remains massless, viz
\[
\tilde{\Gamma}^{(2)}_R (p; \mu, g_R, \Lambda) |_{p^2 = 0} = 0
\]
\[
\frac{d}{dp^2} \tilde{\Gamma}^{(2)}_R (p; \mu, g_R, \Lambda) |_{p^2 = \mu^2} = 1
\]
\[
\tilde{\Gamma}^{(4)}_R (p_1, ..., p_4; \mu, g_R, \Lambda) |_{\text{SP}} = \mu^4 g_R.
\] (4.322)
The renormalized proper vertices are given by
\[
\tilde{\Gamma}_R^{(n)}(p_i; \mu, g_R) = Z^{n/2}(\mu, \frac{\Lambda}{\mu}) \tilde{\Gamma}^{(n)}(p_i; g, \Lambda).
\] (4.323)

The bare proper vertices \( \tilde{\Gamma}^{(n)} \) are precisely the proper vertices of statistical mechanics. Now since the renormalized proper vertices \( \tilde{\Gamma}_R^{(n)} \) are independent of \( \Lambda \) we should have
\[
\left( \Lambda \frac{\partial}{\partial \Lambda} Z^{n/2}\tilde{\Gamma}^{(n)} \right)_{\mu, \beta_R} = 0.
\] (4.324)

We obtain the renormalization group equation
\[
\left( \Lambda \frac{\partial}{\partial \Lambda} + \beta \frac{\partial}{\partial g} - \frac{n}{2} \eta \right) \tilde{\Gamma}^{(n)} = 0.
\] (4.325)

The renormalization functions are now given by
\[
\beta(g) = \left( \Lambda \frac{\partial g}{\partial \Lambda} \right)_{g_R, \mu}.
\] (4.326)
\[
\eta(g) = -\left( \Lambda \frac{\partial}{\partial \Lambda} \ln Z \right)_{g_R, \mu}.
\] (4.327)

Clearly the functions \( \beta \) and \( \eta \) cannot depend on the ratio \( \Lambda / \mu \) since \( \tilde{\Gamma}^{(n)} \) does not depend on \( \mu \). We state the (almost) obvious theorem

- The renormalization group equation (4.325) is a direct consequence of the existence of a renormalized field theory. Conversely the existence of a solution to this renormalization group equation implies the existence of a renormalized theory.

**The fixed point \( g = g_\ast \) and the critical exponent \( \omega \):** The renormalization group equation (4.325) can be solved using the method of characteristics. We introduce a dilatation parameter \( \lambda \), a running coupling constant \( g(\lambda) \) and an auxiliary renormalization function \( Z(\lambda) \) such that
\[
\lambda \frac{d}{d\lambda} \left[ Z^{-n/2}(\lambda) \tilde{\Gamma}^{(n)}(p_i; g(\lambda), \lambda \Lambda) \right] = 0.
\] (4.328)

We can verify that proper vertices \( \tilde{\Gamma}^{(n)}(p_i; g(\lambda), \lambda \Lambda) \) solves the renormalization group equation (4.325) provided that \( \beta \) and \( \eta \) solves the first order differential equations
\[
\beta(g(\lambda)) = \lambda \frac{d}{d\lambda} g(\lambda), \ g(1) = g.
\] (4.329)

\[\text{Exercise: Show this result.}\]
\[ \eta(g(\lambda)) = \lambda \frac{d}{d\lambda} \ln Z(\lambda), \ Z(1) = Z. \quad (4.330) \]

We have the identification
\[ \tilde{\Gamma}^{(n)}(p_i; g, \Lambda) = Z^{-n/2}(\lambda)\tilde{\Gamma}^{(n)}(p_i; g(\lambda), \lambda \Lambda). \quad (4.331) \]

Equivalently
\[ \tilde{\Gamma}^{(n)}(p_i; g, \frac{\Lambda}{\lambda}) = Z^{-n/2}(\lambda)\tilde{\Gamma}^{(n)}(p_i; g(\lambda), \Lambda). \quad (4.332) \]

The limit \( \Lambda \to \infty \) is equivalent to the limit \( \lambda \to 0 \). The functions \( \beta \) and \( \eta \) are assumed to be regular functions for \( g \geq 0 \).

The integration of (4.329) and (4.330) yields the integrated renormalization group equations
\[ \ln \lambda = \int_g^{g(\lambda)} \frac{dx}{\beta(x)}. \quad (4.333) \]
\[ \ln Z(\lambda) = \int_1^\lambda \frac{dx}{x} \eta(g(x)). \quad (4.334) \]

The zeros \( g = g_* \) of the beta function \( \beta \) which satisfy \( \beta(g_*) = 0 \) are of central importance to quantum field theory and critical phenomena. Let us assume that the a zero \( g = g_* \) of the beta function does indeed exist. We observe then that any value of the running coupling constant \( g(\lambda) \) near \( g_* \) will run into \( g_* \) in the limit \( \lambda \to 0 \) regardless of the initial value \( g = g(1) \) which can be either above or below \( g_* \). This can be made precise as follows. We expand \( \beta(g) \) about the zero as follows
\[ \beta(g) = \beta(g_*) + (g - g_*)\omega + \ldots \quad (4.335) \]
\[ \beta(g_*) = 0, \quad \omega = \beta'(g_*). \quad (4.336) \]

We compute
\[ \ln \lambda = \frac{1}{\omega} \frac{g(\lambda) - g_*}{g - g_*} \Rightarrow g(\lambda) - g_* \sim \lambda^\omega, \ \lambda \to 0. \quad (4.337) \]

If \( \omega > 0 \) then \( g(\lambda) \to g_* \) when \( \lambda \to 0 \). The point \( g = g_* \) is then called an attractive or stable infrared (since the limit \( \lambda \to 0 \) is equivalent to the massless limit \( \lambda \Lambda \to 0 \)) fixed point (since \( \frac{d^n g(\lambda)}{d\lambda^n}|_{g_*} = 0 \)). If \( \omega < 0 \) then the point \( g = g_* \) is called a repulsive infrared fixed point or equivalently a stable ultraviolet fixed point since \( g(\lambda) \to g_* \) when \( \lambda \to \infty \).
The slope $\omega = \beta'(g_*)$ is our first critical exponent which controls leading corrections to scaling laws.

As an example let us consider the beta function

$$\beta(g) = -\epsilon g + bg^2, \quad b = \frac{3}{16\pi^2}. \quad (4.338)$$

There are in this case two fixed points the origin and $g_* = \epsilon/b$ with critical exponents $\omega = -\epsilon < 0$ (infrared repulsive) and $\omega = +\epsilon$ (infrared attractive) respectively. We compute immediately

$$\ln \lambda = \frac{1}{\epsilon} \int_{1/g}^{1/g(\lambda)} \frac{dx}{x - 1/g_*} \Rightarrow g(\lambda) = \frac{g_*}{1 + \lambda^\epsilon (g_*/g - 1)}. \quad (4.339)$$

Since $\epsilon = 4 - d > 0$, $g(\lambda) \rightarrow g_*$ when $\lambda \rightarrow 0$ and as a consequence $g_* = \epsilon/b$ is a stable infrared fixed point known as the non trivial (interacting) Wilson-Fisher fixed point. In the limit $\lambda \rightarrow \infty$ we see that $g(\lambda) \rightarrow 0$, i.e. the origin is a stable ultraviolet fixed point which is the famous trivial (free) Gaussian fixed point. See figure 15.

The fact that the origin is a repulsive (unstable) infrared fixed point is the source of the strong infrared divergence found in dimensions $< 4$ since perturbation theory in this case is an expansion around the wrong fixed point. Remark that for $d > 4$ the origin becomes an attractive (stable) infrared fixed point while $g = g_*$ becomes repulsive.

**The critical exponent $\eta$:** Now we solve the second integrated renormalization group equation (4.330). We expand $\eta$ as $\eta(g(\lambda)) = \eta(g_*) + (g(\lambda) - g_*)\eta'(g_*) + \ldots$. In the limit $\lambda \rightarrow 0$ we obtain

$$\ln Z(\lambda) = \eta \ln \lambda + \ldots \Rightarrow Z(\lambda) = \lambda^\eta. \quad (4.340)$$

The critical exponent $\eta$ is defined by

$$\eta = \eta(g_*). \quad (4.341)$$

The proper vertex (4.332) becomes

$$\tilde{\Gamma}^{(n)}(p_i; g, \frac{\Lambda}{\lambda}) = \lambda^{-\frac{n}{2}} \eta \Gamma^{(n)}(p_i; g(\lambda), \Lambda). \quad (4.342)$$

However from dimensional considerations we know the mass dimension of $\tilde{\Gamma}^{(n)}(p_i; g, \frac{\Lambda}{\lambda})$ to be $M^{d-n(d/2-1)}$ and hence the mass dimension of $\Gamma^{(n)}(\lambda p_i; g, \Lambda)$ is $(\lambda M)^{d-n(d/2-1)}$. We get therefore

$$\tilde{\Gamma}^{(n)}(p_i; g, \frac{\Lambda}{\lambda}) = \lambda^{-d+n(d/2-1)} \Gamma^{(n)}(\lambda p_i; g, \lambda). \quad (4.343)$$

By combining these last two equations we obtain the crucial result
\[ \tilde{\Gamma}^{(n)}(\lambda p_i; g, \Lambda) = \lambda^{-d + \frac{n}{2}(d-2+\eta)} \tilde{\Gamma}^{(n)}(p_i; g, \Lambda), \lambda \to 0. \] (4.344)

The critical proper vertices have a power law behavior for small momenta which is independent of the original value \( g \) of the \( \phi^4 \) coupling constant. This in turn is a manifestation of the universality of the critical behavior. The mass dimension of the field \( \phi \) has also changed from the canonical (classical) value \((d-2)/2\) to the anomalous (quantum) value

\[ d_\phi = \frac{1}{2}(d - 2 + \eta). \] (4.345)

In the particular case \( n = 2 \) we have the behavior

\[ \tilde{\Gamma}^{(2)}(\lambda p; g, \Lambda) = \lambda^{\eta-2}\tilde{\Gamma}^{(2)}(p; g, \Lambda), \lambda \to 0. \] (4.346)

Hence the 2-point function must behave as

\[ \tilde{\tilde{G}}^{(2)}(p) \sim \frac{1}{p^{2-\eta}}, p \to 0. \] (4.347)

**The critical exponent \( \nu \):** The full renormalized conditions of the massless (critical) theory when \( K \neq 0 \) are (4.322) plus the two extra conditions

\[ \tilde{\Gamma}^{(1,2)}_R(q; p_1, p_2; \mu, g, R, \Lambda)|_{q^2 = \mu^2, p_1 p_2 = -\frac{1}{2} \mu^2} = 1. \] (4.348)

\[ \tilde{\Gamma}^{(2,0)}_R(q; -q; \mu, g, R, \Lambda)|_{q^2 = 4 \mu^2} = 0. \] (4.349)

The first condition fixes the renormalization constant \( Z_2 \) while the second condition provides a renormalization of the \( <\phi^2 \phi^2> \) correlation function.

The renormalized proper vertices are defined by (with \( l + n > 2 \))

\[ \tilde{\Gamma}^{(l,n)}_R(q_i; p_i; \mu, g R) = Z^{n/2-l}(g, \frac{\Lambda}{\mu}) Z_2^l(g, \frac{\Lambda}{\mu}) \tilde{\Gamma}^{(l,n)}(q_i; p_i; g, \Lambda). \] (4.350)

We have clearly the condition

\[ \left( \Lambda \frac{\partial}{\partial \Lambda} Z^{n/2-l} Z_2^l \tilde{\Gamma}^{(l,n)} \right)_{\mu, g R} = 0. \] (4.351)

We obtain the renormalization group equation

\[ \left( \Lambda \frac{\partial}{\partial \Lambda} + \beta \frac{\partial}{\partial g} - \frac{n}{2} \eta - l \eta \right) \tilde{\Gamma}^{(l,n)} = 0. \] (4.352)
The renormalization functions $\beta$ and $\eta$ are still given by equations (4.326) and (4.327) while the renormalization function $\eta_2$ is defined by

$$\eta_2(g) = -\left(\Lambda \frac{\partial}{\partial \Lambda} \ln \frac{Z_2}{Z}\right)_{g_R, \mu}. \quad (4.353)$$

As before we solve the above renormalization group equation (4.352) by the method of characteristics. We introduce a dilatation parameter $\lambda$, a running coupling constant $g(\lambda)$ and auxiliary renormalization functions $Z(\lambda)$ and $\zeta_2(\lambda)$ such that

$$\lambda \frac{d}{d\lambda} \left[Z^{-n/2}(\lambda)\zeta_2^{-1}(\lambda)\tilde{\Gamma}^{(l,n)}(q_i; p_i; g(\lambda), \lambda\Lambda)\right] = 0. \quad (4.354)$$

We can verify that proper vertices $\tilde{\Gamma}^{(l,n)}$ solves the above renormalization group equation (4.352) provided that $\beta$, $\eta$ solve the first order differential equations (4.329) and (4.330) and $\eta_2$ solves the first order differential equation

$$\eta_2(g(\lambda)) = \lambda \frac{d}{d\lambda} \ln \zeta_2(\lambda), \quad \zeta_2(1) = \zeta_2. \quad (4.355)$$

We have the identification

$$\tilde{\Gamma}^{(l,n)}(q_i; p_i; g, \Lambda) = Z^{-n/2}(\lambda)\zeta_2^{-1}(\lambda)\tilde{\Gamma}^{(l,n)}(q_i; p_i; g(\lambda), \lambda\Lambda). \quad (4.356)$$

Equivalently

$$\tilde{\Gamma}^{(l,n)}(q_i; p_i; g, \frac{\Lambda}{\lambda}) = Z^{-n/2}(\lambda)\zeta_2^{-1}(\lambda)\tilde{\Gamma}^{(l,n)}(q_i; p_i; g(\lambda), \Lambda). \quad (4.357)$$

The corresponding integrated renormalization group equation is

$$\ln \zeta_2 = \int_{\lambda}^{1} \frac{dx}{x} \eta_2(g(x)). \quad (4.358)$$

We obtain in the limit $\lambda \to 0$ the behavior

$$\zeta_2(\lambda) = \lambda^{\eta_2}. \quad (4.359)$$

The new critical exponent $\eta_2$ is defined by

$$\eta_2 = \eta_2(g_*). \quad (4.360)$$

We introduce the mass critical exponent $\nu$ by the relation

$$\nu = \nu(g_*) \quad \nu(g) = \frac{1}{2 + \eta_2(g)}. \quad (4.361)$$
We have then the infrared behavior of the proper vertices given by
\[
\tilde{\Gamma}^{(l,n)}(q_i; p_i; g, \Lambda, \lambda) = \lambda^{d - d/4 - l/2} \tilde{\Gamma}^{(l,n)}(q_i; p_i; g(\lambda), \Lambda).
\] (4.362)

From dimensional considerations the mass dimension of the proper vertex \(\tilde{\Gamma}^{(l,n)}(q_i; p_i; g, \Lambda/\lambda)\) is \(M^{d-n(d-2)/2-2l}\) and hence
\[
\tilde{\Gamma}^{(l,n)}(q_i; p_i; g, \Lambda) = \lambda^{-d + \frac{n}{2}(d-2)+2l} \tilde{\Gamma}^{(l,n)}(\lambda q_i; \lambda p_i; g(\lambda), \Lambda).
\] (4.363)

By combining the above two equations we obtain
\[
\tilde{\Gamma}^{(l,n)}(\lambda q_i; \lambda p_i; g, \Lambda) = \lambda^{d - \frac{n}{2}(d-2)+\eta_2} \tilde{\Gamma}^{(l,n)}(q_i; p_i; g_*, \Lambda), \quad \lambda \rightarrow 0.
\] (4.364)

### 4.4.2 Scaling Domain \((T > T_c)\)

In this section we will expand around the critical theory. The proper vertices for \(T > T_c\) can be calculated in terms of the critical proper vertices with \(\phi^2\) insertions.

