# THE QFT NOTES 5

Badis Ydri
Department of Physics, Faculty of Sciences, Annaba University,
Annaba, Algeria.

December 9, 2011

## Contents

**1 Renormalization of QED** **2**
1.1 Example III: $e^- + \mu^- \longrightarrow e^- + \mu^-$ . . . . . . . . . . 2
1.2 Example IV : Scattering From External Electromagnetic Fields . . . . . . . . . . 3
1.3 One-loop Calculation I: Vertex Correction . . . . . . . . . . 6
1.3.1 Feynman Parameters and Wick Rotation . . . . . . . . . . 6
1.3.2 Pauli-Villars Regularization . . . . . . . . . . 11
1.3.3 Renormalization (Minimal Subtraction) and Anomalous Magnetic Moment 13
1.4 Exact Fermion $2-$Point Function . . . . . . . . . . 16
1.5 One-loop Calculation II: Electron Self-Energy . . . . . . . . . . 18
1.5.1 Electron Mass at One-Loop . . . . . . . . . . 18
1.5.2 The Wave-Function Renormalization $Z_2$ . . . . . . . . . . 21
1.5.3 The Renormalization Constant $Z_1$ . . . . . . . . . . 22
1.6 Ward-Takahashi Identities . . . . . . . . . . 24
1.7 One-Loop Calculation III: Vacuum Polarization . . . . . . . . . . 28
1.7.1 The Renormalization Constant $Z_3$ and Renormalization of the Electric Charge . . . . . . . . . . 28
1.7.2 Dimensional Regularization . . . . . . . . . . 30
1.8 Renormalization of QED . . . . . . . . . . 34
1.9 Problems and Exercises . . . . . . . . . . 36

# 1 Renormalization of QED

## 1.1 Example III: $e^- + \mu^- \longrightarrow e^- + \mu^-$

The most important one-loop correction to the probability amplitude of the process $e^- + e^+ \longrightarrow \mu^- + \mu^+$ is given by the Feynamn diagram RAD2. This is known as the vertex correction as it gives quantum correction to the QED interaction vertex $-ie\gamma^\mu$. It has profound observable measurable physical consequences. For example it will lead among other things to the infamous anomalous magnetic moment of the electron. This is a generic effect. Indeed vertex correction should appear in all electromagnetic processes.

Let us consider here as an example the different process

$$e^-(p) + \mu^-(k) \longrightarrow e^-(p^{'}) + \mu^-(k^{'}). \tag{1}$$

This is related to the process $e^- + e^+ \longrightarrow \mu^- + \mu^+$ by the so-called crossing symmetry or substitution law. Remark that the incoming positron became the outgoing electron and the outgoing antimuon became the incoming muon. The substitution law is essentially the statement that the probability amplitudes of these two processes can be obtained from the same Green's function. Instead of following this route we will simply use Feynman rules to write down the probability amplitude of the above process of electron scattering from a heavy particle which is here the muon.

For vertex correction we will need to add the probability amplitudes of the three Feynamn diagrams VERTEX. The tree level contribution (first graph) is (with $q = p - p^{'}$ and $l^{'} = l - q$)

$$(2\pi)^4\delta^4(k + p - k^{'} - p^{'})\frac{ie^2}{q^2}(\bar{u}^{s^{'}}(p^{'})\gamma^\mu u^s(p))(\bar{u}^{r^{'}}(k^{'})\gamma_\mu u^r(k)). \tag{2}$$

The electron vertex correction (the second graph) is

$$(2\pi)^4\delta^4(k + p - k^{'} - p^{'})\frac{-e^4}{q^2}\int\frac{d^4l}{(2\pi)^4}\frac{1}{(l-p)^2 + i\epsilon}\bigg(\bar{u}^{s^{'}}(p^{'})\gamma^\lambda\frac{i(\gamma.l^{'} + m_e)}{l^{'2} - m_e^2 + i\epsilon}\gamma^\mu\frac{i(\gamma.l + m_e)}{l^2 - m_e^2 + i\epsilon}\gamma_\lambda u^s(p)\bigg)$$
$$\times \quad (\bar{u}^{r^{'}}(k^{'})\gamma_\mu u^r(k)). \tag{3}$$

The muon vertex correction (the third graph) is similar to the electron vertex correction but since it will be neglected in the limit $m_\mu \longrightarrow \infty$ we will not write down here.

Adding the three diagrams together we obtain

$$(2\pi)^4\delta^4(k + p - k^{'} - p^{'})\frac{ie^2}{q^2}(\bar{u}^{s^{'}}(p^{'})\Gamma^\mu(p^{'}, p)u^s(p))(\bar{u}^{r^{'}}(k^{'})\gamma_\mu u^r(k)). \tag{4}$$

This is the same as the tree level term with an effective vertex $-ie\Gamma^\mu(p^{'}, p)$ where $\Gamma^\mu(p^{'}, p)$ is given by

$$\Gamma^\mu(p^{'}, p) = \gamma^\mu + ie^2\int\frac{d^4l}{(2\pi)^4}\frac{1}{(l-p)^2 + i\epsilon}\bigg(\gamma^\lambda\frac{i(\gamma.l^{'} + m_e)}{l^{'2} - m_e^2 + i\epsilon}\gamma^\mu\frac{i(\gamma.l + m_e)}{l^2 - m_e^2 + i\epsilon}\gamma_\lambda\bigg). \tag{5}$$

If we did not take the limit $m_\mu \longrightarrow \infty$ the muon vertex would have also been corrected in the same fashion.

The corrections to external legs are given by the four diagrams WAVEFUNCTION. We only write explicitly the first of these diagrams. This is given by

$$(2\pi)^4\delta^4(k+p-k^{'}-p^{'})\frac{e^4}{q^2}\int\frac{d^4l}{(2\pi)^4}\frac{1}{(l-p)^2+i\epsilon}(\bar{u}^{s^{'}}(p^{'})\gamma^\mu\frac{\gamma.p+m_e}{p^2-m_e^2}\gamma^\lambda\frac{\gamma.l+m_e}{l^2-m_e^2}\gamma_\lambda u^s(p))(\bar{u}^{r^{'}}(k^{'})\gamma_\mu u^r(k)). \quad (6)$$

The last diagram contributing to the one-loop radiative corrections is the vacuum polarization diagram shown on figure PHOTONVACUUM. It is given by

$$(2\pi)^4\delta^4(k+p-k^{'}-p^{'})\frac{ie^2}{(q^2)^2}(\bar{u}^{s^{'}}(p^{'})\gamma_\mu u^s(p))\Pi_2^{\mu\nu}(q)(\bar{u}^{r^{'}}(k^{'})\gamma_\nu u^r(k)). \quad (7)$$

$$i\Pi_2^{\mu\nu}(q)=(-1)\int\frac{d^4k}{(2\pi)^4}\mathrm{tr}(-ie\gamma^\mu)\frac{i(\gamma.k+m_e)}{k^2-m_e^2+i\epsilon}(-ie\gamma^\nu)\frac{i(\gamma.(k+q)+m_e)}{(k+q)^2-m_e^2+i\epsilon}. \quad (8)$$

## 1.2 Example IV : Scattering From External Electromagnetic Fields

We will now consider the problem of scattering of electrons from a fixed external electromagnetic field $A_\mu^{\mathrm{backgr}}$, viz

$$e^-(p)\longrightarrow e^-(p^{'}). \quad (9)$$

The transfer momentum which is here $q=p^{'}-p$ is taken by the background electromagnetic field $A_\mu^{\mathrm{backgr}}$. Besides this background field there will also be a fluctuating quantum electromagnetic field $A_\mu$ as usual. This means in particular that the interaction Lagrangian is of the form

$$\mathcal{L}_{\mathrm{in}}=-e\bar{\hat{\psi}}_{\mathrm{in}}\gamma_\mu\hat{\psi}_{\mathrm{in}}(\hat{A}^\mu+A^{\mu,\mathrm{backgr}}). \quad (10)$$

The initial and final states in this case are given by

$$|\vec{p},s\ \mathrm{in}>=\sqrt{2E_{\vec{p}}}\ \hat{b}_{\mathrm{in}}(\vec{p},s)^+|0\ \mathrm{in}>. \quad (11)$$

$$|\vec{p}^{'},s^{'}\ \mathrm{out}>=\sqrt{2E_{\vec{p}^{'}}}\ \hat{b}_{\mathrm{out}}(\vec{p}^{'},s^{'})^+|0\ \mathrm{out}>. \quad (12)$$

The probability amplitude after reducing the initial and final electron states using the appropriate reduction formulas is given by

$$<\vec{p}^{'}s^{'}\ \mathrm{out}|\vec{p}s\ \mathrm{in}> \;=\; -\Big[\bar{u}^{s^{'}}(p^{'})(\gamma.p^{'}-m_e)\Big]_{\alpha^{'}}G_{\alpha^{'}\alpha}(-p^{'},p)\Big[(\gamma.p-m_e)u^s(p)\Big]_\alpha. \quad (13)$$

Here $G_{\alpha^{'}\alpha}(p^{'},p)$ is the Fourier transform of the 2−point Green's function $<0\ \mathrm{out}|T(\hat{\psi}_{\alpha^{'}}(x^{'})\bar{\hat{\psi}}_\alpha(x))|0\ \mathrm{in}>$, viz

$$< 0\ \text{out}|T\Big(\hat{\psi}_{\alpha'}(x')\bar{\hat{\psi}}_{\alpha}(x)\Big)|0\ \text{in} > \ = \ \int \frac{d^4p'}{(2\pi)^4}\int \frac{d^4p}{(2\pi)^4}G_{\alpha',\alpha}(p',p)\ e^{ipx+ip'x'}. \tag{14}$$

By using the Gell-Mann Low formula we get

$$< 0\ \text{out}|T\Big(\hat{\psi}_{\alpha'}(x')\bar{\hat{\psi}}_{\alpha}(x)\Big)|0\ \text{in} > \ = \ < 0\ \text{in}|T\Big(\hat{\psi}_{\alpha',\text{in}}(x')\bar{\hat{\psi}}_{\alpha,\text{in}}(x)S\Big)|0\ \text{in} > . \tag{15}$$

Now we use Wick's theorem. The first term in $S$ leads 0. The second term in $S$ leads to the contribution

$$\begin{aligned}
i\int d^4z < 0\ \text{in}|T\Big(\hat{\psi}_{\alpha',\text{in}}(x')\bar{\hat{\psi}}_{\alpha,\text{in}}(x)\mathcal{L}_{\text{in}}(z)\Big)|0\ \text{in} > \ &= \ (-ie)\int d^4z < 0\ \text{in}|T\Big(\hat{\psi}_{\alpha',\text{in}}(x')\bar{\hat{\psi}}_{\alpha,\text{in}}(x).\bar{\hat{\psi}}_{\text{in}}(z)\gamma_\mu \\
&\times \ \hat{\psi}_{\text{in}}(z)\Big)|0\ \text{in} > A^{\mu,\text{backgr}}(z) \\
&= \ (-ie)\int d^4z\Big(S_F(x'-z)\gamma_\mu S_F(z-x)\Big)^{\alpha'\alpha}A^{\mu,\text{backgr}}(z) \\
&= \ (-ie)\int \frac{d^4p'}{(2\pi)^4}\int \frac{d^4p}{(2\pi)^4}\Big(S(p')\gamma_\mu S(p)\Big)^{\alpha'\alpha}A^{\mu,\text{backgr}}(q) \\
&\times \ e^{ipx-ip'x'}.
\end{aligned} \tag{16}$$

We read from this equation the Fourier transform

$$G_{\alpha'\alpha}(-p',p) \ = \ (-ie)\Big(S(p')\gamma_\mu S(p)\Big)^{\alpha'\alpha}A^{\mu,\text{backgr}}(q). \tag{17}$$

The tree level probability amplitude is therefore given by

$$< \vec{p}'s'\ \text{out}|\vec{p}s\ \text{in} > \ = \ -ie\Big(\bar{u}^{s'}(p')\gamma_\mu u^s(p)\Big)A^{\mu,\text{backgr}}(q). \tag{18}$$

The Fourier transform $A^{\mu,\text{backgr}}(q)$ is defined by

$$A^{\mu,\text{backgr}}(x) = \int \frac{d^4q}{(2\pi)^4}A^{\mu,\text{backgr}}(q)\ e^{-iqx}. \tag{19}$$

This tree level process corresponds to the Feynman diagram EXT-TREE.

The background field is usually assumed to be small. So we will only keep linear terms in $A^{\mu,\text{backgr}}(x)$. The third term in $S$ does not lead to any correction which is linear in $A^{\mu,\text{backgr}}(x)$. The fourth term in $S$ leads to a linear term in $A^{\mu,\text{backgr}}(x)$ given by

$$\begin{aligned}
&\frac{(-ie)^3}{3!}(3)\int d^4z_1\int d^4z_2\int d^4z_3 < 0\ \text{in}|T\Big(\hat{\psi}_{\alpha',\text{in}}(x')\bar{\hat{\psi}}_{\alpha,\text{in}}(x).\bar{\hat{\psi}}_{\text{in}}(z_1)\gamma_\mu\hat{\psi}_{\text{in}}(z_1).\bar{\hat{\psi}}_{\text{in}}(z_2)\gamma_\nu\hat{\psi}_{\text{in}}(z_2).\bar{\hat{\psi}}_{\text{in}}(z_3) \\
\times \ &\gamma_\lambda\hat{\psi}_{\text{in}}(z_3)\Big)|0\ \text{in} >< 0\ \text{out}|T(\hat{A}^\mu(z_1)\hat{A}^\nu(z_2))|0\ \text{in} > A^{\lambda,\text{backgr}}(z_3).
\end{aligned} \tag{20}$$

We use Wick's theorem. For the gauge fields the result is trivial. It is simply given by the photon propagator. For the fermion fields the result is quite complicated. As before there are in total 24 contractions. By dropping those disconnected contractions which contain $S_F(0)$ we will only have 11 contractions left. By further inspection we see that only 8 are really disconnected. By using then the symmetry between the internal points $z_1$ and $z_2$ we obtain the four terms

$$\begin{aligned}
&< 0 \text{ in}|T\Big(\hat{\psi}_{\alpha',\text{in}}(x')\hat{\bar{\psi}}_{\alpha,\text{in}}(x).\hat{\bar{\psi}}_{\text{in}}(z_1)\gamma_\mu\hat{\psi}_{\text{in}}(z_1).\hat{\bar{\psi}}_{\text{in}}(z_2)\gamma_\nu\hat{\psi}_{\text{in}}(z_2).\hat{\bar{\psi}}_{\text{in}}(z_3)\gamma_\lambda\hat{\psi}_{\text{in}}(z_3)\Big)|0 \text{ in}> \\
=& -2\Big[S_F(x'-z_1)\gamma_\mu S_F(z_1-x)\Big]^{\alpha'\alpha}\text{tr}\gamma_\nu S_F(z_2-z_3)\gamma_\lambda S_F(z_3-z_2) \\
+& \ 2\Big[S_F(x'-z_1)\gamma_\mu S_F(z_1-z_2)\gamma_\nu S_F(z_2-z_3)\gamma_\lambda S_F(z_3-x)\Big]^{\alpha'\alpha} \\
+& \ 2\Big[S_F(x'-z_3)\gamma_\lambda S_F(z_3-z_2)\gamma_\nu S_F(z_2-z_1)\gamma_\mu S_F(z_1-x)\Big]^{\alpha'\alpha} \\
+& \ 2\Big[S_F(x'-z_1)\gamma_\mu S_F(z_1-z_3)\gamma_\lambda S_F(z_3-z_2)\gamma_\nu S_F(z_2-x)\Big]^{\alpha'\alpha}. \qquad (21)
\end{aligned}$$

These four terms correspond to the four Feynman diagrams on figure EXT-RAD. Clearly only the last diagram will contribute to the vertex correction so we will only focus on it in the rest of this discussion. The fourth term in $S$ leads therefore to a linear term in the background field $A^{\mu,\text{backgr}}(x)$ given by

$$\begin{aligned}
&(-ie)^3\int d^4z_1\int d^4z_2\int d^4z_3\Big[S_F(x'-z_1)\gamma_\mu S_F(z_1-z_3)\gamma_\lambda S_F(z_3-z_2)\gamma_\nu S_F(z_2-x)\Big]^{\alpha'\alpha} iD_F^{\mu\nu}(z_1-z_2) \\
\times& \ A^{\lambda,\text{backgr}}(z_3) = e^3\int\frac{d^4p'}{(2\pi)^4}\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4k'}{(2\pi)^4}\int\frac{d^4k}{(2\pi)^4}\frac{1}{(p'-k)^2+i\epsilon}\Big(S(p')\gamma_\mu S(k)\gamma_\lambda S(k')\gamma^\mu S(p)\Big)^{\alpha'\alpha} \\
\times& \ A^{\lambda,\text{backgr}}(q)\ (2\pi)^4\delta^4(q-k+k')\ e^{ipx-ip'x'}. \qquad (22)
\end{aligned}$$

The corresponding Fourier transform is

$$G_{\alpha',\alpha}(-p',p) = e^3\int\frac{d^4k}{(2\pi)^4}\frac{1}{(p'-k)^2+i\epsilon}\Big(S(p')\gamma_\mu S(k)\gamma_\lambda S(k-q)\gamma^\mu S(p)\Big)^{\alpha'\alpha}A^{\lambda,\text{backgr}}(q). \qquad (23)$$

The probability amplitude (including also the tree level contribution) is therefore given by

$$\begin{aligned}
< \vec{p}'s' \text{ out}|\vec{p}s \text{ in}> &= -ie\Big(\bar{u}^{s'}(p')\gamma_\lambda u^s(p)\Big)A^{\lambda,\text{backgr}}(q) \\
&+ e^3\int\frac{d^4k}{(2\pi)^4}\frac{1}{(p-k)^2+i\epsilon}\Big(\bar{u}^{s'}(p')\gamma_\mu S(k+q)\gamma_\lambda S(k)\gamma^\mu u^s(p)\Big)A^{\lambda,\text{backgr}}(q) \\
&= -ie\Big(\bar{u}^{s'}(p')\Gamma_\lambda(p',p)u^s(p)\Big)A^{\lambda,\text{backgr}}(q). \qquad (24)
\end{aligned}$$

The effective vertex $\Gamma_\lambda(p',p)$ is given by the same formula as before. This is a general result. The quantum electron vertex at one-loop is always given by the function $\Gamma_\lambda(p',p)$.