The correlation length: In order to allow a large but finite correlation length (non zero renormalized mass) in this massless theory without generating infrared divergences we consider the action
\[
S[\phi] = \beta \mathcal{H}(\phi) = \int d^d x \left[ \frac{1}{2} (\partial_t \phi)^2 + \frac{1}{2} (m_c^2 + K(x))\phi^2 - \frac{\Lambda^4 g}{4!} (\phi^2)^2 \right].
\] (4.365)

We want to set at the end
\[
K(x) = t \propto \frac{T - T_c}{T_c}.
\] (4.366)

The \(n\)-point proper vertices are given by
\[
\tilde{\Gamma}^{(n)}(p_i; K, g, \Lambda) = \sum_{l=0}^{n-1} \frac{1}{l!} \int \frac{d^d q_1}{(2\pi)^d} \cdots \int \frac{d^d q_l}{(2\pi)^d} \tilde{\Gamma}^{(l,n)}(q_i; p_i; g, \Lambda) \tilde{K}(q_1) \cdots \tilde{K}(q_l).
\] (4.367)

We consider the differential operator
\[
D = \Lambda \frac{\partial}{\partial \Lambda} + \beta \frac{\partial}{\partial g} - \frac{n}{2} \eta_1 - \eta_2 \int d^d \Phi \frac{\delta}{\delta \tilde{K}(q)}.
\] (4.368)

We compute
\[
\int d^d q \tilde{K}(q) \frac{\delta}{\delta \tilde{K}(q)} \tilde{\Gamma}^{(n)}(p_i; K, g, \Lambda) = \sum_{l=0}^{n-1} \frac{1}{l!} \int d^d q_1 \cdots \int d^d q_l \tilde{\Gamma}^{(l,n)}(q_i; p_i; g, \Lambda) \\
\times \int d^d q \tilde{K}(q) \frac{\delta}{\delta \tilde{K}(q)} \tilde{K}(q_1) \cdots \tilde{K}(q_l) \\
= \sum_{l=0}^{n-1} \frac{1}{l!} \int d^d q_1 \cdots \int d^d q_l \tilde{\Gamma}^{(l,n)}(q_i; p_i; g, \Lambda) \tilde{K}(q_1) \cdots \tilde{K}(q_l).
\] (4.369)
By using now the Callan-Symanzik equation (4.352) we get

\[
\left( \Lambda \frac{\partial}{\partial \Lambda} + \beta \frac{\partial}{\partial g} - \frac{n}{2} \eta - \eta_2 \int d^d q \tilde{K}(q) \frac{\delta}{\delta K(q)} \right) \tilde{\Gamma}^{(n)}(p_i; K, g, \Lambda) = 0. \tag{4.370}
\]

We now set \( K(x) = t \) or equivalently \( \tilde{K}(q) = t(2\pi)^d \delta^d(q) \) to obtain

\[
\left( \Lambda \frac{\partial}{\partial \Lambda} + \beta \frac{\partial}{\partial g} - \frac{n}{2} \eta - \eta_2 t \frac{\partial}{\partial t} \right) \tilde{\Gamma}^{(n)}(p_i; t, g, \Lambda) = 0. \tag{4.371}
\]

We employ again the method of characteristics in order to solve this renormalization group equation. We introduce a dilatation parameter \( \lambda \), a running coupling constant \( g(\lambda) \), a running mass \( t(\lambda) \) and an auxiliary renormalization functions \( Z(\lambda) \) such that

\[
\lambda \frac{d}{d\lambda} \left[ Z^{-n/2}(\lambda) \tilde{\Gamma}^{(n)}(p_i; t(\lambda), g(\lambda), \lambda \Lambda) \right] = 0. \tag{4.372}
\]

Then \( \tilde{\Gamma}^{(n)}(p_i; t(\lambda), g(\lambda), \lambda \Lambda) \) will solve the renormalization group equation (4.371) provided the renormalization functions \( \beta, \eta \) and \( \eta_2 \) satisfy

\[
\beta(g(\lambda)) = \lambda \frac{d}{d\lambda} g(\lambda) , \quad g(1) = g. \tag{4.373}
\]

\[
\eta(g(\lambda)) = \lambda \frac{d}{d\lambda} \ln Z(\lambda) , \quad Z(1) = Z. \tag{4.374}
\]

\[
\eta_2(g(\lambda)) = -\lambda \frac{d}{d\lambda} \ln t(\lambda) , \quad t(1) = t. \tag{4.375}
\]

The new definition of \( \eta_2 \) given in the last equation is very similar to the definition of \( \gamma_m \) given in equation (4.246). We make the identification

\[
\tilde{\Gamma}^{(n)}(p_i; t, g, \Lambda) = Z^{-n/2}(\lambda) \tilde{\Gamma}^{(n)}(p_i; t(\lambda), g(\lambda), \lambda \Lambda). \tag{4.376}
\]

From dimensional considerations we have

\[
\tilde{\Gamma}^{(n)}(p_i; t(\lambda), g(\lambda), \lambda \Lambda) = (\lambda \Lambda)^{d-\frac{d}{2}(d-2)} \tilde{\Gamma}^{(n)}(p_i; \lambda \frac{t(\lambda)}{m^2}, \lambda^2 \lambda^2, g(\lambda), 1). \tag{4.377}
\]

Thus

\[
\tilde{\Gamma}^{(n)}(p_i; t, g, \Lambda) = Z^{-n/2}(\lambda) m^{d-\frac{d}{2}(d-2)} \tilde{\Gamma}^{(n)}(p_i; \lambda \frac{t(\lambda)}{m^2}, \lambda^2 \lambda^2, g(\lambda), 1). \tag{4.378}
\]

We have used the notation \( m = \lambda \Lambda \). We use the freedom of choice of \( \lambda \) to choose

\[
t(\lambda) = m^2 = \lambda^2 \Lambda^2. \tag{4.379}
\]
The theory at scale $\lambda$ is therefore not critical since the critical regime is defined by the requirement $t << \Lambda^2$.

The integrated form of the renormalization group equation (4.375) is given by

$$ t(\lambda) = t \exp - \int_1^\lambda \frac{dx}{x} \eta_2(g(x)). $$

(4.380)

This can be rewritten as

$$ \ln \frac{t \lambda^2}{t(\lambda)} = \int_1^\lambda \frac{dx}{x} \frac{1}{\nu(g(x))}. $$

(4.381)

Equivalently

$$ \ln \frac{t}{\Lambda^2} = \int_1^\lambda \frac{dx}{x} \frac{1}{\nu(g(x))}. $$

(4.382)

This is an equation for $\lambda$. In the critical regime $\ln t/\Lambda^2 \to -\infty$. For $\nu(g) > 0$ this means that $\lambda \to 0$ and hence $g(\lambda) \to g_*$. By expanding around the fixed point $g(\lambda) = g_*$ we obtain in the limit $\lambda \to 0$ the result

$$ \lambda = \left( \frac{t}{\Lambda^2} \right)^{\nu}. $$

(4.383)

By using this result in (4.378) we conclude that the proper vertices $\tilde{\Gamma}^{(n)}(p_i; t, g, \Lambda = 1)$ must have the infrared ($\lambda \to 0$) scaling

$$ \tilde{\Gamma}^{(n)}(p_i; t, g, \Lambda = 1) = m^{d-\frac{d-2}{2}(d-2+\eta)} \tilde{\Gamma}^{(n)}(\frac{p_i}{m}; 1, g_*, 1), t << 1, p << 1. $$

(4.384)

The mass $m$ is thus the physical mass $\xi^{-1}$ where $\xi$ is the correlation length. We have then

$$ m(\Lambda = 1) = \xi^{-1} = t^{\nu}. $$

(4.385)

Clearly $\xi \to \infty$ when $t \to 0$ or equivalently $T \to T_c$ since $\nu > 0$.

The critical exponents $\alpha$ and $\gamma$: At zero momentum the above proper vertices are finite because of the non zero mass $t$. They have the infrared scaling

$$ \tilde{\Gamma}^{(n)}(0; t, g, \Lambda = 1) = m^{d-\frac{d}{2}(d-2+\eta)} t^{\nu(d-\frac{d-2}{2}(d-2+\eta))}, t << 1, p << 1. $$

(4.386)

The case $n = 2$ is of particular interest since it is related to the inverse susceptibility, viz

$$ \chi^{-1} = \tilde{\Gamma}^{(2)}(0; t, g, \Lambda = 1) = t^{\gamma}. $$

(4.387)
The critical exponent $\gamma$ is given by

$$\gamma = \nu (2 - \eta).$$

(4.388)

The obvious generalization of the renormalization group equation (4.371) is

$$\left( \Lambda \frac{\partial}{\partial \Lambda} + \beta \frac{\partial}{\partial g} - \frac{n}{2} \eta - \eta_2 (l + t \frac{\partial}{\partial t}) \right) \tilde{\Gamma}^{(l,n)}(q_i; p_i; t, g, \Lambda) = 0.$$  

(4.389)

This is valid for all $n + l > 2$. The case $l = 2$, $n = 0$ is special because of the non-multiplicative nature of the renormalization required in this case and as a consequence the corresponding renormalization group equation will be inhomogeneous. However we will not pay attention to this difference since the above renormalization group equation is sufficient to reproduce the leading infrared behavior, and as a consequence the relevant critical exponent, of the proper vertex with $l = 2$ and $n = 0$.

We find after some calculation, similar to the calculation used for the case $l = 0$, the leading infrared ($\lambda \to 0$) behavior

$$\tilde{\Gamma}^{(l,n)}(q; p; t, g, \Lambda = 1) = m^{-\frac{l}{2} + d - \frac{n}{2} (d-2+\eta)} \tilde{\Gamma}^{(l,n)}(\frac{q_i}{m}; \frac{p_i}{m}; 1, g, 1), \ t << 1, \ q, p << 1.$$

(4.390)

By applying this formula naively to the case $l = 2$, $n = 0$ we get the desired leading infrared behavior of $\tilde{\Gamma}^{(2,0)}$ which corresponds to the most infrared singular part of the energy-energy correlation function. We obtain

$$\Gamma^{(2,0)}(q; t, g, \Lambda = 1) = m^{-\frac{2}{2} + d} \Gamma^{(2,0)}(\frac{q}{m}; 1, g, 1), \ t << 1, \ q << 1.$$

(4.391)

By substituting $\tilde{K}(q) = t (2\pi)^d \delta^d(q)$ in (4.367) we obtain

$$\tilde{\Gamma}^{(n)}(p; K, g, \Lambda) = \sum_{l=0}^{\infty} \frac{t^l}{l!} \tilde{\Gamma}^{(l,n)}(0; p; g, \Lambda).$$

(4.392)

Hence

$$\frac{\partial^2 \Gamma(K, g, \Lambda)}{\partial t^2} |_{t=0} = \Gamma^{(2,0)}(0; g, \Lambda).$$

(4.393)

In other words $\tilde{\Gamma}^{(2,0)}$ at zero momentum is the specific heat since $t$ is the temperature and $\Gamma$ is the thermodynamic energy (effective action). The infrared behavior of the specific heat is therefore given by

$$C_v = \tilde{\Gamma}^{(2,0)}(0; t, g, \Lambda = 1) = t^{-\alpha} \tilde{\Gamma}^{(2,0)}(0; 1, g, 1), \ t << 1, \ q << 1.$$

(4.394)

The new critical exponent $\alpha$ is defined by

$$\alpha = 2 - \nu d.$$  

(4.395)
4.4.3 Scaling Below $T_c$

In order to describe in a continuous way the ordered phase corresponding to $T < T_c$ starting from the disordered phase ($T > T_c$) we introduce a magnetic field $B$, i.e. a source $J = B$. The corresponding magnetization $M$ is precisely the classical field $\phi_c = \langle \phi(x) \rangle$, viz

$$M(x) = \langle \phi(x) \rangle. \quad (4.396)$$

The Helmholtz free energy (vacuum energy) will depend on the magnetic field $B$, viz $W = W(B) = -\ln Z(B)$. We know that the magnetization and the magnetic field are conjugate variables, i.e. $M(x) = \partial W(B)/\partial B(x)$. The Gibbs free energy or thermodynamic energy (effective action) is the Legendre transform of $W(B)$, viz $\Gamma(M) = \int d^d x M(x) B(x) - W(B)$. We compute then

$$B(x) = \frac{\partial \Gamma(M)}{\partial M(x)}. \quad (4.397)$$

The effective action can be expanded as

$$\Gamma[M, t, g, \Lambda] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{d^d p_1}{(2\pi)^d} \cdots \int \frac{d^d p_n}{(2\pi)^d} \tilde{\Gamma}^{(n)}(p_i; t, g, \Lambda) M(p_1) \cdots M(p_n). \quad (4.398)$$

Thus

$$B(p) = \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{d^d p_1}{(2\pi)^d} \cdots \int \frac{d^d p_n}{(2\pi)^d} \tilde{\Gamma}^{(n+1)}(p_i, p; t, g, \Lambda) M(p_1) \cdots M(p_n). \quad (4.399)$$

By assuming that the magnetization is uniform we obtain

$$\Gamma[M, t, g, \Lambda] = \sum_{n=0}^{\infty} \frac{M^n}{n!} \tilde{\Gamma}^{(n)}(p_i = 0; t, g, \Lambda). \quad (4.400)$$

$$B[M, t, g, \Lambda] = \sum_{n=0}^{\infty} \frac{M^n}{n!} \tilde{\Gamma}^{(n+1)}(p_i = 0; t, g, \Lambda). \quad (4.401)$$

By employing the renormalization group equation (4.371) we get

$$\left( \Lambda \frac{\partial}{\partial \Lambda} + \beta \frac{\partial}{\partial g} - \frac{n+1}{2} \eta - \eta_2 t \frac{\partial}{\partial t} \right) B = \sum_{n=0}^{\infty} \frac{M^n}{n!} \left( \Lambda \frac{\partial}{\partial \Lambda} + \beta \frac{\partial}{\partial g} - \frac{n+1}{2} \eta - \eta_2 t \frac{\partial}{\partial t} \right) \tilde{\Gamma}^{(n+1)} = 0. \quad (4.402)$$