## 1.3 One-loop Calculation I: Vertex Correction

### 1.3.1 Feynman Parameters and Wick Rotation

We will calculate $\delta\Gamma^{\mu}(p^{'},p)=\Gamma^{\mu}(p^{'},p)-\gamma^{\mu}$. First we use the identities $\gamma^{\nu}\gamma^{\mu}\gamma_{\nu}=-2\gamma^{\mu}$, $\gamma^{\lambda}\gamma^{\rho}\gamma^{\mu}\gamma_{\lambda}=4\eta^{\rho\mu}$ and

$$\begin{aligned}\gamma^{\lambda}\gamma^{\rho}\gamma^{\mu}\gamma^{\sigma}\gamma_{\lambda} &= 2\gamma^{\sigma}\gamma^{\rho}\gamma^{\mu}-2\gamma^{\mu}\gamma^{\rho}\gamma^{\sigma}-2\gamma^{\rho}\gamma^{\mu}\gamma^{\sigma}\\ &= -2\gamma^{\sigma}\gamma^{\mu}\gamma^{\rho}.\end{aligned} \tag{25}$$

We have

$$\begin{aligned}\bar{u}^{s^{'}}(p^{'})\delta\Gamma^{\mu}(p^{'},p)u^{s}(p) &= 2ie^{2}\int\frac{d^{4}l}{(2\pi)^{4}}\frac{1}{((l-p)^{2}+i\epsilon)(l'^{2}-m_{e}^{2}+i\epsilon)(l^{2}-m_{e}^{2}+i\epsilon)}\bar{u}^{s^{'}}(p^{'})\Big((\gamma.l)\gamma^{\mu}(\gamma.l^{'})\\ &+ m_{e}^{2}\gamma^{\mu}-2m_{e}(l+l^{'})^{\mu}\Big)u^{s}(p).\end{aligned} \tag{26}$$

**Feynman Parameters:** Now we note the identity

$$\frac{1}{A_{1}A_{2}...A_{n}}=\int_{0}^{1}dx_{1}dx_{2}...dx_{n}\delta(x_{1}+x_{2}+...+x_{n}-1)\frac{(n-1)!}{(x_{1}A_{1}+x_{2}A_{2}+...+x_{n}A_{n})^{n}}. \tag{27}$$

For $n=2$ this is obvious since

$$\begin{aligned}\frac{1}{A_{1}A_{2}} &= \int_{0}^{1}dx_{1}dx_{2}\delta(x_{1}+x_{2}-1)\frac{1}{(x_{1}A_{1}+x_{2}A_{2})^{2}}\\ &= \int_{0}^{1}dx_{1}\frac{1}{(x_{1}A_{1}+(1-x_{1})A_{2})^{2}}\\ &= \frac{1}{(A_{1}-A_{2})^{2}}\int_{A_{2}/(A_{1}-A_{2})}^{A_{1}/(A_{1}-A_{2})}\frac{dx_{1}}{x_{1}^{2}}.\end{aligned} \tag{28}$$

In general the identity can be proven as follows. Let $\epsilon$ be a small positive real number. We start from the identity

$$\frac{1}{A}=\int_{0}^{\infty}dt\ e^{-t(A+\epsilon)}. \tag{29}$$

Hence

$$\frac{1}{A_{1}A_{2}...A_{n}}=\int_{0}^{\infty}dt_{1}dt_{2}...dt_{n}\ e^{-\sum_{i=1}^{n}t_{i}(A_{i}+\epsilon)}. \tag{30}$$

Since $t_{i}\geq 0$ we have also the identity

$$\int_{0}^{\infty}\frac{d\lambda}{\lambda}\ \delta(1-\frac{1}{\lambda}\sum_{i=1}^{n}t_{i})=1. \tag{31}$$

Inserting (31) into (30) we obtain

$$\frac{1}{A_{1}A_{2}...A_{n}}=\int_{0}^{\infty}dt_{1}dt_{2}...dt_{n}\int_{0}^{\infty}\frac{d\lambda}{\lambda}\ \delta(1-\frac{1}{\lambda}\sum_{i=1}^{n}t_{i})\ e^{-\sum_{i=1}^{n}t_{i}(A_{i}+\epsilon)}. \tag{32}$$

We change variables from $t_i$ to $x_i = t_i/\lambda$. We obtain

$$\frac{1}{A_1 A_2 ... A_n} = \int_0^{\infty} dx_1 dx_2 ... dx_n \int_0^{\infty} d\lambda \lambda^{n-1}\ \delta(1 - \sum_{i=1}^{n} x_i)\ e^{-\lambda \sum_{i=1}^{n} x_i (A_i + \epsilon)}. \tag{33}$$

We use now the integral representation of the gamma function given by (with $\mathrm{Re}(X) > 0$)

$$\Gamma(n) = (n-1)! = X^n \int_0^{\infty} d\lambda \lambda^{n-1}\ e^{-\lambda X}. \tag{34}$$

We obtain

$$\frac{1}{A_1 A_2 ... A_n} = \int_0^{\infty} dx_1 dx_2 ... dx_n\ \delta(1 - \sum_{i=1}^{n} x_i)\ \frac{(n-1)!}{\Big(\sum_{i=1}^{n} x_i (A_i + \epsilon)\Big)^n}. \tag{35}$$

Since $x_i \geq 0$ and $\sum_{i=1}^{n} x_i = 1$ we must have $0 \leq x_i \leq 1$. Thus

$$\frac{1}{A_1 A_2 ... A_n} = \int_0^{1} dx_1 dx_2 ... dx_n\ \delta(1 - \sum_{i=1}^{n} x_i)\ \frac{(n-1)!}{\Big(A_1 x_1 + A_2 x_2 + ... + A_n x_n\Big)^n}. \tag{36}$$

The variables $x_i$ are called Feynman parameters.

This identity will allow us to convert a product of propagators into a single fraction. Let us see how this works in our current case. We have

$$\frac{1}{((l-p)^2 + i\epsilon)(l'^2 - m_e^2 + i\epsilon)(l^2 - m_e^2 + i\epsilon)} = 2 \int_0^1 dx dy dz\ \delta(x + y + z - 1) \frac{1}{D^3}. \tag{37}$$

$$D = x((l-p)^2 + i\epsilon) + y(l'^2 - m_e^2 + i\epsilon) + z(l^2 - m_e^2 + i\epsilon). \tag{38}$$

Let us recall that the variable of integration is the four-momentum $l$. Clearly we must try to complete the square. By using $x + y + z = 1$ we have

$$\begin{aligned} D &= l^2 - 2(xp + yq)l + xp^2 + yq^2 - (y+z)m_e^2 + i\epsilon \\ &= \Big(l - xp - yq\Big)^2 - x^2p^2 - y^2q^2 - 2xypq + xp^2 + yq^2 - (y+z)m_e^2 + i\epsilon \\ &= \Big(l - xp - yq\Big)^2 + xzp^2 + xyp'^2 + yzq^2 - (y+z)m_e^2 + i\epsilon. \end{aligned} \tag{39}$$

Since this will act on $u^s(p)$ and $\bar{u}^{s'}(p')$ and since $p^2 u^s(p) = m_e^2 u^s(p)$ and $p'^2 \bar{u}^{s'}(p') = m_e^2 \bar{u}^{s'}(p')$ we can replace both $p^2$ and $p'^2$ in $D$ with their on-shell value $m_e^2$. We get then

$$D = \Big(l - xp - yq\Big)^2 + yzq^2 - (1-x)^2 m_e^2 + i\epsilon. \tag{40}$$

We will define

$$\Delta = -yzq^2 + (1-x)^2 m_e^2. \tag{41}$$

This is always positive since $q^2 < 0$ for scattering processes. We shift the variable $l$ as $l \longrightarrow L = l - xp - yq$. We get

$$D = L^2 - \Delta + i\epsilon. \tag{42}$$

Plugging this result into our original integral we get

$$\begin{aligned} \bar{u}^{s'}(p')\delta\Gamma^{\mu}(p',p)u^s(p) &= 4ie^2 \int_0^1 dxdydz\ \delta(x+y+z-1) \int \frac{d^4L}{(2\pi)^4} \frac{1}{(L^2-\Delta+i\epsilon)^3} \bar{u}^{s'}(p') \Big( (\gamma.l)\gamma^{\mu}(\gamma.l') \\ &+ m_e^2\gamma^{\mu} - 2m_e(l+l')^{\mu} \Big) u^s(p). \end{aligned} \tag{43}$$