Clearly

$$M \frac{\partial}{\partial M} B = \sum_{n=0}^{\infty} \frac{M^n}{n!} \tilde{\Gamma}^{(n+1)}(p_i = 0; t, g, \Lambda). \quad (4.403)$$
Hence the magnetic field obeys the renormalization group equation
\[
\left( \Lambda \frac{\partial}{\partial \Lambda} + \beta \frac{\partial}{\partial g} - \frac{1}{2} (1 + M \frac{\partial}{\partial M}) \eta - \eta_2 t \frac{\partial}{\partial t} \right) B = 0. \tag{4.404}
\]
By using the method of characteristics we introduce as before a running coupling constant \( g(\lambda) \), a running mass \( t(\lambda) \) and an auxiliary renormalization functions \( Z(\lambda) \) such as equations (4.373), (4.374) and (4.375) are satisfied. However in this case we need also to introduce a running magnetization \( M(\lambda) \) such that
\[
\lambda \frac{d}{d\lambda} \ln M(\lambda) = -\frac{1}{2} \eta[g(\lambda)]. \tag{4.405}
\]
By comparing (4.374) and (4.405) we obtain
\[
M(\lambda) = M Z^{-\frac{1}{2}}(\lambda) \tag{4.406}
\]
We must impose
\[
\lambda \frac{d}{d\lambda} \left[ Z^{-1/2}(\lambda) B(M(\lambda), t(\lambda), g(\lambda), \Lambda) \right] = 0. \tag{4.407}
\]
In other words we make the identification
\[
B(M, t, g, \Lambda) = Z^{-1/2}(\lambda) B(M(\lambda), t(\lambda), g(\lambda), \Lambda). \tag{4.408}
\]
From dimensional analysis we know that \([\Gamma^{(n)}] = M^{d-n(d-2)/2}\) and \([M] = M^{(d-2)/2} = M^{1-\epsilon/2}\) and hence \([B] = M^{(d+2)/2} = M^{3-\epsilon/2}\). Hence
\[
B(M, t, g, \Lambda) = \Lambda^{3-\epsilon/2} B\left( \frac{M}{\Lambda^{1-\epsilon/2}}, \frac{t}{\Lambda^2}, g, 1 \right). \tag{4.409}
\]
By combining the above two equations we get
\[
B(M, t, g, \Lambda) = Z^{-1/2}(\lambda) (\Lambda \lambda)^{3-\epsilon/2} B\left( \frac{M(\lambda)}{(\lambda \Lambda)^{1-\epsilon/2}}, \frac{t(\lambda)}{\lambda^2 \Lambda^2}, g(\lambda), 1 \right). \tag{4.410}
\]
Again we use the arbitrariness of \( \lambda \) to make the theory non critical and as a consequence avoid infrared divergences. We choose \( \lambda \) such that
\[
\frac{M(\lambda)}{(\lambda \Lambda)^{1-\epsilon/2}} = 1. \tag{4.411}
\]
The solution of equation (4.405) then reads
\[
\ln \frac{M(\lambda)}{M} = -\frac{1}{2} \int_1^\lambda \frac{dx}{x} \eta(g(x)) \Rightarrow \ln \frac{M}{\Lambda^{1-\epsilon/2}} = \frac{1}{2} \int_1^\lambda \frac{dx}{x} [d - 2 + \eta(g(x))]. \tag{4.412}
\]
The critical domain is defined obviously by $M \ll \Lambda^{1-\epsilon/2}$. For $d-2+\eta$ positive we conclude that $\lambda$ must be small and thus $g(\lambda)$ is close to the fixed point $g_\ast$. This equation then leads to the infrared behavior

$$
\frac{M}{\Lambda^{1-\epsilon/2}} = \lambda^{d-2+\eta/2}.
$$

From equation (4.381) we get the infrared behavior

$$
\frac{t(\lambda)}{\lambda^2} = \lambda^{-\frac{1}{2}}.
$$

We know also the infrared behavior

$$
Z(\lambda) = \lambda^\eta.
$$

The infrared behavior of equation (4.410) is therefore given by

$$
B(M, t, g, \Lambda) = \lambda^{2d-6} \Lambda^{3-\epsilon/2} B(1, \frac{t}{\Lambda^2} \lambda^{-1/\nu}, g_\ast, 1).
$$

This can also be rewritten as

$$
B(M, t, g, 1) = M^\delta f(tM^{1/\beta}).
$$

This is the equation of state. The two new critical exponents $\beta$ and $\delta$ are defined by

$$
\beta = \frac{\nu}{2}(d-2+\eta),
$$

$$
\delta = \frac{d+2-\eta}{d-2+\eta}.
$$

From equations (4.413) and (4.414) we observe that

$$
M = t^\beta \left( \frac{\lambda^2}{t(\lambda)} \right)^\beta.
$$

For negative $t$ ($T < T_c$) the appearance of a spontaneous magnetization $M \neq 0$ at $B = 0$ means that the function $f(x)$, where $x = tM^{-1/\beta}$, admits a negative zero $x_0$. Indeed the condition $B = 0$, $M \neq 0$ around $x = x_0$ reads explicitly

$$
0 = f(x_0) + (x - x_0)f'(x_0) + ..
$$

This is equivalent to

$$
M = |x_0|^{-\beta} (-t)^\beta.
$$

We state without proof that correlation functions below $T_c$ have the same scaling behavior as above $T_c$. In particular the critical exponents $\nu, \gamma$ and $\alpha$ below $T_c$ are the same as those defined earlier above $T_c$. We only remark that in the presence of a magnetic field $B$ we have two mass scales $t^\nu$ (as before) and $m = M^{\nu/\beta}$ where $M$ is the magnetization which is the correct choice in this phase. In the limit $B \rightarrow 0$ (with $T < T_c$) the magnetization becomes spontaneous and $m$ becomes the physical mass.
4.4.4 Critical Exponents from 2–Loop and Comparison with Experiment

The most important critical exponents are the mass critical exponent $\nu$ and the anomalous dimension $\eta$. As we have shown these two critical exponents define the infrared behavior of proper vertices. At $T = T_c$ we find the scaling

$$\tilde{\Gamma}^{(l,n)}(\lambda q_i; \lambda p_i; g, \Lambda) = \lambda^{d-\frac{2(d-2+\eta)}{2}} \tilde{\Gamma}^{(l,n)}(q_i; p_i; g, \Lambda), \quad \lambda \to 0. \quad (4.423)$$

The critical exponent $\eta$ provides the quantum mass dimension of the field operator, viz

$$[\phi] = M^{d\phi}, \quad d\phi = \frac{1}{2}(d - 2 + \eta). \quad (4.424)$$

The scaling of the wave function renormalization is also determined by the anomalous dimension, viz

$$Z(\lambda) \simeq \lambda^\eta. \quad (4.425)$$

The 2–point function at $T = T_c$ behaves therefore as

$$G^{(2)}(p) = \frac{1}{p^{d-\eta}} \Leftrightarrow G^{(2)}(r) = \frac{1}{r^{d-2+\eta}}. \quad (4.426)$$

The critical exponent $\nu$ determines the scaling behavior of the correlation length. For $T > T_c$ we find the scaling

$$\tilde{\Gamma}^{(n)}(p_i; t, g, \Lambda = 1) = m^{d-\frac{2(d-2+\eta)}{2}} F^{(n)}(\frac{p_i}{m}), \quad t = \frac{T - T_c}{T_c} << 1, \quad p_i << 1. \quad (4.427)$$

The mass $m$ is proportional to the mass scale $t^\nu$. From this equation we see that $m$ is the physical mass $\xi^{-1}$ where $\xi$ is the correlation length $\xi$. We have then

$$m = \xi^{-1} \sim t^\nu. \quad (4.428)$$

The 2–point function for $T > T_c$ behaves therefore as $^{12}$

$$G^{(2)}(r) = \frac{1}{r^{d-2+\eta}} \exp(-r/\xi). \quad (4.429)$$

The scaling behavior of correlation functions for $T < T_c$ is the same as for $T > T_c$ except that there exists a non zero spontaneous magnetization $M$ in this regime which sets an extra mass scale given by $M^{1/\beta}$ besides $t^\nu$. The exponent $\beta$ is another critical exponent associated with the magnetization $M$ given by the scaling law

$$\beta = \frac{\nu}{2}(d - 2 + \eta). \quad (4.430)$$

---

$^{12}$Exercise: Give an explicit proof.
In other words for $T$ close to $T_c$ from below we must have

$$M \sim (-t)\beta$$  \hspace{1cm} (4.431)

For $T < T_c$ the physical mass $m$ is given by

$$m = \xi^{-1} \sim M^{\nu/\beta} \sim (-t)^\nu.$$  \hspace{1cm} (4.432)

There are three more critical exponents $\alpha$ (associated with the specific heat), $\gamma$ (associated with the susceptibility) and $\delta$ (associated with the equation of state) which are not independent but given by the scaling laws

$$\alpha = 2 - \nu d.$$  \hspace{1cm} (4.433)

$$\gamma = \nu(2 - \eta).$$  \hspace{1cm} (4.434)

$$\delta = \frac{d + 2 - \eta}{d - 2 + \eta}.$$  \hspace{1cm} (4.435)

The last critical exponent of interest is $\omega$ which is given by the slope of the beta function at the fixed point and measures the approach to scaling.

The beta function at two-loop order of the $O(N)$ sigma model is given by

$$\beta(g_R) = -\epsilon g_R + \frac{N + 8}{6} g_R^2 N_d - \frac{3N + 14}{12} g_R^3 N_d^2.$$  \hspace{1cm} (4.436)

The fixed point $g_*$ is defined by

$$\beta(g_R) = 0 \Rightarrow \frac{3N + 14}{12} g_R^2 N_d^2 - \frac{N + 8}{6} g_R N_d + \epsilon = 0.$$ \hspace{1cm} (4.437)

The solution must be of the form

$$g_R N_d = a \epsilon + b \epsilon^2 + ...$$  \hspace{1cm} (4.438)

We find the solution

$$a = \frac{6}{N + 8}, \quad b = \frac{18(3N + 14)}{(N + 8)^3}.$$  \hspace{1cm} (4.439)

Thus

$$g_R N_d = \frac{6}{N + 8} \epsilon + \frac{18(3N + 14)}{(N + 8)^3} \epsilon^2 + ...$$  \hspace{1cm} (4.440)

The critical exponent $\omega$ is given by

$$\omega = \beta'(g_R^*) = \epsilon - \frac{3(3N + 14)}{(N + 8)^2} \epsilon^2 + ...$$  \hspace{1cm} (4.441)
The critical exponent $\eta$ is given by

$$\eta = \eta(g_R).$$  \hfill (4.442)

The renormalization function $\eta(g)$ is given by

$$\eta(g_R) = \frac{N + 2}{72} g_R^2 N_d^2.$$  \hfill (4.443)

We substitute now the value of the fixed point. We obtain immediately

$$\eta = \frac{N + 2}{2(N + 8)^2} \epsilon^2.$$  \hfill (4.444)

The critical exponent $\nu$ is given by

$$\nu = \frac{1}{2 + \eta_2},$$  \hfill (4.445)

$$\nu = \nu(g_R), \; \eta_2 = \eta_2(g_R).$$  \hfill (4.446)

The renormalization function $\eta_2(g)$ is given by

$$\eta_2 = - \frac{N + 2}{6} N_d g_R + \frac{5(N + 2)^2}{72} g_R^2 N_d^2.$$  \hfill (4.447)

By substituting the value of the fixed point we compute immediately

$$\eta_2 = - \frac{N + 2}{N + 8} \epsilon - \frac{(N + 2)(13N + 44)}{2(N + 8)^3} \epsilon^2 + \ldots.$$  \hfill (4.448)

$$\nu = \frac{1}{2} + \frac{N + 2}{4(N + 8)} \epsilon + \frac{(N + 2)(N^2 + 23N + 60)}{8(N + 8)^3} \epsilon^2 + \ldots.$$  \hfill (4.449)

All critical exponents can be determined in terms of $\nu$ and $\eta$. They only depend on the dimension of space $d$ and on the dimension of the symmetry space $N$ which is precisely the statement of universality. The epsilon expansion is divergent for all $\epsilon$ and as a consequence a resummation is required before we can coherently compare with experiments. This is a technical exercise which we will not delve into here and content ourselves by using what we have already established and also by quoting some results.

The most important predictions (in our view) correspond to $d = 3$ ($\epsilon = 1$) and $N = 1, 2, 3$.

- The case $N = 1$ describes Ising-like systems such as the liquid-vapor transitions in classical fluids. Experimentally we observe

$$\nu = 0.625 \pm 0.006$$

$$\gamma = 1.23 - 1.25.$$  \hfill (4.450)
The theoretical calculation gives

\[ \nu = \frac{1}{2} + \frac{1}{12} + \frac{7}{162} + ... = \frac{203}{324} + ... = 0.6265 \pm 0.0432 \]

\[ \eta = \frac{1}{54} + .... = 0.019 \Leftrightarrow \gamma = \nu(2 - \eta) = 1.241. \tag{4.451} \]

The agreement for \( \nu \) and \( \eta \) up to order \( \epsilon^2 \) is very reasonable and is a consequence of the asymptotic convergence of the \( \epsilon \) series. The error is estimated by the last term available in the epsilon expansion.

• The case \( N = 2 \) corresponds to the Helium superfluid transition. This system allows precise measurement near \( T_c \) of \( \nu \) and \( \alpha \) given by

\[ \nu = 0.672 \pm 0.001 \]
\[ \alpha = -0.013 \pm 0.003. \tag{4.452} \]

The theoretical calculation gives

\[ \nu = \frac{1}{2} + \frac{1}{10} + \frac{11}{200} + ... = \frac{131}{200} + ... = 0.6550 \pm 0.0550 \]
\[ \alpha = 2 - \nu \eta = -0.035. \tag{4.453} \]

Here the agreement up to order \( \epsilon^2 \) is not very good. After proper resummation of the \( \epsilon \) expansion we find excellent agreement with the experimental values. We quote the improved theoretical predictions

\[ \nu = 0.664 - 0.671 \]
\[ \alpha = -(0.008 - 0.013). \tag{4.454} \]

• The case \( N = 3 \) corresponds to magnetic systems. The experimental values are

\[ \nu = 0.7 - 0.725 \]
\[ \gamma = 1.36 - 1.42. \tag{4.455} \]

The theoretical calculation gives

\[ \nu = \frac{1}{2} + \frac{5}{44} + \frac{345}{5324} + ... = \frac{903}{1331} + ... = 0.6874 \pm 0.0648 \]
\[ \eta = \frac{5}{242} = 0.021 \Leftrightarrow \gamma = 1.36. \tag{4.456} \]

There is a very good agreement.
4.5 The Wilson Approximate Recursion Formulas

4.5.1 Kadanoff-Wilson Phase Space Analysis

We start by describing a particular phase space cell decomposition due to Wilson which is largely motivated by Kadanoff block spins.

We assume a hard cutoff $2\Lambda$. Thus if $\phi(x)$ is the field (spin) variable and $\tilde{\phi}(k)$ is its Fourier transform we will assume that $\tilde{\phi}(k)$ is zero for $k > 2\Lambda$.

We expand the field as

$$\phi(x) = \sum_{\vec{m}} \sum_{l=0}^{\infty} \tilde{\psi}_{\vec{m}l}(x) \phi_{\vec{m}l}. \quad (4.457)$$

The wave functions $\psi_{\vec{m}l}(x)$ satisfy the orthonormality condition

$$\int d^d x \tilde{\psi}_{\vec{m}_1 l_1}^* (x) \tilde{\psi}_{\vec{m}_2 l_2} (x) = \delta_{\vec{m}_1 \vec{m}_2} \delta_{l_1 l_2}. \quad (4.458)$$

The Fourier transform $\tilde{\psi}_{\vec{m}l}(k)$ is defined by

$$\tilde{\psi}_{\vec{m}l}(k) = \int d^d x \tilde{\psi}_{\vec{m}l}(x) e^{ikx}. \quad (4.459)$$

The interpretation of $l$ and $\vec{m}$ is as follows. We decompose momentum space into thin spherical shells, i.e. logarithmically as

$$\frac{1}{2^l} \leq \frac{|k|}{\Lambda} \leq \frac{1}{2^{l-1}}. \quad (4.460)$$

The functions $\tilde{\psi}_{\vec{m}l}(k)$ for a fixed $l$ are non zero only inside the shell $l$, i.e. for $1/2^l \leq |k|/\Lambda \leq 1/2^{l-1}$. We will assume furthermore that the functions $\tilde{\psi}_{\vec{m}l}(k)$ are constant within its shell and satisfy the normalization condition

$$\int \frac{d^d k}{(2\pi)^d} |\tilde{\psi}_{\vec{m}l}(k)|^2 = 1. \quad (4.461)$$

The functions $\tilde{\psi}_{\vec{m}l}(k)$ and $\psi_{\vec{m}l}(x)$ for a fixed $l$ and a fixed $\vec{m}$ should be thought of as minimal wave packets, i.e. if $\Delta k$ is the width of $\tilde{\psi}_{\vec{m}l}(k)$ and $\Delta x$ is the width of $\psi_{\vec{m}l}(x)$ then one must have by the uncertainty principle the requirement $\Delta x \Delta k = (2\pi)^d$. Thus for each shell we divide position space into blocks of equal size each with volume inversely proportional to the volume of the corresponding shell. The volume of the $l$th momentum shell is proportional to $R^d$ where $R = 1/2^l$, viz $\Delta k = (2\pi)^d 2^{-ld} w$ where $w$ is a constant. Hence the volume of the corresponding position space box is $\Delta x = 2^{ld} w^{-1}$. The functions $\psi_{\vec{m}l}(x)$ are non zero (constant) only inside this box by construction. This position space box is characterized by the index $\vec{m}$ as is obvious from the normalization condition

$$\int d^d x |\psi_{\vec{m}l}(x)|^2 = \int_{\vec{x} \in \text{box} \ \vec{m}} d^d x |\psi_{\vec{m}l}(x)|^2 = 1. \quad (4.462)$$
In other words

\[
\int d^d x = \sum_{\vec{m}} \int_{\vec{x} \in \text{box } \vec{m}} d^d x. \tag{4.463}
\]

The normalization conditions in momentum and position spaces lead to the relations

\[
|\tilde{\psi}_{\vec{m}l}(k)| = 2^{ld/2} w^{-1/2}. \tag{4.464}
\]

\[
|\psi_{\vec{m}l}(x)| = 2^{-ld/2} w^{1/2}. \tag{4.465}
\]

Obviously \(|\tilde{\psi}_{\vec{m}l}(0)| = 0\). Thus \(\int d^d x \psi_{\vec{m}l} = \tilde{\psi}_{\vec{m}l}(0) = 0\) and as a consequence we will assume that \(\psi_{\vec{m}l}(x)\) is equal to \(+2^{-ld/2} w^{1/2}\) in one half of the box and \(-2^{-ld/2} w^{1/2}\) in the other half.