In this equation $l = L + xp + yq$ and $l' = L + xp + (y-1)q$. By dropping odd terms in $L$ which must vanish by summetry we get

$$\begin{aligned} \bar{u}^{s'}(p')\delta\Gamma^{\mu}(p',p)u^s(p) &= 4ie^2 \int_0^1 dxdydz\ \delta(x+y+z-1) \int \frac{d^4L}{(2\pi)^4} \frac{1}{(L^2-\Delta+i\epsilon)^3} \bar{u}^{s'}(p') \Big( (\gamma.L)\gamma^{\mu}(\gamma.L) \\ &+ m_e^2\gamma^{\mu} + (x\gamma.p + y\gamma.q)\gamma^{\mu}(x\gamma.p + (y-1)\gamma.q) - 2m_e(2xp + (2y-1)q)^{\mu} \Big) u^s(p). \end{aligned} \tag{44}$$

Again by using symmetry considerations quadratic terms in $L$ must be given by

$$\int \frac{d^4L}{(2\pi)^4} \frac{L^{\mu}L^{\nu}}{(L^2-\Delta+i\epsilon)^3} = \int \frac{d^4L}{(2\pi)^4} \frac{\frac{1}{4}\eta^{\mu\nu}L^2}{(L^2-\Delta+i\epsilon)^3} \tag{45}$$

Thus

$$\begin{aligned} \bar{u}^{s'}(p')\delta\Gamma^{\mu}(p',p)u^s(p) &= 4ie^2 \int_0^1 dxdydz\ \delta(x+y+z-1) \int \frac{d^4L}{(2\pi)^4} \frac{1}{(L^2-\Delta+i\epsilon)^3} \bar{u}^{s'}(p') \Big( -\frac{1}{2}\gamma^{\mu}L^2 \\ &+ m_e^2\gamma^{\mu} + (x\gamma.p + y\gamma.q)\gamma^{\mu}(x\gamma.p + (y-1)\gamma.q) - 2m_e(2xp + (2y-1)q)^{\mu} \Big) u^s(p). \end{aligned} \tag{46}$$

By using $\gamma.pu^s(p) = m_e u^s(p)$, $\bar{u}^{s'}(p')\gamma.p' = m_e\bar{u}^{s'}(p')$ and $\gamma.p\gamma^{\mu} = 2p^{\mu} - \gamma^{\mu}\gamma.p$, $\gamma^{\mu}\gamma.p' = 2p'^{\mu} - \gamma.p'\gamma^{\mu}$ we can make the replacement

$$\begin{aligned} \bar{u}^{s'}(p')\Big[(x\gamma.p + y\gamma.q)\gamma^{\mu}(x\gamma.p + (y-1)\gamma.q)\Big]u^s(p) &\longrightarrow \bar{u}^{s'}(p')\Big[\Big((x+y)\gamma.p - ym_e\Big)\gamma^{\mu}\Big((x+y-1)m_e \\ &- (y-1)\gamma.p'\Big)\Big]u^s(p) \\ &\longrightarrow \bar{u}^{s'}(p')\Big[m_e(x+y)(x+y-1)(2p^{\mu} - m_e\gamma^{\mu}) \\ &- (x+y)(y-1)\Big(2m_e(p+p')^{\mu} + q^2\gamma^{\mu} - 3m_e^2\gamma^{\mu}\Big) \\ &- m_e^2 y(x+y-1)\gamma^{\mu} + m_e y(y-1)(2p'^{\mu} - m_e\gamma^{\mu})\Big] \\ &\times u^s(p). \end{aligned} \tag{47}$$

After some more algebra we obtain the result

$$\begin{aligned}\bar{u}^{s'}(p')\delta\Gamma^{\mu}(p',p)u^s(p) &= 4ie^2\int_0^1 dxdydz\ \delta(x+y+z-1)\int\frac{d^4L}{(2\pi)^4}\frac{1}{(L^2-\Delta+i\epsilon)^3}\bar{u}^{s'}(p')\Big[\gamma^{\mu}\Big(-\frac{1}{2}L^2\\ &+ (1-z)(1-y)q^2+(1-x^2-2x)m_e^2\Big)+m_e x(x-1)(p+p')^{\mu}\\ &+ m_e(x-2)(x+2y-1)m_e q^{\mu}\Big]u^s(p).\end{aligned} \tag{48}$$

The term proportional to $q^{\mu}=p^{\mu}-p^{'\mu}$ is zero because it is odd under the exchange $y\leftrightarrow z$ since $x+2y-1=y-z$. This is our first manifestation of the so-called Ward identity. In other words we have

$$\begin{aligned}\bar{u}^{s'}(p')\delta\Gamma^{\mu}(p',p)u^s(p) &= 4ie^2\int_0^1 dxdydz\ \delta(x+y+z-1)\int\frac{d^4L}{(2\pi)^4}\frac{1}{(L^2-\Delta+i\epsilon)^3}\bar{u}^{s'}(p')\Big[\gamma^{\mu}\Big(-\frac{1}{2}L^2\\ &+ (1-z)(1-y)q^2+(1-x^2-2x)m_e^2\Big)+m_e x(x-1)(p+p')^{\mu}\Big]u^s(p).\end{aligned} \tag{49}$$

Now we use the so-called Gordon's identity given by (with the spin matrices $\sigma^{\mu\nu}=2\Gamma^{\mu\nu}=i[\gamma^{\mu},\gamma^{\nu}]/2$)

$$\bar{u}^{s'}(p')\gamma^{\mu}u^s(p)=\frac{1}{2m_e}\bar{u}^{s'}(p')\Big[(p+p')^{\mu}-i\sigma^{\mu\nu}q_{\nu}\Big]u^s(p). \tag{50}$$

This means that we can make the replacement

$$\bar{u}^{s'}(p')(p+p')^{\mu}u^s(p)\longrightarrow\bar{u}^{s'}(p')\Big[2m_e\gamma^{\mu}+i\sigma^{\mu\nu}q_{\nu}\Big]u^s(p). \tag{51}$$

Hence we get

$$\begin{aligned}\bar{u}^{s'}(p')\delta\Gamma^{\mu}(p',p)u^s(p) &= 4ie^2\int_0^1 dxdydz\ \delta(x+y+z-1)\int\frac{d^4L}{(2\pi)^4}\frac{1}{(L^2-\Delta+i\epsilon)^3}\bar{u}^{s'}(p')\Big[\gamma^{\mu}\Big(-\frac{1}{2}L^2\\ &+ (1-z)(1-y)q^2+(1+x^2-4x)m_e^2\Big)+im_e x(x-1)\sigma^{\mu\nu}q_{\nu}\Big]u^s(p).\end{aligned} \tag{52}$$

**Wick Rotation:** The natural step at this stage is to actually do the 4−dimensional integral over $L$. Towards this end we will perform the so-called Wick rotation of the real integration variable $L^0$ to a pure imaginary variable $L^4=-iL^0$ which will allow us to convert the Minkowskian signature of the metric into an Euclidean signature. Indeed the Minkowski line element $dL^2=(dL^0)^2-(dL^i)^2$ becomes under Wick rotation the Euclid line element $dL^2=-(dL^4)^2-(dL^i)^2$. In a very profound sense the quantum field theory integral becomes under Wick rotation a statistical mechanics integral. This is of course possible because of the location of the poles $\sqrt{\vec{L}^2+\Delta}-i\epsilon'$ and $-\sqrt{\vec{L}^2+\Delta}+i\epsilon'$ of the $L^0$ integration and because the integral over $L^0$ goes to 0 rapidly enough for large positive $L^0$. Note that the prescription $L^4=-iL^0$ corresponds to a rotation by $\pi/2$ counterclockwise of the $L^0$ axis.

Let us now compute

$$\int \frac{d^4L}{(2\pi)^4}\frac{(L^2)^n}{(L^2-\Delta+i\epsilon)^m} = \frac{i}{(2\pi)^4}\frac{(-1)^n}{(-1)^m}\int d^4L_E\frac{(L_E^2)^n}{(L_E^2+\Delta)^m}. \tag{53}$$

In this equation $\vec{L}_E=(L^1,L^2,L^3,L^4)$. Since we are dealing with Euclidean coordinates in four dimensions we can go to spherical coordinates in four dimensions defined by (with $0\leq r\leq\infty$, $0\leq\theta\leq\pi$, $0\leq\phi\leq 2\pi$ and $0\leq\omega\leq\pi$)

$$\begin{aligned} L^1 &= r\sin\omega\sin\theta\cos\phi \\ L^2 &= r\sin\omega\sin\theta\sin\phi \\ L^3 &= r\sin\omega\cos\theta \\ L^4 &= r\cos\omega. \end{aligned} \tag{54}$$

We also know that

$$d^4L_E = r^3\sin^2\omega\sin\theta dr d\theta d\phi d\omega. \tag{55}$$

We calculate then

$$\begin{aligned} \int \frac{d^4L}{(2\pi)^4}\frac{(L^2)^n}{(L^2-\Delta+i\epsilon)^m} &= \frac{i}{(2\pi)^4}\frac{(-1)^n}{(-1)^m}\int\frac{r^{2n+3}dr}{(r^2+\Delta)^m}\int\sin^2\omega\sin\theta d\theta d\phi d\omega \\ &= \frac{2i\pi^2}{(2\pi)^4}\frac{(-1)^n}{(-1)^m}\int\frac{r^{2n+3}dr}{(r^2+\Delta)^m}. \end{aligned} \tag{56}$$

The case $n=0$ is easy. We have

$$\begin{aligned} \int \frac{d^4L}{(2\pi)^4}\frac{1}{(L^2-\Delta+i\epsilon)^m} &= \frac{2i\pi^2}{(2\pi)^4}\frac{1}{(-1)^m}\int\frac{r^3dr}{(r^2+\Delta)^m} \\ &= \frac{i\pi^2}{(2\pi)^4}\frac{1}{(-1)^m}\int_\Delta^\infty\frac{(x-\Delta)dx}{x^m} \\ &= \frac{i}{(4\pi)^2}\frac{(-1)^m}{(m-2)(m-1)}\frac{1}{\Delta^{m-2}}. \end{aligned} \tag{57}$$

The case $n=1$ turns out to be divergent

$$\begin{aligned} \int \frac{d^4L}{(2\pi)^4}\frac{L^2}{(L^2-\Delta+i\epsilon)^m} &= \frac{2i\pi^2}{(2\pi)^4}\frac{-1}{(-1)^m}\int\frac{r^5dr}{(r^2+\Delta)^m} \\ &= \frac{i\pi^2}{(2\pi)^4}\frac{-1}{(-1)^m}\int_\Delta^\infty\frac{(x-\Delta)^2dx}{x^m} \\ &= \frac{i\pi^2}{(2\pi)^4}\frac{-1}{(-1)^m}\Big(\frac{x^{3-m}}{3-m}-2\Delta\frac{x^{2-m}}{2-m}+\Delta^2\frac{x^{1-m}}{1-m}\Big)_\Delta^\infty \\ &= \frac{i}{(4\pi)^2}\frac{(-1)^{m+1}}{(m-3)(m-2)(m-1)}\frac{2}{\Delta^{m-3}}. \end{aligned} \tag{58}$$

This does not make sense for $m=3$ which is the case of interest.

### 1.3.2 Pauli-Villars Regularization

We will now show that this divergence is ultraviolet in the sense that it is coming from integrating arbitrarily high momenta in the loop integral. We will also show the existence of an infrared divergence coming from integrating arbitrarily small momenta in the loop integral. In order to control these infinities we need to regularize the loop integral in one way or another. We adopt here the so-called Pauli-Villars regularization. This is given by making the following replacement

$$\frac{1}{(l-p)^2+i\epsilon}\longrightarrow\frac{1}{(l-p)^2-\mu^2+i\epsilon}-\frac{1}{(l-p)^2-\Lambda+i\epsilon}. \tag{59}$$

The infrared cutoff $\mu$ will be taken to zero at the end and thus it should be thought of as a small mass for the physical photon. The ultraviolet cutoff $\Lambda$ will be taken to $\infty$ at the end. The UV cutoff $\Lambda$ does also look like a a very large mass for a fictitious photon which becomes infinitely heavy and thus unobservable in the limit $\Lambda\longrightarrow\infty$.

Now it is not difficult to see that

$$\frac{1}{((l-p)^2-\mu^2+i\epsilon)(l'^2-m_e^2+i\epsilon)(l^2-m_e^2+i\epsilon)}=2\int_0^1 dxdydz\;\delta(x+y+z-1)\frac{1}{D_{\mu}^3}. \tag{60}$$

$$D_{\mu}=D-\mu^2x=L^2-\Delta_{\mu}+i\epsilon\;,\;\Delta_{\mu}=\Delta+\mu^2x. \tag{61}$$

$$\frac{1}{((l-p)^2-\Lambda^2+i\epsilon)(l'^2-m_e^2+i\epsilon)(l^2-m_e^2+i\epsilon)}=2\int_0^1 dxdydz\;\delta(x+y+z-1)\frac{1}{D_{\Lambda}^3}. \tag{62}$$

$$D_{\Lambda}=D-\Lambda^2x=L^2-\Delta_{\Lambda}+i\epsilon\;,\;\Delta_{\Lambda}=\Delta+\Lambda^2x. \tag{63}$$

The result (52) becomes

$$\begin{aligned}\bar{u}^{s'}(p')\delta\Gamma^{\mu}(p',p)u^s(p) &= 4ie^2\int_0^1 dxdydz\;\delta(x+y+z-1)\int\frac{d^4L}{(2\pi)^4}\Big[\frac{1}{(L^2-\Delta_{\mu}+i\epsilon)^3}-\frac{1}{(L^2-\Delta_{\Lambda}+i\epsilon)^3}\Big]\\ &\times\;\bar{u}^{s'}(p')\Big[\gamma^{\mu}\Big(-\frac{1}{2}L^2+(1-z)(1-y)q^2+(1+x^2-4x)m_e^2\Big)+im_ex(x-1)\sigma^{\mu\nu}q_{\nu}\Big]\\ &\times\;u^s(p).\end{aligned} \tag{64}$$

We compute now (after Wick rotation)

$$\begin{aligned}\int\frac{d^4L}{(2\pi)^4}\Big[\frac{L^2}{(L^2-\Delta_{\mu}+i\epsilon)^3}-\frac{L^2}{(L^2-\Delta_{\Lambda}+i\epsilon)^3}\Big] &= \frac{2i}{(4\pi)^2}\Big[\int\frac{r^5dr}{(r^2+\Delta_{\mu})^3}-\int\frac{r^5dr}{(r^2+\Delta_{\Lambda})^3}\Big]\\ &= \frac{i}{(4\pi)^2}\Big[\int_{\Delta_{\mu}}^{\infty}\frac{(x-\Delta_{\mu})^2dx}{x^3}-\int_{\Delta_{\Lambda}}^{\infty}\frac{(x-\Delta_{\Lambda})^2dx}{x^3}\Big]\\ &= \frac{i}{(4\pi)^2}\ln\frac{\Delta_{\Lambda}}{\Delta_{\mu}}.\end{aligned} \tag{65}$$

Clearly in the limit $\Lambda \longrightarrow \infty$ this goes as $\ln \Lambda^2$. This shows explicitly that the divergence problem seen earlier is a UV one,i.e. coming from high momenta. Also we compute

$$\int \frac{d^4L}{(2\pi)^4}\Big[\frac{1}{(L^2-\Delta_\mu+i\epsilon)^3}-\frac{1}{(L^2-\Delta_\Lambda+i\epsilon)^3}\Big] = -\frac{2i}{(4\pi)^2}\Big[\int\frac{r^3dr}{(r^2+\Delta_\mu)^3}-\int\frac{r^3dr}{(r^2+\Delta_\Lambda)^3}\Big]$$
$$= -\frac{i}{(4\pi)^2}\Big[\int_{\Delta_\mu}^\infty\frac{(x-\Delta_\mu)dx}{x^3}-\int_{\Delta_\Lambda}^\infty\frac{(x-\Delta_\Lambda)dx}{x^3}\Big]$$
$$= -\frac{i}{2(4\pi)^2}\Big(\frac{1}{\Delta_\mu}-\frac{1}{\Delta_\Lambda}\Big). \quad (66)$$

The second term vanishes in the limit $\Lambda \longrightarrow \infty$. We get then the result

$$\bar{u}^{s'}(p')\delta\Gamma^\mu(p',p)u^s(p) = (4ie^2)(-\frac{i}{2(4\pi)^2})\int_0^1 dxdydz\ \delta(x+y+z-1)\bar{u}^{s'}(p')\Big[\gamma^\mu\Big(\ln\frac{\Delta_\Lambda}{\Delta_\mu}$$
$$+ \frac{(1-z)(1-y)q^2+(1+x^2-4x)m_e^2}{\Delta_\mu}\Big)+\frac{i}{\Delta_\mu}m_e x(x-1)\sigma^{\mu\nu}q_\nu\Big]u^s(p)$$
$$= \bar{u}^{s'}(p')\Big(\gamma^\mu(F_1(q^2)-1)-\frac{i\sigma^{\mu\nu}q_\nu}{2m_e}F_2(q^2)\Big)u^s(p). \quad (67)$$

$$F_1(q^2) = 1+\frac{\alpha}{2\pi}\int_0^1 dxdydz\ \delta(x+y+z-1)\Big(\ln\frac{\Lambda^2x}{\Delta_\mu}+\frac{(1-z)(1-y)q^2+(1+x^2-4x)m_e^2}{\Delta_\mu}\Big). \quad (68)$$

$$F_2(q^2) = \frac{\alpha}{2\pi}\int_0^1 dxdydz\ \delta(x+y+z-1)\frac{2m_e^2x(1-x)}{\Delta_\mu}. \quad (69)$$