The meaning of the index \(\vec{m}\) which labels the position space boxes can be clarified further by the following argument. By an appropriate scale transformation in momentum space we can scale the momenta such that the \(l\)th shell becomes the largest shell \(l = 0\). Clearly the correct scale transformation is \(k \rightarrow 2^l k\) since \(1 \leq |2^l k|/\Lambda \leq 2\). This corresponds to a scale transformation in position space of the form \(x \rightarrow x/2^l\). We obtain therefore the relation \(\psi_{\vec{m}l}(x) = \psi_{\vec{0}0}(x/2^l)\). Next we perform an appropriate translation in position space to bring the box \(\vec{m}\) to the box \(\vec{0}\). This is clearly given by the translation \(\vec{x} \rightarrow \vec{x} - a_0 \vec{m}\). We obtain therefore the relation

\[
\psi_{\vec{m}l}(x) = \psi_{\vec{0}0}(x/2^l - a_0 \vec{m}). \tag{4.466}
\]

The functions \(\tilde{\psi}_{\vec{m}l}(k)\) and \(\psi_{\vec{m}l}(x)\) correspond to a single degree of freedom in phase space occupying a volume \((2\pi)^d\), i.e. a single cell in phase space is characterized by \(l\) and \(\vec{m}\). Each momentum shell \(l\) corresponds to a lattice in the position space with a lattice spacing given by

\[
a_l = (\Delta x)^{1/d} = 2^l a_0, \quad a_0 = w^{-1/d}. \tag{4.467}
\]

The largest shell \(l = 0\) correspond to a lattice spacing \(a_0\) and each time \(l\) is increased by 1 the lattice spacing gets doubled which is the original spin blocking idea of Kadanoff.

We are interested in integrating out only the \(l = 0\) modes. We write then

\[
\phi(x) = \sum_{\vec{m}} \psi_{\vec{0}0}(x) \phi_{\vec{0}0} + \phi_1(x). \tag{4.468}
\]

\[
\phi_1(x) = \sum_{\vec{m}} \sum_{l=1}^{\infty} \psi_{l\vec{m}}(x) \phi_{l\vec{m}}. \tag{4.469}
\]

From the normalization (4.465) and the scaling law (4.466) we have

\[
\psi_{l\vec{m}}(x) = 2^{-d/2} \psi_{l-1\vec{m}}(x/2). \tag{4.470}
\]
We define $\phi^\prime (x/2)$ by

$$
\phi_1(x) = 2^{-d/2}\alpha_0\phi^\prime (x/2).
$$

(4.471)

In other words

$$
\phi^\prime (x/2) = \sum_{\vec{m}}\sum_{l=1}^{\infty} \psi_{\vec{m}l-1}(x/2)\phi_{\vec{m}l-1}^\prime, \quad \phi_{\vec{m}l-1}^\prime = \alpha_0^{-1}\phi_{\vec{m}l}.
$$

(4.472)

We have then

$$
\phi(x) = \sum_{\vec{m}} \psi_{\vec{m}0}(x)\phi_{\vec{m}0} + 2^{-d/2}\alpha_0\phi^\prime (x/2).
$$

(4.473)

### 4.5.2 Recursion Formulas

We will be interested in actions of the form

$$
S_0(\phi(x)) = \frac{K}{2}\int d^d x R_0(\phi(x))\partial_\mu \phi(\vec{x})\partial^\mu \phi(\vec{x}) + \int d^d x P_0(\phi(x)).
$$

(4.474)

We will assume that $P_0$ and $R_0$ are even polynomials of the field and that $dR_0/d\phi$ is much smaller than $P_0$ for all relevant configurations. The partition function is

$$
Z_0 = \int \mathcal{D}\phi(x) e^{-S_0(\phi(x))}
= \int \mathcal{D}\phi^\prime (x/2) \prod_{\vec{m}} d\phi_{\vec{m}0} e^{-S_0(\phi(x))}.
$$

(4.475)

The degrees of freedom contained in the fluctuation $\phi^\prime (x/2)$ correspond to momentum shells $l \geq 1$ and thus correspond to position space wave packets larger than the box $\vec{m}$ by at least a factor of 2. We can thus assume that $\phi^\prime (x/2)$ is almost constant over the box $\vec{m}$. If $x_0$ is the center of the box $\vec{m}$ we can expand $\phi^\prime (x/2)$ as

$$
\phi^\prime (x/2) = \phi^\prime (x_0/2) + (x-x_0)^\mu \partial_\mu \phi^\prime (x_0/2) + \frac{1}{2}(x-x_0)^\mu (x-x_0)^\nu \partial_\mu \partial_\nu \phi^\prime (x_0/2) + ...
$$

(4.476)

The partition function becomes

$$
Z_0 = \int \mathcal{D}\phi^\prime (x_0/2) \prod_{\vec{m}} d\phi_{\vec{m}0} e^{-S_0(\phi(x))}.
$$

(4.477)
We note again that within the box \( \vec{m} \int \vec{m} \). Next we want to compute the Interaction Term: 

Thus the third term in the above equation can be approximated by (dropping also higher derivative corrections and defining \( u_\vec{m} = 2^{-d/2} \alpha_0 \phi'(x_0/2) \))

The kinetic term:

The contribution of \( \partial_\mu \psi_{\vec{n}_0}(x) \) is however only appreciable when \( \psi_{\vec{n}_0}(x) = 0 \), i.e., at the center of the box. In the first and second terms of the above equation we can then replace \( R_0(\phi(x)) \) by \( R_0(u_\vec{m}) \). The second term in the above equation (4.478) vanishes by conservation of momentum. In the first term we can neglect all the coupling terms \( \phi_{\vec{n}_0}\phi'_{\vec{n}_0'} \) with \( \vec{m} \neq \vec{m}' \) since \( \psi_{\vec{m}_0'}(x) \) is zero inside the box \( \vec{m} \). We define the integral

This is independent of \( \vec{m} \) because of the relation (4.466). The first term becomes therefore

The interaction term: Next we want to compute

We note again that within the box \( \vec{m} \) the field is given by

\[
\phi^{(m)}(x) = \psi_{\vec{n}_0}(x)\phi_{\vec{n}_0} + 2^{-d/2} \alpha_0 \phi'(x/2).
\]
Introduce $z_0 = \psi_{\bar{m}0}(x)\phi_{\bar{m}0} + u_{\bar{m}}$. We have then

$$
\int_{\vec{x} \in \text{box} \bar{m}} d^d x P_0(\phi^{(m)}(x)) = \int_{\vec{x} \in \text{box} \bar{m}} d^d x \left[ P_0(z_0) + \frac{1}{2} (x - x_0)^\mu \partial_\mu u_{\bar{m}} \right] \times \partial_\nu u_{\bar{m}} \frac{dP_0}{dz_0} |_{z_0} + \ldots \right].
$$

(4.484)

As stated earlier $\psi_{\bar{m}0}(x)$ is approximated by $+\frac{1}{2}w$ in one half of the box and by $-\frac{1}{2}w$ in the other half. Also recall that the volume of the box is $w^{-1}$. The first term can be approximated by

$$
\int_{\vec{x} \in \text{box} \bar{m}} d^d x P_0(z_0) = \frac{w^{-1}}{2} \left[ P_0(w^{1/2}\phi_{\bar{m}0} + u_{\bar{m}}) + P_0(-w^{1/2}\phi_{\bar{m}0} + u_{\bar{m}}) \right].
$$

(4.485)

We compute now the third term in (4.484). We start from the obvious identity

$$
\int_{\text{box}} d^d x \frac{1}{2} (x - x_0)^\mu (x - x_0)^\nu = \int_{\text{box}+} d^d x \frac{1}{2} (x - x_0)^\mu (x - x_0)^\nu + \int_{\text{box}-} d^d x \frac{1}{2} (x - x_0)^\mu (x - x_0)^\nu.
$$

(4.486)

We will think of the box $\bar{m}$ as a sphere of volume $w^{-1}$. Thus

$$
\int_{\text{box}} d^d x \frac{1}{2} (x - x_0)^\mu (x - x_0)^\nu = \frac{1}{2} V_\eta^{\mu\nu}.
$$

(4.487)

We have the definitions

$$
V = \frac{1}{d} \int_{\text{box}} r^2 d^d x, \quad w^{-1} = \int_{\text{box}} d^d x.
$$

(4.488)

Explicitly we have

$$
V = \frac{1}{d + 2} \left( \frac{dw^{-1}}{\Omega_{d-1}} \right)^{(d+2)/d} \frac{\Omega_{d-1}}{d}.
$$

(4.489)

The above identity becomes then

$$
\frac{1}{2} V_\eta^{\mu\nu} = \int_{\text{box}+} d^d x \frac{1}{2} (x - x_0)^\mu (x - x_0)^\nu + \int_{\text{box}-} d^d x \frac{1}{2} (x - x_0)^\mu (x - x_0)^\nu.
$$

(4.490)

The sum of these two integrals is rotational invariant. As stated before we think of the box as a sphere divided into two regions of equal volume. The first region (the first half of the box) is a concentric smaller sphere whereas the second region (the second half of the box) is a thin spherical shell. Both regions are spherically symmetric and thus we can assume that.
\[
\int_{\text{box}_{+}} d^4x \frac{1}{2} (x - x_0)^\mu (x - x_0)\nu = \frac{1}{2} V_+ \eta^{\mu\nu}.
\] (4.491)

Clearly \( V = V_+ + V_- \). The integral of interest is

\[
\int_{\xi \in \text{box}_{-}} d^4x \frac{1}{2} (x - x_0)^\mu (x - x_0)\nu \partial_\mu \partial_\nu u_m \frac{dP_0}{dz} \bigg|_{z_0} = \frac{dP_0}{dz} \bigg|_{w^{1/2} \phi_{m0} + u_m} \partial_\mu \partial_v u_m \int_{\text{box}_{+}} d^4x \frac{1}{2} (x - x_0)^\mu (x - x_0)\nu \\
+ \frac{dP_0}{dz} \bigg|_{-w^{1/2} \phi_{m0} + u_m} \partial_\mu \partial_v u_m \int_{\text{box}_{-}} d^4x \frac{1}{2} (x - x_0)^\mu (x - x_0)\nu \\
= \frac{V_+}{2} \frac{dP_0}{dz} \bigg|_{w^{1/2} \phi_{m0} + u_m} \partial_\mu \partial^\mu u_m + \frac{V_-}{2} \frac{dP_0}{dz} \bigg|_{-w^{1/2} \phi_{m0} + u_m} \partial_\mu \partial^\mu u_m \\
= 2^{-1-d/2} \alpha_0 V_+ \frac{dP_0}{dz} \bigg|_{w^{1/2} \phi_{m0} + u_m} \partial_\mu \partial^\mu \phi' (x_0/2) \\
+ 2^{-1-d/2} \alpha_0 V_- \frac{dP_0}{dz} \bigg|_{-w^{1/2} \phi_{m0} + u_m} \partial_\mu \partial^\mu \phi' (x_0/2). \tag{4.492}
\]

Similarly the fourth term in (4.484) is computed as follows. We have

\[
\int_{\xi \in \text{box}_{-}} d^4x \frac{1}{2} (x - x_0)^\mu (x - x_0)\nu \partial_\mu u_m \partial_\nu u_m \frac{d^2P_0}{dz^2} \bigg|_{z_0} = \frac{d^2P_0}{dz^2} \bigg|_{w^{1/2} \phi_{m0} + u_m} \partial_\mu u_m \partial_\nu u_m \int_{\text{box}_{+}} d^4x \frac{1}{2} (x - x_0)^\mu \\
\times (x - x_0)\nu \\
+ \frac{d^2P_0}{dz^2} \bigg|_{-w^{1/2} \phi_{m0} + u_m} \partial_\mu u_m \partial_\nu u_m \int_{\text{box}_{-}} d^4x \frac{1}{2} (x - x_0)^\mu \\
\times (x - x_0)\nu \\
= \frac{V_+}{2} \frac{d^2P_0}{dz^2} \bigg|_{w^{1/2} \phi_{m0} + u_m} \partial_\mu u_m \partial^\mu u_m \\
+ \frac{V_-}{2} \frac{d^2P_0}{dz^2} \bigg|_{-w^{1/2} \phi_{m0} + u_m} \partial_\mu u_m \partial^\mu u_m \\
= 2^{-1-d} \alpha_0 V_+ \frac{d^2P_0}{dz^2} \bigg|_{w^{1/2} \phi_{m0} + u_m} \partial_\mu \phi' (x_0/2) \partial^\mu \phi' (x_0/2) \\
+ 2^{-1-d} \alpha_0 V_- \frac{d^2P_0}{dz^2} \bigg|_{-w^{1/2} \phi_{m0} + u_m} \partial_\mu \phi' (x_0/2) \partial^\mu \phi' (x_0/2). \tag{4.493}
\]

Finally we compute the first term in (4.484). We have

\[
\int_{\xi \in \text{box}_{-}} d^4x (x - x_0)^\mu \partial_\nu u_m \frac{dP_0}{dz} \bigg|_{z_0} = \frac{dP_0}{dz} \bigg|_{w^{1/2} \phi_{m0} + u_m} \partial_\mu u_m \int_{\text{box}_{+}} d^4x (x - x_0)^\mu \\
+ \frac{dP_0}{dz} \bigg|_{-w^{1/2} \phi_{m0} + u_m} \partial_\mu u_m \int_{\text{box}_{-}} d^4x (x - x_0)^\mu. \tag{4.494}
\]
Clearly we must have
\[ \int_{\text{box}^+} d^d x (x - x_0)^\mu + \int_{\text{box}^-} d^d x (x - x_0)^\mu = 0. \] (4.495)

We will assume that both integrals vanish again by the same previous argument. The linear term is therefore 0, viz
\[ \int x \in \text{box} \ d^d x (x - x_0)^\mu \partial_\mu u_\vec{m} \frac{dP_0}{dz} = 0. \] (4.496)

The final result is
\[ \int x \in \text{box} \ d^d x P_0(\phi^{(m)}(x)) = \frac{w^{-1}}{2} \left[ P_0(w^{1/2} \phi_\vec{m} + u_\vec{m}) + P_0(-w^{1/2} \phi_\vec{m} + u_\vec{m}) \right] + 2^{-1-d/2} \alpha_0 \left[ V_+ \frac{dP_0}{dz} |_{w^{1/2} \phi_\vec{m} + u_\vec{m}} + V_- \frac{dP_0}{dz} |_{-w^{1/2} \phi_\vec{m} + u_\vec{m}} \right] \partial_\mu \partial^\mu \phi' (x_0/2) + 2^{-1-d/2} \alpha_0 \left[ V_+ \frac{d^2 P_0}{dz^2} |_{w^{1/2} \phi_\vec{m} + u_\vec{m}} + V_- \frac{d^2 P_0}{dz^2} |_{-w^{1/2} \phi_\vec{m} + u_\vec{m}} \right] \partial_\mu \partial^\mu \phi' (x_0/2). \] (4.497)

**The Action:** By putting all the previous results together we obtain the expansion of the action. We get
\[ S_0(\phi(x)) = \frac{K}{2} \int d^d x R_0(\phi(x)) \partial_\mu \phi(x) \partial^\mu \phi(x) + \int d^d x P_0(\phi(x)) \]
\[ = \frac{K \rho}{2} \sum_{\vec{m}} R_0(u_{\vec{m}}) \phi_{\vec{m}0}^2 + 2^{-d-2} \frac{\alpha_0}{w^{-1}} \sum_{\vec{m}} \left[ R_0(w^{1/2} \phi_\vec{m} + u_\vec{m}) + R_0(-w^{1/2} \phi_\vec{m} + u_\vec{m}) \right] \partial_\mu \phi' (x_0/2) \partial^\mu \phi' (x_0/2) + \frac{w^{-1}}{2} \sum_{\vec{m}} \left( P_0(w^{1/2} \phi_\vec{m} + u_\vec{m}) + P_0(-w^{1/2} \phi_\vec{m} + u_\vec{m}) \right) \]
\[ + 2^{-1-d/2} \alpha_0 \sum_{\vec{m}} \left( V_+ \frac{dP_0}{dz} |_{w^{1/2} \phi_\vec{m} + u_\vec{m}} + V_- \frac{dP_0}{dz} |_{-w^{1/2} \phi_\vec{m} + u_\vec{m}} \right) \partial_\mu \partial^\mu \phi' (x_0/2) + 2^{-1-d/2} \alpha_0 \sum_{\vec{m}} \left( V_+ \frac{d^2 P_0}{dz^2} |_{w^{1/2} \phi_\vec{m} + u_\vec{m}} + V_- \frac{d^2 P_0}{dz^2} |_{-w^{1/2} \phi_\vec{m} + u_\vec{m}} \right) \partial_\mu \partial^\mu \phi' (x_0/2). \] (4.498)

**The Path Integral:** We need now to evaluate the path integral
\[ \int \prod_{\vec{m}} d\phi_{\vec{m}0} e^{-S_0(\phi(x))}. \] (4.499)
We introduce the variables
\[ u_{\tilde{m}} \rightarrow z_{\tilde{m}} = \left( \frac{K\rho}{2w} \right)^{1/2} u_{\tilde{m}}. \] (4.500)

\[ \phi_{\tilde{m}0} \rightarrow y_{\tilde{m}} = \left( \frac{K\rho}{2} \right)^{1/2} \phi_{\tilde{m}0}. \] (4.501)

\[ R_0 \rightarrow W_0(x) = R_0 \left( \frac{2w}{K\rho} \right)^{1/2} x). \] (4.502)

\[ P_0 \rightarrow Q_0(x) = w^{-1} P_0 \left( \frac{2w}{K\rho} \right)^{1/2} x). \] (4.503)

We compute then
\[
\int \prod_{\tilde{m}} d\phi_{\tilde{m}0} e^{-S_0(\phi(x))} = \prod_{\tilde{m}} \left( \frac{2}{K\rho} \right)^{1/2} \int dy_{\tilde{m}} \exp \left( -y_{\tilde{m}}^2 W_0(z_{\tilde{m}}) - \frac{1}{2} Q_0(y_{\tilde{m}} + z_{\tilde{m}}) - \frac{1}{2} Q_0(-y_{\tilde{m}} + z_{\tilde{m}}) \right)
\[
- 2^{-3+d/2} \alpha_0 (K\rho w)^{1/2} \left[ V_+ \frac{dQ_0}{dy_{\tilde{m}}}(y_{\tilde{m}} + z_{\tilde{m}}) + V_- \frac{dQ_0}{dy_{\tilde{m}}}(-y_{\tilde{m}} + z_{\tilde{m}}) \right] \partial_\mu \partial^\mu \phi'(x_0/2)
\[
- 2^{-d-2} \alpha_0^2 K\rho \left[ V_+ \frac{d^2Q_0}{dy_{\tilde{m}}^2}(y_{\tilde{m}} + z_{\tilde{m}}) + V_- \frac{d^2Q_0}{dy_{\tilde{m}}^2}(-y_{\tilde{m}} + z_{\tilde{m}}) \right] \partial_\mu \phi'(x_0/2) \partial^\mu \phi'(x_0/2)
\[
- 2^{-d-2} \alpha_0^2 K w^{-1} \left[ W_0(y_{\tilde{m}} + z_{\tilde{m}}) + W_0(-y_{\tilde{m}} + z_{\tilde{m}}) \right] \partial_\mu \phi'(x_0/2) \partial^\mu \phi'(x_0/2). \] (4.504)

Define
\[
M_0(z_{\tilde{m}}) = \int dy_{\tilde{m}} \exp \left( -y_{\tilde{m}}^2 W_0(z_{\tilde{m}}) - \frac{1}{2} Q_0(y_{\tilde{m}} + z_{\tilde{m}}) - \frac{1}{2} Q_0(-y_{\tilde{m}} + z_{\tilde{m}}) \right)
\[
- 2^{-3+d/2} \alpha_0 (K\rho w)^{1/2} \left[ V_+ \frac{dQ_0}{dy_{\tilde{m}}}(y_{\tilde{m}} + z_{\tilde{m}}) + V_- \frac{dQ_0}{dy_{\tilde{m}}}(-y_{\tilde{m}} + z_{\tilde{m}}) \right] \partial_\mu \partial^\mu \phi'(x_0/2)
\[
- 2^{-d-2} \alpha_0^2 K\rho \left[ V_+ \frac{d^2Q_0}{dy_{\tilde{m}}^2}(y_{\tilde{m}} + z_{\tilde{m}}) + V_- \frac{d^2Q_0}{dy_{\tilde{m}}^2}(-y_{\tilde{m}} + z_{\tilde{m}}) \right] \partial_\mu \phi'(x_0/2) \partial^\mu \phi'(x_0/2)
\[
- 2^{-d-2} \alpha_0^2 K w^{-1} \left[ W_0(y_{\tilde{m}} + z_{\tilde{m}}) + W_0(-y_{\tilde{m}} + z_{\tilde{m}}) \right] \partial_\mu \phi'(x_0/2) \partial^\mu \phi'(x_0/2). \] (4.505)

In this equation \( \phi'(x_0/2) \) is given in terms of \( z_{\tilde{m}} \) by
\[ \phi'(x_0/2) = 2^{d/2} \alpha_0 \left( \frac{2w}{K\rho} \right)^{1/2} z_{\tilde{m}}. \] (4.506)
The remaining dependence on the box \( \vec{m} \) is only through the center of the box \( x_0 \). Then

\[
\int \prod_{\vec{m}} d\phi_{\vec{m}0} \ e^{-S_0(\phi(x))} = \prod_{\vec{m}} \left( \frac{2}{K \rho} \right)^{1/2} M_0(z_{\vec{m}})
\]

\[
= \prod_{\vec{m}} \left( \frac{2}{K \rho} \right)^{1/2} \exp(\ln M_0(z_{\vec{m}}))
\]

\[
= \exp \left( \sum_{\vec{m}} \ln \frac{M_0(z_{\vec{m}})}{I_0(0)} \right) \prod_{\vec{m}} \left( \frac{2}{K \rho} \right)^{1/2} I_0(0).
\] (4.507)

The function \( I_0(z) \) is defined by

\[
I_0(z) = \int dy \exp \left( -y^2 W_0(z) - \frac{1}{2} Q_0(y + z) - \frac{1}{2} Q_0(-y + z) \right).
\] (4.508)

In order to compute \( M_0 \) we will assume that the derivative terms are small and expand the exponential around the ultra local approximation. We compute

\[
M_0(z_{\vec{m}}) = I_0(z_{\vec{m}}) \left[ 1 - 2^{-(3+d)/2} \alpha_0 (K \nu w)^{1/2} V - \frac{dQ_0}{dy_{\vec{m}}} (y_{\vec{m}} + z_{\vec{m}}) + \partial_\mu \partial^\mu \phi' (x_0/2)
\]

\[
- 2^{-d-1} \alpha_0^2 K \left[ \frac{\rho V}{2} < \frac{d^2 Q_0}{dy_{\vec{m}}^2} (y_{\vec{m}} + z_{\vec{m}}) > + w^{-1} < W_0(y_{\vec{m}} + z_{\vec{m}}) > \right] \partial_\mu \phi' (x_0/2) \partial^\mu \phi' (x_0/2) + ... \right].
\] (4.509)

The path integral (4.499) becomes

\[
\int \prod_{\vec{m}} d\phi_{\vec{m}0} \ e^{-S_0(\phi(x))} = \prod_{\vec{m}} \left( \frac{2}{K \rho} \right)^{1/2} I_0(0) \times \exp \left( \sum_{\vec{m}} \ln \frac{I_0(z_{\vec{m}})}{I_0(0)} - 2^{-(3+d)/2} \alpha_0 (K \nu w)^{1/2} V \sum_{\vec{m}} \left[ \frac{\rho V}{2} < \frac{d^2 Q_0}{dy_{\vec{m}}^2} (y_{\vec{m}} + z_{\vec{m}}) > + w^{-1} < W_0(y_{\vec{m}} + z_{\vec{m}}) > \right] \partial_\mu \phi' (x_0/2) \partial^\mu \phi' (x_0/2) \right).
\] (4.510)

We make the change of variable \( x_0/2 \rightarrow x \) (this means that the position space wave packet with \( l = 1 \) corresponding to the highest not integrated momentum will now fit into the box) to obtain

\[
\int \prod_{\vec{m}} d\phi_{\vec{m}0} \ e^{-S_0(\phi(x))} = \prod_{\vec{m}} \left( \frac{2}{K \rho} \right)^{1/2} I_0(0) \times \exp \left( \sum_{\vec{m}} \ln \frac{I_0(z_{\vec{m}})}{I_0(0)} - 2^{-(3+d)/2} \alpha_0 (K \nu w)^{1/2} V \frac{4}{\alpha_0^2 K} \sum_{\vec{m}} \left[ \frac{\rho V}{2} < \frac{d^2 Q_0}{dy_{\vec{m}}^2} (y_{\vec{m}} + z_{\vec{m}}) > + w^{-1} < W_0(y_{\vec{m}} + z_{\vec{m}}) > \right] \partial_\mu \phi' (x_0/2) \partial^\mu \phi' (x_0/2) \right).
\] (4.511)
Now \( z_\tilde{m} \) is given by \( z_\tilde{m} = (K \rho / 2w)^{1/2} 2^{-d/2} \alpha_0 \phi'(x) \). It is clear that \( dQ_0(y_\tilde{m} + z_\tilde{m})/dy_\tilde{m} = dQ_0(y_\tilde{m} + z_\tilde{m})/dz_\tilde{m} \), etc. Recall that the volume of the box \( \tilde{m} \) is \( w^{-1} \) and thus we can make the identification \( w \int d^dx = \sum \tilde{m} \). However we have also made the rescaling \( x_0 \to 2x \) and hence we must make instead the identification \( 2^d w \int d^dx = \sum \tilde{m} \). We obtain

\[
\int \prod_{\tilde{m}} d\phi_\tilde{m} e^{-S_0(\phi(x))} = \prod_{\tilde{m}} \left( \frac{2}{K \rho} \right)^{1/2} I_0(0) \times \exp \left( 2^d w \int d^dx \ln \frac{I_0(z)}{I_0(0)} - 2^{-3(3-d)/2} \frac{\alpha_0 (K \rho w)^{1/2} V}{4} 2^d w \right)
\]

\[
\times \int d^dx < \frac{dQ_0}{dz}(z) > \partial_\mu \phi'(x) - 2^{-d-1} \frac{\alpha_0^2 K w}{4} 2^d w \int d^dx \left[ \frac{\rho V}{2} < \frac{d^2 Q_0}{dz^2}(z) > \right]
\]

\[
+ w^{-1} < W_0(z) > \left[ \partial_\mu \phi'(x) \partial^\mu \phi'(x) \right].
\]  

(4.512)

Now \( z \) is given by \( z = (K \rho / 2w)^{1/2} 2^{-d/2} \alpha_0 \phi'(x) \). The expectation values \( < O^n(z) > \) are defined by

\[
< O^n(z) > = \frac{1}{I(z)} \int dy \left( \frac{O(y + z) + O(-y + z)}{2} \right)^n \exp \left(-y^2 W_0(z) - \frac{1}{2} Q_0(y + z) - \frac{1}{2} Q_0(-y + z) \right).
\]  

(4.513)

Next we derive in a straightforward way the formula

\[
\frac{d}{dz} < \frac{dQ_0}{dz}(z) > = < \frac{d^2 Q_0}{dz^2}(z) > - < \left( \frac{dQ_0}{dz}(z) \right)^2 > + < \frac{dQ_0}{dz}(z) >^2
\]

\[
+ \frac{dW_0}{dz} \left( < \frac{dQ_0}{dz}(z) > - \frac{1}{I_0(z)} \int dy y^2 e^{-} - \frac{1}{I_0(z)} \int dy \frac{dQ_0}{dz}(y + z) y^2 e^{-} \right).
\]  

(4.514)

By integrating by part the second term in (4.512) we can see that the first term in (4.514) cancels the 3rd term in (4.512). The last term can be neglected if we assume that \( dW_0/dz \) is much smaller than \( Q_0 \). We obtain then

\[
\int \prod_{\tilde{m}} d\phi_\tilde{m} e^{-S_0(\phi(x))} = \prod_{\tilde{m}} \left( \frac{2}{K \rho} \right)^{1/2} I_0(0) \times \exp \left( 2^d w \int d^dx \ln \frac{I_0(z)}{I_0(0)} - \frac{\alpha_0^2 K w}{8} \int d^dx \left[ \frac{\rho V}{2} < \left( \frac{dQ_0}{dz} \right)^2(z) > - < \frac{dQ_0}{dz}(z) >^2 \right] + w^{-1} < W_0(z) > \right)
\]

\[
\times \left[ \partial_\mu \phi'(x) \partial^\mu \phi'(x) \right].
\]  

(4.515)

**The Recursion Formulas:** The full path integral is therefore given by

\[
Z_0 = \int \mathcal{D} \phi'(x_0/2) \int \prod_{\tilde{m}} d\phi_\tilde{m} e^{-S_0(\phi(x))}
\]

\[
\times \int \mathcal{D} \phi'(x) \int \exp \left( 2^d w \int d^dx \ln \frac{I_0(z)}{I_0(0)} - \frac{\alpha_0^2 K w}{8} \int d^dx \left[ \frac{\rho V}{2} < \left( \frac{dQ_0}{dz} \right)^2(z) > - < \frac{dQ_0}{dz}(z) >^2 \right] + w^{-1} < W_0(z) > \right)
\]

\[
+ w^{-1} < W_0(z) > \left[ \partial_\mu \phi'(x) \partial^\mu \phi'(x) \right].
\]  

(4.516)
We write this as
\[ Z_1 = \int D\phi'(x) e^{-S_1(\phi'(x))}. \] (4.517)