The functions $F_1(q^2)$ and $F_2(q^2)$ are known as the form factors of the electron. The form factor $F_1(q^2)$ is logarithmically UV divergent and requires a redefinition which is termed a renormalization. This will be done in the next section. This form factor is also IR divergent. To see this recall that $\Delta_\mu = -yzq^2+(1-x)^2m_e^2+\mu^2x$. Now set $q^2=0$ and $\mu^2=0$. The term proportional to $1/\Delta_\mu$ is

$$F_1(0) = ...+\frac{\alpha}{2\pi}\int_0^1 dx\int_0^1 dy\int_0^1 dz\ \delta(x+y+z-1)\frac{1+x^2-4x}{(1-x)^2}$$
$$= ...+\frac{\alpha}{2\pi}\int_0^1 dx\int_0^1 dy\int_0^{1-y} dt\ \delta(x-t)\frac{1+x^2-4x}{(1-x)^2}$$
$$= ...+\frac{\alpha}{2\pi}\int_0^1 dy\int_0^{1-y} dt\ \frac{1+t^2-4t}{(1-t)^2}$$
$$= ...-\frac{\alpha}{2\pi}\int_0^1 dy\int_1^y dt\ (1+\frac{2}{t}-\frac{2}{t^2})$$
$$= ...-\frac{\alpha}{2\pi}\int_0^1 dy\ (y+2\ln y+\frac{2}{y}-3). \quad (70)$$

As it turns out this infrared divergence will cancel exactly the infrared divergence coming from bremsstrahlung diagrams. Bremsstrahlung is scattering with radiation, i.e. scattering with emission of very low energy photons which can not be detected.

### 1.3.3 Renormalization (Minimal Subtraction) and Anomalous Magnetic Moment

**Electric Charge and Magnetic Moment of the Electron:** The form factors $F_1(q^2)$ and $F_2(q^2)$ define the charge and the magnetic moment of the electron. To see this we go to the problem of scattering of electrons from an external electromagnetic field. The probability amplitude is given by equation (24) with $q = p^{'} - p$. Thus

$$\begin{aligned} <\vec{p}^{'}s^{'}\ \text{out}|\vec{p}s\ \text{in}> &= -ie\bar{u}^{s^{'}}(p^{'})\Gamma_\lambda(p^{'},p)u^s(p).A^{\lambda,\text{backgr}}(q) \\ &= -ie\bar{u}^{s^{'}}(p^{'})\Big[\gamma_\lambda F_1(q^2) + \frac{i\sigma_{\lambda\gamma}q^\gamma}{2m_e}F_2(q^2)\Big]u^s(p).A^{\lambda,\text{backgr}}(q). \end{aligned} \quad (71)$$

Firstly we will consider an electrostatic potential $\phi(\vec{x})$, viz $A^{\lambda,\text{backgr}}(q) = (2\pi\delta(q^0)\phi(\vec{q}), 0)$. We have then

$$<\vec{p}^{'}s^{'}\ \text{out}|\vec{p}s\ \text{in}> = -ieu^{s^{'}+}(p^{'})\Big[F_1(-\vec{q}^2) + \frac{F_2(-\vec{q}^2)}{2m_e}\gamma^i q^i\Big]u^s(p).2\pi\delta(q^0)\phi(\vec{q}). \quad (72)$$

We will assume that the electrostatic potential $\phi(\vec{x})$ is slowly varying over a large region so that $\phi(\vec{q})$ is concentrated around $\vec{q} = 0$. In other words the momentum $\vec{q}$ can be treated as small and as a consequence the momenta $\vec{p}$ and $\vec{p}^{'}$ are also small.

In the nonrelativistic limit the spinor $u^s(p)$ behaves as (recall that $\sigma_\mu p^\mu = E - \vec{\sigma}\vec{p}$ and $\bar{\sigma}_\mu p^\mu = E + \vec{\sigma}\vec{p}$)

$$u^s(p) = \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\xi^s \\ \sqrt{\bar{\sigma}_\mu p^\mu}\xi^s \end{pmatrix} = \sqrt{m_e}\begin{pmatrix} (1 - \frac{\vec{\sigma}\vec{p}}{2m_e} + O(\frac{\vec{p}^2}{m_e^2}))\xi^s \\ (1 + \frac{\vec{\sigma}\vec{p}}{2m_e} + O(\frac{\vec{p}^2}{m_e^2}))\xi^s \end{pmatrix}. \quad (73)$$

We remark that the nonrelativistic limit is equivalent to the limit of small momenta. Thus by dropping all terms which are at least linear in the momenta we get

$$\begin{aligned} <\vec{p}^{'}s^{'}\ \text{out}|\vec{p}s\ \text{in}> &= -ieu^{s^{'}+}(p^{'})F_1(0)u^s(p).2\pi\delta(q^0)\phi(\vec{q}) \\ &= -ieF_1(0).2m_e\xi^{s^{'}+}\xi^s.2\pi\delta(q^0)\phi(\vec{q}) \\ &= -ieF_1(0)\phi(\vec{q}).2m_e\delta^{s^{'}s}.2\pi\delta(q^0). \end{aligned} \quad (74)$$

The corresponding $T-$matrix element is thus

$$<\vec{p}^{'}s^{'}\ \text{in}|iT|\vec{p}s\ \text{in}> = -ieF_1(0)\phi(\vec{q}).2m_e\delta^{s^{'}s}. \quad (75)$$

This should be compared with the Born approximation of the probability amplitude of scattering from a potential $V(\vec{x})$ (with $V(\vec{q}) = \int d^3x V(\vec{x})e^{-i\vec{q}\vec{x}}$)

$$<\vec{p}^{'}\ \text{in}|iT|\vec{p}\ \text{in}> = iV(\vec{q}). \quad (76)$$

The factor $2m_e$ should not bother us because it is only due to our normalization of spinors and so it should be omitted in the comparison. The Kronecker's delta $\delta^{s^{'}s}$ coincides with the prediction of nonrelativistic quantum mechanics. Thus the problem is equivalent to scattering from the potential

$$V(\vec{x}) = -eF_1(0)\phi(\vec{x}). \tag{77}$$

The charge of the electron in units of $-e$ is precisely $F_1(0)$.

Next we will consider a vector potential $\vec{A}(\vec{x})$, viz $A^{\lambda,\mathrm{backgr}}(q) = (0, 2\pi\delta(q^0)\vec{A}(\vec{q}))$. We have

$$< \vec{p}^{'} s^{'} \text{ in}|iT|\vec{p}s \text{ in}> \;=\; -ie\bar{u}^{s^{'}}(p^{'})\Big[\gamma_i F_1(-\vec{q}^2) + \frac{i\sigma_{ij}q^j}{2m_e}F_2(-\vec{q}^2)\Big]u^s(p).A^{i,\mathrm{backgr}}(\vec{q}). \tag{78}$$

We will keep up to the linear term in the momenta. Thus

$$< \vec{p}^{'} s^{'} \text{ in}|iT|\vec{p}s \text{ in}> \;=\; -ieu^{s^{'}+}(p^{'})\gamma^0\Big[\gamma_i F_1(0) - \frac{[\gamma_i,\gamma_j]q^j}{4m_e}F_2(0)\Big]u^s(p).A^{i,\mathrm{backgr}}(\vec{q}). \tag{79}$$

We compute

$$\begin{aligned} u^{s^{'}+}(p^{'})\gamma^0\gamma_i u^s(p) &= m_e\xi^{s^{'}+}\Big((1-\frac{\vec{\sigma}\vec{p}^{'}}{2m_e})\sigma^i(1-\frac{\vec{\sigma}\vec{p}}{2m_e}) - (1+\frac{\vec{\sigma}\vec{p}^{'}}{2m_e})\sigma^i(1+\frac{\vec{\sigma}\vec{p}}{2m_e})\Big)\xi^s \\ &= \xi^{s^{'}+}\Big(-(p+p^{'})^i + i\epsilon^{ijk}q^j\sigma^k\Big)\xi^s. \end{aligned} \tag{80}$$

$$u^{s^{'}+}(p^{'})\gamma^0[\gamma_i,\gamma_j]q^j u^s(p) \;=\; 2m_e\xi^{s^{'}+}\Big(-2i\epsilon^{ijk}q^j\sigma^k\Big)\xi^s. \tag{81}$$

We get then

$$\begin{aligned} < \vec{p}^{'} s^{'} \text{ in}|iT|\vec{p}s \text{ in}> &= -ie\xi^{s^{'}+}\Big[-(p^i+p^{'i})F_1(0)\Big]\xi^s.A^{i,\mathrm{backgr}}(\vec{q}) \\ &- ie\xi^{s^{'}+}\Big[i\epsilon^{ijk}q^j\sigma^k(F_1(0)+F_2(0))\Big]\xi^s.A^{i,\mathrm{backgr}}(\vec{q}). \end{aligned} \tag{82}$$