The new action \( S_1 \) has the same form as the action \( S_0 \), viz
\[ S_1(\phi'(x)) = \frac{K}{2} \int d^d x R_1(\phi'(x))\partial_\mu \phi'(x)\partial^\mu \phi'(x) + \int d^d x P_1(\phi'(x)). \] (4.518)

The new polynomials \( P_1 \) and \( R_1 \) (or equivalently \( Q_1 \) and \( W_1 \)) are given in terms of the old ones \( P_0 \) and \( R_0 \) (or equivalently \( Q_0 \) and \( W_0 \)) by the relations
\[ W_1(2^{d/2} \alpha_0^{-1} z) = R_1(\phi'(x)) = \frac{\alpha_0^2 C_d}{8} \left( < (\frac{dQ_0}{dz})^2(z) > - < \frac{dQ_0}{dz}(z) >^2 \right) + \frac{\alpha_0^2}{8} < W_0(z) >. \] (4.519)

\[ Q_1(2^{d/2} \alpha_0^{-1} z) = w^{-1} P_1(\phi'(x)) = -2^d \ln \frac{I_0(z)}{I_0(0)}. \] (4.520)

The constant \( C_d \) is given by
\[ C_d = w \rho V. \] (4.521)

Before we write down the recursion formulas we also introduce the notation
\[ \phi^{(0)} = \phi, \phi^{(1)} = \phi'. \] (4.522)

The above procedure can be repeated to integrate out the momentum shell \( l = 1 \) and get from \( S_1 \) to \( S_2 \). The modes with \( l \geq 2 \) will involve a new constant \( \alpha_1 \) and instead of \( P_0, R_0, S_0 \) and \( I_0 \) we will have \( P_1, R_1, S_1 \) and \( I_1 \). Aside from this trivial relabeling everything else will be the same including the constants \( w \), \( V \) and \( C_d \) since \( l = 1 \) can be mapped to \( l = 0 \) due to the scaling \( x_0 \rightarrow 2x \) (see equations (4.470) and (4.472)). This whole process can be repeated an arbitrary number of times to get a renormalization group flow of the action given explicitly by the sequences \( P_0 \rightarrow P_1 \rightarrow P_2 \rightarrow ... \rightarrow P_i \) and \( R_0 \rightarrow R_1 \rightarrow R_2 \rightarrow ... \rightarrow R_i \). By assuming that \( dR_i/d\phi^{(i)} \) is sufficiently small compared to \( P_i \) for all \( i \) the recursion formulas which relates the different operators at the renormalization group steps \( i \) and \( i + 1 \) are obviously given by
\[ W_{i+1}(2^{d/2} \alpha_1^{-1} z) = R_{i+1}(\phi^{(i+1)}(x)) = \frac{\alpha_1^2 C_d}{8} \left( < \left( \frac{dQ_1}{dz} \right)^2(z) > - < \frac{dQ_1}{dz}(z) >^2 \right) + \frac{\alpha_1^2}{8} < W_i(z) >. \] (4.523)
\[ Q_{i+1}(2^{d/2} \alpha_{i}^{-1} z) = w^{-1} P_{i+1}(\phi^{(i+1)}(x)) \]
\[ = -2^{d} \ln \frac{I_{i}(z)}{I_{i}(0)} \]  

(4.524)

The function \( I_{i}(z) \) is given by the same formula (4.508) with the substitutions \( I_{0} \rightarrow I_{i}, W_{0} \rightarrow W_{i} \) and \( Q_{0} \rightarrow Q_{i} \).

The field \( \phi^{(i+1)} \) and the variable \( z \) are related by \( z = (K \rho/2w)^{1/2} 2^{-d/2} \alpha_{i} \phi^{(i+1)} \). The full action at the renormalization group step \( i \) is

\[ S_{i}(\phi^{(i)}(x)) = \frac{K}{2} \int d^{d} x R_{i}(\phi^{(i)}(x)) \partial_{\mu} \phi^{(i)}(x) \partial^{\mu} \phi^{(i)}(x) + \int d^{d} x P_{i}(\phi^{(i)}(x)). \]  

(4.525)

The constants \( \alpha_{i} \) will be determined from the normalization condition

\[ W_{i+1}(0) = 1. \]  

(4.526)

Since \( Q \) is even this normalization condition is equivalent to

\[ \frac{\alpha_{i}^{2}}{4} < W_{i}(z) > |_{z=0} = 1. \]  

(4.527)

**The Ultra Local Recursion Formula:** This corresponds to keeping in the expansion (4.476) only the first term. The resulting recursion formula is obtained from the above recursion formulas by dropping from equation (4.523) the fluctuation term

\[ < \left( \frac{dQ_{i}}{dz} \right)^{2}(z) > - < \frac{dQ_{i}}{dz}(z) >^{2}. \]  

(4.528)

The recursion formula (4.523) becomes

\[ W_{i+1}(2^{d/2} \alpha_{i}^{-1} z) = \frac{\alpha_{i}^{2}}{4} < W_{i}(z) >. \]  

(4.529)

The solution of this equation together with the normalization condition (4.527) is given by

\[ W_{i} = 1, \alpha_{i} = 2. \]  

(4.530)

The remaining recursion formula is given by

\[ Q_{i+1}(2^{d/2} 2^{-1} z) = -2^{d} \ln \frac{I_{i}(z)}{I_{i}(0)}. \]  

(4.531)

We state without proof that the use of this recursion formula is completely equivalent to the use in perturbation theory of the Polyakov-Wilson rules given by the approximations:

- We replace every internal propagator \( 1/(k^{2} + r_{0}^{2}) \) by \( 1/(\Lambda^{2} + r_{0}^{2}) \).
- We replace every momentum integral \( \int_{\Lambda/2}^{\Lambda} d^{d} p/(2\pi)^{d} \) by the volume \( c/4 \) where \( c = 4\Omega_{d-1}\Lambda^{d}(1 - 2^{-d})/(d(2\pi)^{d}) \).
4.5.3 The Wilson-Fisher Fixed Point

Let us start with a $\phi^4$ action given by

$$S_0[\phi_0] = \int d^d x \left( \frac{1}{2} (\partial_\mu \phi_0)^2 + \frac{1}{2} r_0 \phi_0^2 + u_0 \phi_0^4 \right). \quad (4.532)$$

The Fourier transform of the field is given by

$$\phi_0(x) = \int_0^\Lambda \frac{d^d p}{(2\pi)^d} \tilde{\phi}_0(p) e^{ipx} \quad (4.533)$$

We will decompose the field as

$$\phi_0(x) = \phi_1'(x) + \Phi(x). \quad (4.534)$$

The background field $\phi_1'(x)$ corresponds to the low frequency modes $\tilde{\phi}_0(p)$ where $0 \leq p \leq \Lambda/2$ whereas the fluctuation field $\Phi(x)$ corresponds to high frequency modes $\tilde{\phi}_0(p)$ where $\Lambda/2 < p \leq \Lambda$, viz

$$\phi_1'(x) = \int_0^{\Lambda/2} \frac{d^d p}{(2\pi)^d} \tilde{\phi}_0(p) e^{ipx}, \quad \Phi(x) = \int_{\Lambda/2}^\Lambda \frac{d^d p}{(2\pi)^d} \tilde{\phi}_0(p) e^{ipx}. \quad (4.535)$$

The goal is to integrate out the high frequency modes from the partition function. The partition function is given by

$$Z_0 = \int D\phi_0 \ e^{-S_0[\phi_0]} = \int D\phi_1' \int D\Phi \ e^{-S_0[\phi_1' + \Phi]} = \int D\phi_1' \ e^{-S_1[\phi_1']}. \quad (4.536)$$

The first goal is to determine the action $S_1[\phi_1']$. We have

$$e^{-S_1[\phi_1']} = \int d\Phi \ e^{-S_0[\phi_1' + \Phi]} = e^{-S_0[\phi_1']} \int d\Phi \ e^{-S_0[\Phi]} e^{-u_0 \int d^d x [4\Phi^3 \phi_1' + 4\Phi \phi_1'^3 + 6\Phi^2 \phi_1'^2]}. \quad (4.537)$$

We will expand in the field $\phi_1'$ up to the fourth power. We define expectation values with respect to the partition function

$$Z = \int d\Phi \ e^{-S_0[\Phi]}. \quad (4.538)$$
Let us also introduce

\[ V_1 = -4u_0 \int d^d x \Phi^3 \phi_1' \]
\[ V_2 = -6u_0 \int d^d x \Phi^2 \phi_1'' \]
\[ V_3 = -4u_0 \int d^d x \Phi \phi_1''' . \]  

(4.539)

Then we compute

\[ e^{-S_1[\phi_1']} = Ze^{-S_0[\phi_1']} < 1 + V_1 + V_2 + V_3 + \frac{1}{2}(V_1^2 + 2V_1V_2 + V_2^2 + 2V_1V_3) + \frac{1}{6}(V_1^3 + 3V_1^2V_2) + \frac{1}{24}V_1^4 > . \]

(4.540)

By using the symmetry \( \Phi \rightarrow -\Phi \) we obtain

\[ e^{-S_1[\phi_1']} = Ze^{-S_0[\phi_1']} < 1 + V_2 + \frac{1}{2}(V_1^2 + V_2^2 + 2V_1V_3) + \frac{1}{6}(3V_1^2V_2) + \frac{1}{24}V_1^4 > . \]

(4.541)

The term \( < V_1V_3 > \) vanishes by momentum conservation. We rewrite the different expectation values in terms of connected functions. We have

\[ < V_2 > = < V_2 >_c o \]
\[ < V_1^2 > = < V_1^2 >_c o \]
\[ < V_2^2 > = < V_2^2 >_c o + < V_2 >^2_c o \]
\[ < V_1^2V_2 > = < V_1^2V_2 >_c o + < V_1^2 >_c o < V_2 >_c o \]
\[ < V_1^4 > = < V_1^4 >_c o + 3 < V_1^2 >^2_2 c o . \]

(4.542)

By using these results the partition function becomes

\[ e^{-S_1[\phi_1']} = Ze^{-S_0[\phi_1']} e^{<V_2>_c o + \frac{1}{2}(<V_1^2>_c o + <V_2^2>_c o) + \frac{1}{2}V_1^2V_2>_c o + \frac{1}{24}V_1^4>_c o .} \]

(4.543)

In other words the partition function is expressible only in terms of irreducible connected functions. This is sometimes known as the cumulant expansion. The action \( S_1[\phi_1'] \) is given by

\[ S_1[\phi_1'] = S_0[\phi_1'] - < V_2 >_c o - \frac{1}{2}(< V_1^2 >_c o + < V_2^2 >_c o) - \frac{1}{2} < V_1^2V_2 >_c o - \frac{1}{24} < V_1^4 >_c o . \]

(4.544)

We need the propagator

\[ < \Phi(x)\Phi(y) >_c o = < \Phi(x)\Phi(y) >_0 - 12u_0 \int d^d z < \Phi(x)\Phi(z) >_0 < \Phi(y)\Phi(z) >_0 < \Phi(z)\Phi(z) >_0 + O(u_0^2) . \]

(4.545)
The free propagator is obviously given by

\[ < \Phi(x)\Phi(y) >_0 = \int_{\Lambda/2}^{\Lambda} \frac{d^dp}{(2\pi)^d} \frac{1}{p^2 + r_0} e^{ip(x-y)}. \]  

(4.546)

Thus

\[ < \Phi(x)\Phi(y) >_{co} = \left[ \int \frac{d^dp_1}{(2\pi)^d} \frac{1}{p_1^2 + r_0} - 12u_0 \int \frac{d^dp_1}{(2\pi)^d} \frac{d^dp_2}{(2\pi)^d} \frac{1}{(p_1^2 + r_0)^2(p_2^2 + r_0)} + O(u_0^2) \right] e^{ip_1(x-y)}. \]  

(4.547)

We can now compute

\[- < V_2 >_{co} = 6u_0 \int d^dx \phi_1^2(x) < \Phi^2(x) >_{co} \]

\[ = 6u_0 \int \frac{d^dp}{(2\pi)^d} |\tilde{\phi}_1(p)|^2 \left[ \int \frac{d^dp_1}{(2\pi)^d} \frac{1}{p_1^2 + r_0} - 12u_0 \int \frac{d^dp_1}{(2\pi)^d} \frac{d^dp_2}{(2\pi)^d} \frac{1}{(p_1^2 + r_0)^2(p_2^2 + r_0)} + O(u_0^2) \right]. \]  

(4.548)

\[- \frac{1}{2} < V_1^2 >_{co} = -8u_0^2 \int d^dx_1d^dx_2 \phi_1^2(x_1)\phi_1^2(x_2) < \Phi^3(x_1)\Phi^3(x_2) >_{co} \]

\[ = -8u_0^2 \int d^dx_1d^dx_2 \phi_1^2(x_1)\phi_1^2(x_2) \left[ 6 < \Phi(x_1)\Phi(x_2) >_0^3 + 3 < \Phi(x_1)\Phi(x_2) >_0 < \Phi(x_1)\Phi(x_1) >_0 \right. \]

\[ \times < \Phi(x_2)\Phi(x_2) >_0 \]

\[ = -8u_0^2 \int \frac{d^dp}{(2\pi)^d} |\tilde{\phi}_1(p)|^2 \left[ 6 \int \frac{d^dp_1}{(2\pi)^d} \frac{d^dp_2}{(2\pi)^d} \frac{1}{(p_1^2 + r_0)(p_2^2 + r_0)((p + p_1 + p_2)^2 + r_0)} + O(u_0) \right]. \]  

(4.549)

The second term of the second line of the above equation did not contribute because of momentum conservation. Next we compute

\[- \frac{1}{2} < V_2^2 >_{co} = -18u_0^2 \int d^dx_1d^dx_2 \phi_1^2(x_1)\phi_1^2(x_2) < \Phi^2(x_1)\Phi^2(x_2) >_{co} \]

\[ = -36u_0^2 \int d^dx_1d^dx_2 \phi_1^2(x_1)\phi_1^2(x_2) < \Phi(x_1)\Phi(x_2) >^2_0 \]

\[ = -12u_0^2 \int \frac{d^dp_1}{(2\pi)^d} \frac{d^dp_3}{(2\pi)^d} \frac{1}{(p_1^2 + r_0)^2(p_2^2 + r_0)^2} + 2 \text{ permutations} + O(u_0) \].
The last two terms of the cumulant expansion are of order $u_0^3$ and $u_0^4$ respectively which we are not computing. The action $S_1[\phi'_1]$ reads then explicitly

$$S_1[\phi'_1] = \frac{1}{2} \int_0^{\Lambda/2} \frac{d^d p}{(2\pi)^d} \left[ \bar{\phi}(p) \right]^2 \left\{ \frac{p^2}{4} + r_0 + 12u_0 \int \frac{d^d k_1}{(2\pi)^d k_1^2} \frac{1}{k_1^2 + r_0} - 144u_0^2 \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \frac{1}{k_1^2 + r_0} + O(u_0^3) \right\}$$

$$- 96u_0^3 \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \frac{1}{(k_1^2 + r_0)(k_2^2 + r_0)((p + k_1 + k_2)^2 + r_0)} + O(u_0^3)$$

$$+ \int_0^{\Lambda/2} \frac{d^d p_1}{(2\pi)^d} \ldots \frac{d^d p_3}{(2\pi)^d} \bar{\phi}(p_1) \cdots \bar{\phi}(p_3) \bar{\phi}_1(-p_1 - p_2 - p_3) \left[ u_0 - 12u_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + r_0} \right] \times \frac{1}{(k + p_1 + p_2)^2 + r_0 + 2 \text{ permutations}} + O(u_0^3) \right\}.$$  \hspace{1cm} (4.551)

The Fourier mode $\bar{\phi}_1(p)$ is of course equal $\bar{\phi}_0(p)$ for $0 \leq p \leq \Lambda/2$ and 0 otherwise. We scale now the field as

$$\bar{\phi}_1(p) = \alpha_0 \bar{\phi}_1(2p).$$  \hspace{1cm} (4.552)

The action becomes

$$S_1[\phi'_1] = \frac{1}{2} \alpha_0^2 2^{-d} \int_0^{\Lambda} \frac{d^d p}{(2\pi)^d} \left[ \bar{\phi}(p) \right]^2 \left\{ \frac{p^2}{4} + r_0 + 12u_0 \int \frac{d^d k_1}{(2\pi)^d k_1^2} \frac{1}{k_1^2 + r_0} - 144u_0^2 \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \frac{1}{k_1^2 + r_0} \right\}$$

$$- 96u_0^3 \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \frac{1}{(k_1^2 + r_0)(k_2^2 + r_0)((\frac{1}{2}p + k_1 + k_2)^2 + r_0)} + O(u_0^3)$$

$$+ \alpha_0^2 2^{-3d} \int_0^{\Lambda} \frac{d^d p_1}{(2\pi)^d} \ldots \frac{d^d p_3}{(2\pi)^d} \bar{\phi}(p_1) \cdots \bar{\phi}(p_3) \bar{\phi}_1(-p_1 - p_2 - p_3) \left[ u_0 - 12u_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + r_0} \right] \times \frac{1}{(k + p_1 + p_2)^2 + r_0 + 2 \text{ permutations}} + O(u_0^3) \right\}.$$  \hspace{1cm} (4.553)

In the above equation the internal momenta $k_i$ are still unscaled in the interval $[\Lambda/2, \Lambda]$. The one-loop truncation of this result is given by

$$S_1[\phi'_1] = \frac{1}{2} \alpha_0^2 2^{-d} \int_0^{\Lambda} \frac{d^d p}{(2\pi)^d} \left[ \bar{\phi}(p) \right]^2 \left\{ \frac{p^2}{4} + r_0 + 12u_0 \int \frac{d^d k_1}{(2\pi)^d k_1^2} \frac{1}{k_1^2 + r_0} + O(u_0^3) \right\}$$

$$+ \alpha_0^2 2^{-3d} \int_0^{\Lambda} \frac{d^d p_1}{(2\pi)^d} \ldots \frac{d^d p_3}{(2\pi)^d} \bar{\phi}(p_1) \cdots \bar{\phi}(p_3) \bar{\phi}_1(-p_1 - p_2 - p_3) \left[ u_0 - 12u_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + r_0} \right] \times \frac{1}{(k + p_1 + p_2)^2 + r_0 + 2 \text{ permutations}} + O(u_0^3) \right\}.$$  \hspace{1cm} (4.554)

We bring the kinetic term to the canonical form by choose $\alpha_0$ as

$$\alpha_0 = 2^{1+d/2}.$$  \hspace{1cm} (4.555)

Furthermore we truncate the interaction term in the action by setting the external momenta to zero since we are only interested in the renormalization group flow of the
operators present in the original action. We get then

\[ S_1[\phi_1] = \frac{1}{2} \int_0^\Lambda \frac{d^dp}{(2\pi)^d} |\tilde{\phi}_1(p)|^2 (p^2 + r_1) + u_1 \int_0^\Lambda \frac{d^dp_1}{(2\pi)^d} ... \frac{d^dp_3}{(2\pi)^d} \tilde{\phi}_1(p_1) ... \tilde{\phi}_1(p_3) \tilde{\phi}_1(-p_1 - p_2 - p_3). \]

(4.556)

The new mass parameter \( r_1 \) and the new coupling constant \( u_1 \) are given by

\[ r_1 = 4r_0 + 48u_0 \int \frac{d^dk_1}{(2\pi)^d} \frac{1}{k_1^2 + r_0} + O(u_0^2). \]

(4.557)

\[ u_1 = 2^{4-d} \left[ u_0 - 36u_0^2 \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2 + r_0)^2} + O(u_0^3) \right]. \]

(4.558)

Now we employ the Wilson-Polyakov rules corresponding to the ultra local Wilson recursion formula (4.531) consisting of making the following approximations:

- We replace every internal propagator \( 1/(k^2 + r_0^2) \) by \( 1/(\Lambda^2 + r_0^2) \).
- We replace every momentum integral \( \int_0^\Lambda \frac{d^dp}{(2\pi)^d} \) by the volume \( c/4 \) where \( c = 4\Omega_{d-1} \Lambda^d (1 - 2^{-d})/(d(2\pi)^d) \).

The mass parameter \( r_1 \) and the coupling constant \( u_1 \) become

\[ r_1 = 4 \left[ r_0 + 3c \frac{u_0}{\Lambda^2 + r_0} + O(u_0^2) \right]. \]

(4.559)

\[ u_1 = 2^{4-d} \left[ u_0 - 9c \frac{u_0^2}{(\Lambda^2 + r_0)^2} + O(u_0^3) \right]. \]

(4.560)

This is the result of our first renormalization group step. Since the action \( S_1[\phi_1] \) is of the same form as the action \( S_0[\phi_0] \) the renormalization group calculation can be repeated without any change to go from \( r_1 \) and \( u_1 \) to a new mass parameter \( r_2 \) and a new coupling constant \( u_2 \). This whole process can evidently be iterated an arbitrary number of times to define a renormalization group flow \( (r_0, u_0) \rightarrow (r_1, u_1) \rightarrow ... (r_l, u_l) \rightarrow (r_{l+1}, u_{l+1}) ... \). The renormalization group recursion equations relating \( (r_{l+1}, u_{l+1}) \) to \( (r_l, u_l) \) are given precisely by the above equations, viz

\[ r_{l+1} = 4 \left[ r_l + 3c \frac{u_l}{\Lambda^2 + r_l} \right]. \]

(4.561)

\[ u_{l+1} = 2^{4-d} \left[ u_l - 9c \frac{u_l^2}{(\Lambda^2 + r_l)^2} \right]. \]

(4.562)
The fixed points of the renormalization group equations is define obviously by

\[ r^*_\pm = \frac{4}{3c} \frac{u_\pm}{\Lambda^2 + r^*_\pm}. \]

(4.563)

\[ u^*_\pm = 2^{4-d} \left[ u_\pm - 9c \frac{u^2_\pm}{(\Lambda^2 + r^*_\pm)^2}\right]. \]

(4.564)

We find the solutions

Gaussian fixed point : \( r^*_g = 0, u^*_g = 0, \)

and (by assuming that \( u_* \) is sufficiently small)

Wilson – Fisher fixed point : \( r^*_w = -\frac{4c^2}{\Lambda^2}, u^*_w = \frac{\Lambda^4}{9c}(1 - 2^{d-4}). \)

(4.566)

For \( \epsilon = 4 - d \) small the non trivial (interacting) Wilson-Fisher fixed point approaches the trivial (free) Gaussian fixed point as

\[ r_* = -\frac{4}{9} \Lambda^2 \epsilon \ln 2, \quad u_* = \frac{\Lambda^4}{9c} \epsilon \ln 2. \]

(4.567)

The value \( u_* \) controls the strength of the interaction of the low energy (infrared) physics of the system.

4.5.4 The Critical Exponents \( \nu \)

In the Gaussian model the recursion formula reads simply \( r_{l+1} = 4r_l \) and hence we have two possible solutions. At \( T = T_c \) the mass parameter \( r_0 \) must be zero and hence \( r_l = 0 \) for all \( l \), i.e. \( r_0 = 0 \) is a fixed point. For \( T \neq T_c \) the mass parameter \( r_0 \) is non zero and hence \( r_l = 4^l r_0 \to \infty \) for \( l \to \infty \) (\( r_0 = \infty \) is the second fixed point). For \( T \) near \( T_c \) the mass parameter \( r_0 \) is linear in \( T - T_c \).

In the \( \phi^4 \) model the situation is naturally more complicated. We can be at the critical temperature \( T = T_c \) without having the parameters \( r_0 \) and \( u_0 \) at their fixed point values. Indeed, as we have already seen, for any value \( u_0 \) there will be a critical value \( r_0_c = r_0_c(u_0) \) of \( r_0 \) corresponding to \( T = T_c \). At \( T = T_c \) we have \( r_l \to r_* \) and \( u_l \to u_* \) for \( l \to \infty \). For \( T \neq T_c \) we will have in general a different limit for large \( l \).

The critical exponent \( \nu \) can be calculated by studying the behavior of the theory only for \( T \) near \( T_c \). As stated above \( r_l(T_c) \to r_* \) and \( u_l(T_c) \to u_* \) for \( l \to \infty \). From the analytic property of the recursion formulas we conclude that \( r_l(T) \) and \( u_l(T) \) are analytic functions of the temperature and hence near \( T_c \) we should have \( r_l(T) = r_l(T_c) + (T - T_c) r'_l(T_c) + \ldots \) and \( u_l(T) = u_l(T_c) + (T - T_c) u'_l(T_c) + \ldots \) and as a consequence \( r_l(T) \) and \( u_l(T) \) are close to the fixed point values for sufficiently large \( l \) and sufficiently small \( T - T_c \). We are thus led in a natural way to studying the recursion formulas only around the fixed point, i.e. to studying the linearized recursion formulas.
The Linearized Recursion Formulas: Now we linearize the recursion formulas around the fixed point. We find without any approximation

\[ r_{l+1} - r_* = \left[ 4 - \frac{12c u_*}{(\Lambda^2 + r_l)(\Lambda^2 + r_*)} \right] (r_l - r_*) + \frac{12c}{\Lambda^2 + r_*} (u_l - u_*) + 12c u_* \left[ \frac{1}{\Lambda^2 + r_l} - \frac{1}{\Lambda^2 + r_*} \right]. \]  
(4.568)

\[ u_{l+1} - u_* = 2^{4-d} \frac{9c u_*^2}{(\Lambda^2 + r_*)^2} (r_l - r_*) + 2^{4-d} \left[ 1 - \frac{9c}{(\Lambda^2 + r_*)^2} (u_l + u_*) \right] (u_l - u_*). \]  
(4.569)

Keeping only linear terms we find

\[ r_{l+1} - r_* = \left[ 4 - \frac{12c u_*}{(\Lambda^2 + r_*)^2} \right] (r_l - r_*) + \frac{12c}{\Lambda^2 + r_*} (u_l - u_*). \]  
(4.570)

\[ u_{l+1} - u_* = 2^{4-d} \frac{18c u_*^2}{(\Lambda^2 + r_*)^3} (r_l - r_*) + 2^{4-d} \left[ 1 - \frac{18c u_*}{(\Lambda^2 + r_*)^2} \right] (u_l - u_*). \]  
(4.571)

This can be put into the matrix form

\[ \begin{pmatrix} r_{l+1} - r_* \\ u_{l+1} - u_* \end{pmatrix} = M \begin{pmatrix} r_l - r_* \\ u_l - u_* \end{pmatrix}. \]  
(4.572)

The matrix \( M \) is given by

\[ M = \begin{pmatrix} 4 - \frac{12c u_*}{(\Lambda^2 + r_*)^2} & \frac{12c}{\Lambda^2 + r_*} \\ 2^{4-d} \frac{18c u_*^2}{(\Lambda^2 + r_*)^3} & 2^{4-d} \left[ 1 - \frac{18c u_*}{(\Lambda^2 + r_*)^2} \right] \end{pmatrix} = \begin{pmatrix} 4 - \frac{4\epsilon}{3} \ln 2 & \frac{12c}{\Lambda^2} (1 + \frac{4\epsilon}{3} \ln 2) \\ 0 & 1 - \epsilon \ln 2 \end{pmatrix}. \]  
(4.573)

After \( n \) steps of the renormalization group we will have

\[ \begin{pmatrix} r_{l+n} - r_* \\ u_{l+n} - u_* \end{pmatrix} = M^n \begin{pmatrix} r_l - r_* \\ u_l - u_* \end{pmatrix}. \]  
(4.574)

In other words for large \( n \) the matrix \( M^n \) is completely dominated by the largest eigenvalue of \( M \).

Let \( \lambda_1 \) and \( \lambda_2 \) be the eigenvalues of \( M \) with eigenvectors \( w_1 \) and \( w_2 \) respectively such that \( \lambda_1 > \lambda_2 \). Clearly for \( u_* = 0 \) we have

\[ \lambda_1 = 4, \quad w_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \]  
(4.575)

\[ \lambda_2 = 1, \quad w_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]  
(4.575)
The matrix $M$ is not symmetric and thus diagonalization is achieved by an invertible (and not an orthogonal) matrix $U$. We write
\[ M = UDU^{-1}. \] (4.576)

The eigenvalues $\lambda_1$ and $\lambda_2$ can be determined from the trace and determinant which are given by
\[ \lambda_1 + \lambda_2 = M_{11} + M_{22}, \quad \lambda_1 \lambda_2 = M_{11}M_{22} - M_{12}M_{21}. \] (4.577)

We obtain immediately
\[ \lambda_1 = 4 - \frac{4}{3} \epsilon \ln 2, \quad \lambda_2 = 1 - \epsilon \ln 2. \] (4.578)

The corresponding eigenvectors are
\[ w_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad w_2 = \begin{pmatrix} -\frac{4}{3}\epsilon(1 + \frac{5}{9}\epsilon \ln 2) \\ 1 \end{pmatrix}. \] (4.579)

We write the equation $Mw_k = \lambda_k w_k$ as (with $(w_k)_j = w_{jk}$)
\[ M_{ij}w_{jk} = \lambda_k w_{ik}. \] (4.580)

The identity $M = \sum_k \lambda_k |\lambda_k>\langle\lambda_k|$ can be rewritten as
\[ M_{ij} = \sum_k \lambda_k w_{ik} v_{kj} = \lambda_1 w_{i1} v_{1j} + \lambda_2 w_{i2} v_{2j}. \] (4.581)

The vectors $v_k$ are the eigenvectors of $M^T$ with eigenvalues $\lambda_k$ respectively, viz (with $(v_k)_j = v_{kj}$)
\[ M_{ij}^Tv_{kj} = v_{kj} M_{ji} = \lambda_k v_{ki}. \] (4.582)

We find explicitly
\[ v_1 = \begin{pmatrix} \frac{4}{3}\epsilon(1 + \frac{5}{9}\epsilon \ln 2) \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \] (4.583)

The orthonormality condition is then
\[ \sum_j v_{kj} w_{jl} = \delta_{kl}. \] (4.584)

From the result (4.581) we deduce immediately that
\[ M_{ij}^n = \lambda_1^n w_{i1} v_{1j} + \lambda_2^n w_{i2} v_{2j} \approx \lambda_1^n w_{i1} v_{1j}. \] (4.585)
After each renormalization group step we scale the momenta as
\( r \rightarrow r_\ast \) and the distances as \( x \rightarrow x/2 \) which corresponds to scaling the momenta as \( p \rightarrow 2p \). The linearized recursion formulas become
\[
\begin{align*}
 r_{l+n} - r_\ast &\simeq \lambda_1^n w_{11} \left( v_{11}(r_l - r_\ast) + v_{12}(u_l - u_\ast) \right), \\
u_{l+n} - u_\ast &\simeq \lambda_2^n w_{21} \left( v_{11}(r_l - r_\ast) + v_{12}(u_l - u_\ast) \right).
\end{align*}
\]
(4.586)
(4.587)

Since \( r_l = r_l(T) \) and \( u_l = u_l(T) \) are close to the fixed point values for sufficiently large \( l \) and sufficiently small \( T - T_c \) we conclude that \( r_l - r_\ast \) and \( u_l - u_\ast \) are both linear in \( T - T_c \) and as a consequence
\[
v_{11}(r_l - r_\ast) + v_{12}(u_l - u_\ast) = c_l(T - T_c).
\]
(4.588)