The first term corresponds to the interaction term $\vec{\hat{p}}\vec{\hat{A}} + \vec{\hat{A}}\vec{\hat{p}}$ in the Schrödinger equation. The second term is the magnetic moment interaction. Thus

$$\begin{aligned} < \vec{p}^{'} s^{'} \text{ in}|iT|\vec{p}s \text{ in}>_{\text{magn moment}} &= -ie\xi^{s^{'}+}\Big[i\epsilon^{ijk}q^j\sigma^k(F_1(0)+F_2(0))\Big]\xi^s.A^{i,\mathrm{backgr}}(\vec{q}) \\ &= -ie\xi^{s^{'}+}\Big[\sigma^k(F_1(0)+F_2(0))\Big]\xi^s.B^{k,\mathrm{backgr}}(\vec{q}) \\ &= -i < \mu^k > .B^{k,\mathrm{backgr}}(\vec{q}).2m_e \\ &= iV(\vec{q}).2m_e. \end{aligned} \tag{83}$$

The magnetic field is defined by $\vec{B}^{\mathrm{backgr}}(\vec{x}) = \vec{\nabla}\times\vec{A}^{\mathrm{backgr}}(\vec{x})$ and thus $B^k(\vec{q}) = i\epsilon^{ijk}q^jA^{i,\mathrm{backgr}}(\vec{q})$. The magnetic moment is defined by

$$< \mu^k >= \frac{e}{m_e}\xi^{s^{'}+}\Big[\frac{\sigma^k}{2}(F_1(0)+F_2(0))\Big]\xi^s \Leftrightarrow \mu^k = g\frac{e}{2m_e}\frac{\sigma^k}{2}. \tag{84}$$

The gyromagnetic ratio (Landé g-factor) is then given by

$$g = 2(F_1(0)+F_2(0)). \tag{85}$$

**Renormalization:** We have found that the charge of the electron is $-eF_1(0)$ and not $-e$. This is a tree level result. Thus one must have $F_1(0)=1$. Substituting $q^2=0$ in (68) we get

$$F_1(0) = 1+\frac{\alpha}{2\pi}\int_0^1 dxdydz\ \delta(x+y+z-1)\Big(\ln\frac{\Lambda^2 x}{\Delta_\mu(0)}+\frac{(1+x^2-4x)m_e^2}{\Delta_\mu(0)}\Big). \tag{86}$$

This is clearly not equal 1. In fact $F_1(0)\longrightarrow\infty$ logarithmically when $\Lambda\longrightarrow\infty$. We need to redefine (renormalize) the value of $F_1(q^2)$ in such a way that $F_1(0)=1$. We adopt here a prescription termed minimal subtraction which consists in subtracting from $\delta F_1(q^2)=F_1(q^2)-1$ (which is the actual one-loop correction to the vertex) the divergence $\delta F_1(0)$. We define

$$\begin{aligned}F_1^{\rm ren}(q^2) &= F_1(q^2)-\delta F_1(0)\\ &= 1+\frac{\alpha}{2\pi}\int_0^1 dxdydz\ \delta(x+y+z-1)\Big(\ln\frac{\Delta_\mu(0)}{\Delta_\mu(q^2)}+\frac{(1-z)(1-y)q^2}{\Delta_\mu(q^2)}+\frac{(1+x^2-4x)m_e^2}{\Delta_\mu(q^2)}\\ &- \frac{(1+x^2-4x)m_e^2}{\Delta_\mu(0)}\Big).\end{aligned} \tag{87}$$

This formula satisfies automatically $F_1^{\rm ren}(0)=1$.

The form factor $F_2(0)$ is UV finite since it does not depend on $\Lambda$. It is also as point out earlier IR finite and thus one can simply set $\mu=0$ in this function. The magnetic moment of the electron is proportional to the gyromagnetic ratio $g=2F_1(0)+2F_2(0)$. Since $F_1(0)$ was renormalized to $F_1^{\rm ren}(0)$ the renormalized magnetic moment of the electron will be proportional to the gyromagnetic ratio

$$\begin{aligned}g^{\rm ren} &= 2F_1^{\rm ren}(0)+2F_2(0)\\ &= 2+2F_2(0).\end{aligned} \tag{88}$$

The first term is precisely the prediction of the Dirac theory (tree level). The second term which is due to the quantum one-loop effect will lead to the so-called anomalous magnetic moment. This is given by

$$\begin{aligned}F_2(0) &= \frac{\alpha}{\pi}\int_0^1 dx\int_0^1 dy\int_0^1 dz\ \delta(x+y+z-1)\frac{x}{1-x}\\ &= \frac{\alpha}{\pi}\int_0^1 dx\int_0^1 dy\int_{-y}^{1-y} dt\ \delta(x-t)\frac{x}{1-x}\\ &= \frac{\alpha}{\pi}\int_0^1 dx\int_0^1 dy\int_0^{1-y} dt\ \delta(x-t)\frac{x}{1-x}\\ &= \frac{\alpha}{\pi}\int_0^1 dy\int_0^{1-y} dt\ \frac{t}{1-t}\\ &= \frac{\alpha}{\pi}\int_0^1 dy(y-1-\ln y)\\ &= \frac{\alpha}{\pi}\Big(\frac{1}{2}(y-1)^2+y-y\ln y\Big)_0^1\\ &= \frac{\alpha}{2\pi}.\end{aligned} \tag{89}$$

## 1.4 Exact Fermion 2−Point Function

For simplicity we will consider in this section a scalar field theory and then we will generalize to a spinor field theory. As we have already seen the free 2−point function $<0|T(\hat{\phi}_{\rm in}(x)\hat{\phi}_{\rm in}(y))|0>$ is the probability amplitude for a free scalar particle to propagate from a spacetime point $y$ to a spacetime $x$. In the interacting theory the 2−point function is $<\Omega|T(\hat{\phi}(x)\hat{\phi}(y))|\Omega>$ where $|\Omega>=|0>/\sqrt{<0|0>}$ is the ground state of the full Hamiltonian $\hat{H}$.

The full Hamiltonian $\hat{H}$ commutes with the full momentum operator $\vec{\hat{P}}$. Let $|\lambda_0>$ be an eigenstate of $\hat{H}$ with momentum $\vec{0}$. There could be many such states corresponding to one-particle states with mass $m_r$ and 2−particle and multiparticle states which have a continuous mass spectrum starting at $2m_r$. By Lorentz invariance a generic state of $\hat{H}$ with a momentum $\vec{p}\neq 0$ can be obtained from one of the $|\lambda_0>$ by the application of a boost. Generic eigenstates of $\hat{H}$ are denoted $|\lambda_p>$ and they have energy $E_p(\lambda)=\sqrt{\vec{p}^2+m_\lambda^2}$ where $m_\lambda$ is the energy of the corresponding $|\lambda_0>$. We have the completeness relation in the full Hilbert space

$$\mathbf{1}=|\Omega><\Omega|+\sum_\lambda\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p(\lambda)}|\lambda_p><\lambda_p|. \tag{90}$$

The sum over $\lambda$ runs over all the 0−momentum eigenstates $|\lambda_0>$. Compare this with the completeness relation of the free one-particle states given by

$$\mathbf{1}=\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}|\vec{p}><\vec{p}|\ ,\ E_p=\sqrt{\vec{p}^2+m^2}. \tag{91}$$

By inserting the completeness relation in the full Hilbert space, the full 2−point function becomes (for $x^0>y^0$)

$$\begin{aligned}<\Omega|T(\hat{\phi}(x)\hat{\phi}(y))|\Omega> &= <\Omega|\hat{\phi}(x)|\Omega><\Omega|\hat{\phi}(y)|\Omega>\\ &+ \sum_\lambda\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p(\lambda)}<\Omega|\hat{\phi}(x)|\lambda_p><\lambda_p|\hat{\phi}(y)|\Omega>.\end{aligned} \tag{92}$$

The first term vanishes by symmetry (scalar field) or by Lorentz invariance (spinor and gauge fields). By translation invariance $\hat{\phi}(x)=\exp(iPx)\hat{\phi}(0)\exp(-iPx)$. Furthermore $|\lambda_P>=U|\lambda_0>$ where $U$ is the unitary transformation which implements the Lorentz boost which takes the momentum $\vec{0}$ to the momentum $\vec{p}$. Also we recall that the field operator $\hat{\phi}(0)$ and the ground state $|\Omega>$ are both Lorentz invariant. By using all these facts we can verify that $<\Omega|\hat{\phi}(x)|\lambda_p>=e^{-ipx}<\Omega|\hat{\phi}(0)|\lambda_0>$. We get then

$$<\Omega|T(\hat{\phi}(x)\hat{\phi}(y))|\Omega>=\sum_\lambda\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p(\lambda)}\ e^{-ip(x-y)}|<\Omega|\hat{\phi}(0)|\lambda_0>|^2. \tag{93}$$