The linearized recursion formulas take then the form
\[
\begin{align*}
 r_{l+n} - r_\ast &\simeq c_l \lambda_1^n w_{11}(T - T_c), \\
u_{l+n} - u_\ast &\simeq c_l \lambda_2^n w_{21}(T - T_c).
\end{align*}
\]
(4.589)
(4.590)

**The Critical Exponent \( \nu \):** The correlation length corresponding to the initial action is given by
\[
\xi_0(T) = X(r_0(T), u_0(T)).
\]
(4.591)

After \( l + n \) renormalization group steps the correlation length becomes
\[
\xi_{l+n}(T) = X(r_{l+n}(T), u_{l+n}(T)).
\]
(4.592)

At each renormalization group step we scale the momenta as \( p \rightarrow 2p \) which corresponds to scaling the distances as \( x \rightarrow x/2 \). The correlation length is a measure of distance and thus one must have
\[
X(r_{l+n}, u_{l+n}) = 2^{-l-n} X(r_0, u_0).
\]
(4.593)

From equations (4.589) and (4.590) we have
\[
(r_{l+n+1} - r_\ast)_{T = T_c + \tau/\lambda_1} = (r_{l+n} - r_\ast)_{T = T_c + \tau}, \quad (u_{l+n+1} - u_\ast)_{T = T_c + \tau/\lambda_1} = (u_{l+n} - u_\ast)_{T = T_c + \tau}.
\]
(4.594)

Hence
\[
X(r_{l+n+1}, u_{l+n+1})_{T = T_c + \tau/\lambda_1} = X(r_{l+n}, u_{l+n})_{T = T_c + \tau}.
\]
(4.595)

By using the two results (4.593) and (4.595) we obtain
\[
2^{-l-n-1} \xi_0(T_c + \tau/\lambda_1) = 2^{-l-n} \xi_0(T_c + \tau).
\]
(4.596)

We expect
\[
\xi_0(T_c + \tau) \propto \tau^{-\nu}.
\]
(4.597)

In other words
\[
\frac{1}{2} \left( \frac{\tau}{\lambda_1} \right)^{-\nu} = \tau^{-\nu} \iff \lambda_1^\nu = 2 \iff \nu = \frac{\ln 2}{\ln \lambda_1}.
\]
(4.598)
4.5.5 The Critical Exponent $\eta$

The ultra local recursion formula (4.531) used so far do not lead to a wave function renormalization since all momentum dependence of Feynman diagrams has been dropped and as a consequence the value of the anomalous dimension $\eta$ within this approximation is 0. This can also be seen from the field scaling (4.552) with the choice (4.555) which are made at every renormalization group step and hence the wave function renormalization is independent of the momentum.

In any case we can see from equation (4.554) that the wave function renormalization at the first renormalization group step is given by

$$Z = \frac{\alpha_0^2}{2^{d-2}}. \quad (4.599)$$

From the other hand we have already established that the scaling behavior of $Z(\lambda)$ for small $\lambda$ (the limit in which we approach the infrared stable fixed point) is $\lambda^\eta$. In our case $\lambda = 1/2$ and hence we must have

$$Z = 2^{-\eta}. \quad (4.600)$$

Let $\alpha_s$ be the fixed value of the sequence $\alpha_i$. Then from the above two equations we obtain the formula

$$\eta = \frac{-\ln Z}{\ln 2} = d + 2 - \frac{2 \ln \alpha_s}{\ln 2}. \quad (4.601)$$

As discussed above since $\alpha_s = 2^{1+d/2}$ for the ultra local recursion formula (4.531) we get immediately $\eta = 0$.

To incorporate a non zero value of the critical exponent $\eta$ we must go to the more accurate yet more complicated recursion formulas (4.523) and (4.524). The field scaling at each renormalization group step is a different number $\alpha_i$. These numbers are determined from the normalization condition (4.527).

Recall that integrating out the momenta $1 \leq |k|/\Lambda \leq 2$ resulted in the field $\phi_1(x) = \sum_m \sum_{l=0}^{\infty} \psi_{m,l}(x)\phi_{m,l}$ which was expressed in terms of the field $\phi'(x) = \phi^{(1)}$ which appears in the final action as $\phi_1(x) = 2^{-d/2}a_0\phi'(x/2)$. After $n$ renormalization group steps we integrate out the momenta $2^{1-n} \leq |k|/\Lambda \leq 2$ which results in the field $\phi_n(x) = \sum_m \sum_{l=n}^{\infty} \psi_{m,l}(x)\phi_{m,l}$. However the action will be expressed in terms of the field $\phi' = \phi^{(n)}$ defined by

$$\phi_n(x) = 2^{-nd/2}a_0a_1...a_{n-1}\phi^{(n)}(x/2^n). \quad (4.602)$$

We are interested in the 2–point function

$$<\phi_{m,l}\phi_{m',l'}> = \frac{1}{Z} \int \prod_{l_i=0}^{\infty} \prod_{m_i} d\phi_{m_i,l_i}\phi_{m_i,l_i}' e^{-S_0[\phi]}. \quad (4.603)$$
Let us concentrate on the integral with \( l_1 = l \) and \( \bar{m}_1 = \bar{m} \) and assume that \( l' > l \). We have then the integral

\[
\cdots \int d\phi_{\bar{m}l}d\bar{m}_{\bar{m}l} \prod_{l_1=0}^{l-1} \prod_{\bar{m}_1} d\phi_{\bar{m}_1l_1} e^{-S_0[\phi]} = \cdots \int d\phi_{\bar{m}l}d\bar{m}_{\bar{m}l} e^{-S_l[\phi']}. \tag{4.604}
\]

We have \( \phi' = \phi^{(l)} \) where \( \phi^{(l)} \) contains the momenta \(|k|/\Lambda \leq 1/2^{l-1} \). Since \( \phi_{\bar{m}l} \) is not integrated we have \( \phi_{\bar{m}l} = \alpha_0...\alpha_{l-1}(K\rho/2)^{-1/2}\bar{y}_{\bar{m}} \) which is the generalization of \( \phi_{\bar{m}l} = \alpha_0\phi_{\bar{m}l-1}^{\prime} \). We want now to further integrate \( \phi_{\bar{m}l}^{\prime} \). The final result is similar to (4.504) except that we have an extra factor of \( \bar{y}_{\bar{m}} \) and \( z_{\bar{m}} \) contains all the modes with \( l_1 > l \). The integral thus clearly vanishes because it is odd under \( \bar{y}_{\bar{m}} \rightarrow -\bar{y}_{\bar{m}} \).

We conclude that we must have \( l' = l \) and \( \bar{m}' = \bar{m} \) otherwise the above 2-point function vanishes. After few more calculations we obtain

\[
< \phi_{\bar{m}l}\phi_{\bar{m}'l'} > = \delta_{l'}\delta_{\bar{m}m} - \alpha_0^2...\alpha_{l-1}^2(\frac{K\rho}{2})^{-1} \frac{\prod_{l_1=0}^{l-1} \prod_{\bar{m}_1} d\phi_{\bar{m}_1l_1} \prod_{\bar{m}_1} M_l(z_{\bar{m}_1}) \cdot R_l(z_{\bar{m}})}{\prod_{l_1=0}^{l} \prod_{\bar{m}_1} d\phi_{\bar{m}_1l_1} \prod_{\bar{m}_1} M_l(z_{\bar{m}_1})}. \tag{4.605}
\]

The function \( M_l \) is given by the same formula (4.505) with the substitutions \( M_0 \rightarrow M_l \), \( W_0 \rightarrow W_l \) and \( Q_0 \rightarrow Q_l \). The variable \( z_{\bar{m}} \) is given explicitly by

\[
z_{\bar{m}} = \left\{ \frac{K\rho}{2}\right\}^{1/2} 2^{-d/2} \alpha_l \phi^{(l+1)}(x_0/2). \tag{4.606}
\]

The function \( R_l(z) \) is defined by

\[
R_l(z) = M_l^{-1}(z) \int dy y^2 \exp (...) \tag{4.607}
\]

The exponent is given by the same exponent of equation (4.505) with the substitutions \( M_0 \rightarrow M_l \), \( W_0 \rightarrow W_l \) and \( Q_0 \rightarrow Q_l \).

An order of magnitude formula for the 2-point function can be obtained by replacing the function \( R_l(z) \) by \( R_l(0) \). We obtain then

\[
< \phi_{\bar{m}l}\phi_{\bar{m}'l'} > = \delta_{l'}\delta_{\bar{m}m} - \alpha_0^2...\alpha_{l-1}^2(\frac{K\rho}{2})^{-1} R_l(0). \tag{4.608}
\]

At a fixed point of the recursion formulas we must have

\[
W_l \rightarrow W_s, Q_l \rightarrow Q_s \leftrightarrow R_l \rightarrow R_s, \tag{4.609}
\]

and

\[
\alpha_l \rightarrow \alpha_s. \tag{4.610}
\]

The 2-point function is therefore given by

\[
< \phi_{\bar{m}l}\phi_{\bar{m}'l'} > \propto \delta_{l'}\delta_{\bar{m}m} - \alpha_s^2(\frac{K\rho}{2})^{-1} R_s(0). \tag{4.611}
\]
The modes $< \phi_{\vec{m}l} >$ correspond to the momentum shell $2^{-l} \leq |k|/\Lambda \leq 2^{1-l}$, i.e. $k \sim \Lambda 2^{-l}$. From the other hand the $2-$point function is expected to behave as

$$< \phi_{\vec{m}l} \phi_{\vec{m}'l'} > \propto \delta_{m'}^m \delta_{l'}^l \frac{1}{k^{2-\eta}},$$

where $\eta$ is precisely the anomalous dimension. By substituting $k \sim \Lambda 2^{-l}$ in this last formula we obtain

$$< \phi_{\vec{m}l} \phi_{\vec{m}'l'} > \propto \delta_{m'}^m \Lambda^{\eta - 2l(2-\eta)}. $$

By comparing the $l-$dependent bits in (4.611) and (4.613) we find that the anomalous dimension is given by

$$2^{2-\eta} = \alpha_s^2 \Rightarrow \eta = 2 - \frac{2 \ln \alpha_s}{\ln 2}. $$
Exercises

Power Counting Theorems for Dirac and Vector Fields

• Derive power counting theorems for theories involving scalar as well as Dirac and vector fields by analogy with what we have done for pure scalar field theories.

• What are renormalizable field theories in $d = 4$ dimensions involving spin 0, 1/2 and 1 particles.

• Discuss the case of QED.

Renormalization Group Analysis for The Effective Action

• In order to study the system in the broken phase we must perform a renormalization group analysis of the effective action and study its behavior as a function of the mass parameter. Carry out explicitly this program.
Solve 3 exercises out of 6 as follows:

- Choose between 1 and 2.
- Choose between 3 and 4.
- Choose between 5 and 6.

**Exercise 1:** We consider the two Euclidean integrals

\[
I(m^2) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2}.
\]

\[
J(p^2, m^2) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2} \frac{1}{(p - k)^2 + m^2}.
\]

- Determine in each case the divergent behavior of the integral.
- Use dimensional regularization to compute the above integrals. Determine in each case the divergent part of the integral. In the case of \(J(p^2, m^2)\) assume for simplicity zero external momentum \(p = 0\).

**Exercise 2:** The two integrals in exercise 1 can also be regularized using a cutoff \(\Lambda\). First we perform Laplace transform as follows

\[
\frac{1}{k^2 + m^2} = \int_0^\infty d\alpha e^{-\alpha(k^2 + m^2)}.
\]

- Do the integral over \(k\) in \(I(m^2)\) and \(J(p^2, m^2)\). In the case of \(J(p^2, m^2)\) assume for simplicity zero external momentum \(p = 0\).
- The remaining integral over \(\alpha\) is regularized by replacing the lower bound \(\alpha = 0\) by \(\alpha = 1/\Lambda^2\). Perform the integral over \(\alpha\) explicitly. Determine the divergent part in each case.

Hint: Use the exponential-integral function

\[
Ei(-x) = \int_{-\infty}^{-x} \frac{e^t}{t} dt = C + \ln x + \int_0^x dt \frac{e^{-t} - 1}{t}.
\]
Exercise 3: Let \( z_i \) be a set of complex numbers, \( \theta_i \) be a set of anticommuting Grassmann numbers and let \( M \) be a hermitian matrix. Perform the following integrals

\[
\int \prod_i dz_i^+ dz_i e^{-M_{ij}z_i^+z_j-z_i^+j_i-z_ij_i^+z_i}.
\]

\[
\int \prod_i d\theta_i^+ d\theta_i e^{-M_{ij}\theta_i^+\theta_j-\theta_i^+\eta_i-\eta_i^+\theta_i}.
\]

Exercise 4: Let \( S(r, \theta) \) be an action dependent on two degrees of freedom \( r \) and \( \theta \) which is invariant under 2-dimensional rotations, i.e. \( \vec{r} = (r, \theta) \). We propose to gauge fix the following 2-dimensional path integral

\[
W = \int e^{iS(\vec{r})} d^2 \vec{r}.
\]

We will impose the gauge condition

\[
g(r, \theta) = 0.
\]

- Show that

\[
\left. \frac{\partial g(r, \theta)}{\partial \theta} \right|_{g=0} \int d\phi \delta (g(r, \theta + \phi)) = 1.
\]

- Use the above identity to gauge fix the path integral \( W \).

Exercise 5: The gauge fixed path integral of quantum electrodynamics is given by

\[
Z[J] = \int \prod \mu D A_\mu \exp \left( -i \int d^4 x \frac{(\partial_\mu A^\mu)^2}{2\xi} \right. \left. - \frac{i}{4} \int d^4 x F_{\mu\nu} F^{\mu\nu} - i \int d^4 x J_\mu A^\mu \right).
\]

- Derive the equations of motion.
- Compute \( Z[J] \) in a closed form.
- Derive the photon propagator.

Exercise 6: We consider phi-four interaction in 4 dimensions. The action is given by

\[
S[\phi] = \int d^4 x \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} (\phi^2)^2 \right].
\]

- Write down Feynman rules in momentum space.
- Use Feynman rules to derive the 2-point proper vertex \( \Gamma^2(p) \) up to the one-loop order. Draw the corresponding Feynman diagrams.
• Use Feynman rules to derive the 4-point proper vertex $\Gamma^4(p_1, p_2, p_3, p_4)$ up to the one-loop order. Draw the corresponding Feynman diagrams.

• By assuming that the momentum loop integrals are regularized perform one-loop renormalization of the theory. Impose the two conditions

$$\Gamma^2(0) = m^2_R, \quad \Gamma^4(0,0,0,0) = \lambda_R.$$ 

Determine the bare coupling constants $m^2$ and $\lambda$ in terms of the renormalized coupling constants $m^2_R$ and $\lambda_R$.

• Determine $\Gamma^2(p)$ and $\Gamma^4(p_1, p_2, p_3, p_4)$ in terms of the renormalized coupling constants.
Exercise 1: We consider the two Euclidean integrals

\[ I(m^2) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2}. \]

\[ J(p^2, m^2) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2} \frac{1}{(p - k)^2 + m^2}. \]

- Determine in each case the divergent behavior of the integral.
- Use dimensional regularization to compute the above integrals. Determine in each case the divergent part of the integral. In the case of \( J(p^2, m^2) \) assume for simplicity zero external momentum \( p = 0 \).

Exercise 2: The gauge fixed path integral of quantum electrodynamics is given by

\[ Z[J] = \int \prod \mathcal{D}A_\mu \exp \left(-i \int d^4x \frac{\left(\partial_\mu A_\mu^\prime \right)^2}{2\xi} - \frac{i}{4} \int d^4xF_{\mu\nu}F^{\mu\nu} - i \int d^4xA_\mu A^\prime_\mu \right). \]

- Derive the equations of motion.
- Compute \( Z[J] \) in a closed form.
- Derive the photon propagator.