In this expression $p^0=E_p(\lambda)$. We use the identity (the contour is closed below since $x^0>y^0$)

$$\int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m_\lambda^2+i\epsilon}\ e^{-ip(x-y)}=\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p(\lambda)}\ e^{-ip(x-y)}. \tag{94}$$

Hence we get

$$\begin{aligned} <\Omega|T(\hat{\phi}(x)\hat{\phi}(y))|\Omega> &= \sum_\lambda \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m_\lambda^2+i\epsilon}\; e^{-ip(x-y)}|<\Omega|\hat{\phi}(0)|\lambda_0>|^2 \\ &= \sum_\lambda D_F(x-y;m_\lambda)|<\Omega|\hat{\phi}(0)|\lambda_0>|^2. \end{aligned} \quad (95)$$

We get the same result for $x^0 < y^0$. We put this result into the suggestive form

$$<\Omega|T(\hat{\phi}(x)\hat{\phi}(y))|\Omega>=\int_0^\infty \frac{dM^2}{2\pi}D_F(x-y;M)\rho(M^2). \quad (96)$$

$$\rho(M^2)=\sum_\lambda(2\pi)\delta(M^2-m_\lambda^2)|<\Omega|\hat{\phi}(0)|\lambda_0>|^2. \quad (97)$$

The distribution $\rho(M^2)$ is called Källén-Lehmann spectral density. The one-particle states will contribute to the spectral density only a delta function corresponding to the pole at the exact or physical mass $m_r$ of the scalar $\phi$ particle, viz

$$\rho(M^2)=(2\pi)\delta(M^2-m_r^2)Z+.... \quad (98)$$

We note that the mass $m$ appearing in the Lagrangian (the bare mass) is generally different from the physical mass. The coefficient $Z$ is the so-called field-strength or wave function renormalization and it is equal to the corresponding probability $|<\Omega|\hat{\phi}(0)|\lambda_0>|^2$. We have then

$$<\Omega|T(\hat{\phi}(x)\hat{\phi}(y))|\Omega>=ZD_F(x-y;m_r)+\int_{4m_r^2}^\infty \frac{dM^2}{2\pi}D_F(x-y;M)\rho(M^2). \quad (99)$$

The lower bound $4m_r^2$ comes from the fact that there will be essentially nothing else between the one-particle states at the simple pole $p^2=m_r^2$ and the $2-$particle and multiparticle continuum states starting at $p^2=4m_r^2$ which correspond to a branch cut. Indeed by taking the Fourier transform of the above equation we get

$$\int d^4x e^{ip(x-y)}\;\; <\Omega|T(\hat{\phi}(x)\hat{\phi}(y))|\Omega>=\frac{iZ}{p^2-m_r^2+i\epsilon}+\int_{4m_r^2}^\infty \frac{dM^2}{2\pi}\frac{i}{p^2-M^2+i\epsilon}\rho(M^2). \quad (100)$$

For a spinor field the same result holds. The Fourier transform of the full $2-$point function $<\Omega|T(\hat{\psi}(x)\bar{\hat{\psi}}(y))|\Omega>$ is precisely given by the free Dirac propagator in momentum space with the physical mass $m_r$ instead of the bare mass $m$ times a field-strength normalization $Z_2$. In other words

$$\int d^4x e^{ip(x-y)}\;\; <\Omega|T(\hat{\psi}(x)\bar{\hat{\psi}}(y))|\Omega>=\frac{iZ_2(\gamma.p+m_r)}{p^2-m_r^2+i\epsilon}+... \quad (101)$$

## 1.5 One-loop Calculation II: Electron Self-Energy

### 1.5.1 Electron Mass at One-Loop

From our discussion of the processes $e^-+e^+\longrightarrow\mu^-+\mu^+$, $e^-+\mu^-\longrightarrow e^-+\mu^-$ and electron scattering from an external electromagnetic field we know that there are radiative corrections to the probability amplitudes which involve correction to the external legs. From the corresponding Feynman diagrams we can immediately infer that the first two terms (tree level+one-loop) in the perturbative expansion of the fermion 2−point function $\int d^4x e^{ip(x-y)} <\Omega|T(\hat{\psi}(x)\bar{\hat{\psi}}(y))|\Omega>$ is given by the two diagrams 2POINTFER. By using Feynamn rules we find the expression

$$\begin{aligned}\int d^4x e^{ip(x-y)} <\Omega|T(\hat{\psi}(x)\bar{\hat{\psi}}(y))|\Omega> &= \frac{i(\gamma.p+m_e)}{p^2-m_e^2+i\epsilon}+\frac{i(\gamma.p+m_e)}{p^2-m_e^2+i\epsilon}(-ie\gamma^\mu)\\ &\times\int\frac{d^4k}{(2\pi)^4}\frac{i(\gamma.k+m_e)}{k^2-m_e^2+i\epsilon}\frac{-i\eta_{\mu\nu}}{(p-k)^2+i\epsilon}(-ie\gamma^\nu)\frac{i(\gamma.p+m_e)}{p^2-m_e^2+i\epsilon}\\ &= \frac{i(\gamma.p+m_e)}{p^2-m_e^2+i\epsilon}+\frac{i(\gamma.p+m_e)}{p^2-m_e^2+i\epsilon}(-i\Sigma_2(p))\frac{i(\gamma.p+m_e)}{p^2-m_e^2+i\epsilon}.\end{aligned} \tag{102}$$

The second term is the so-called self-energy of the electron. It is given in terms of the loop integral $\Sigma_2(p)$ which in turn is given by

$$-i\Sigma_2(p) = (-ie)^2\int\frac{d^4k}{(2\pi)^4}\gamma^\mu\frac{i(\gamma.k+m_e)}{k^2-m_e^2+i\epsilon}\gamma_\mu\frac{-i}{(p-k)^2+i\epsilon}. \tag{103}$$

Sometimes we will also call this quantity the electron self-energy. The two-point function $\int d^4x e^{ip(x-y)} <\Omega|T(\hat{\psi}(x)\bar{\hat{\psi}}(y))|\Omega>$ is not of the form (101). To see this more clearly we rewrite the above equation in the form

$$\begin{aligned}\int d^4x e^{ip(x-y)} <\Omega|T(\hat{\psi}(x)\bar{\hat{\psi}}(y))|\Omega> &= \frac{i}{\gamma.p-m_e}+\frac{i}{\gamma.p-m_e}(-i\Sigma_2(p))\frac{i}{\gamma.p-m_e}\\ &= \frac{i}{\gamma.p-m_e}\Big[1+\Sigma_2(p)\frac{1}{\gamma.p-m_e}\Big].\end{aligned} \tag{104}$$

By using now the fact that $\Sigma_2(p)$ commutes with $\gamma.p$ (see below) and the fact that it is supposed to be small of order $e^2$ we rewrite this equation in the form

$$\int d^4x e^{ip(x-y)} <\Omega|T(\hat{\psi}(x)\bar{\hat{\psi}}(y))|\Omega> = \frac{i}{\gamma.p-m_e-\Sigma_2(p)}. \tag{105}$$

This is almost of the desired form (101). The loop-integral $\Sigma_2(p)$ is precisely the one-loop correction to the electron mass.

Physically what we have done here is to add together all the Feynman diagrams with an arbitrary number of insertions of the loop integral $\Sigma_2(p)$. These are given by the Feynman diagrams SELF. By using Feynamn rules we find the expression

$$\begin{aligned}
\int d^4x e^{ip(x-y)} \;<\Omega|T(\hat{\psi}(x)\bar{\hat{\psi}}(y))|\Omega> &= \frac{i}{\gamma.p-m_e}+\frac{i}{\gamma.p-m_e}(-i\Sigma_2(p))\frac{i}{\gamma.p-m_e}\\
&+ \frac{i}{\gamma.p-m_e}(-i\Sigma_2(p))\frac{i}{\gamma.p-m_e}(-i\Sigma_2(p))\frac{i}{\gamma.p-m_e}\\
&+ ...\\
&= \frac{i}{\gamma.p-m_e}\Big[1+\Sigma_2(p)\frac{1}{\gamma.p-m_e}+(\Sigma_2(p)\frac{1}{\gamma.p-m_e})^2+...\Big].
\end{aligned} \tag{106}$$

This is a geometric series. The summation of this geometric series is precisely (105).

The loop integral $-i\Sigma_2(p)$ is an example of a one-particle irreducible (1PI) diagram. The one-particle irreducible diagrams are those diagrams which can not be split in two by cutting a single internal line. The loop integral $-i\Sigma_2(p)$ is the first 1PI diagram (order $e^2$) in the sum $-i\Sigma(p)$ of all 1PI diagrams with 2 fermion lines shown on ONEPARTICLE. Thus the full two-point function $\int d^4x e^{ip(x-y)} \;<\Omega|T(\hat{\psi}(x)\bar{\hat{\psi}}(y))|\Omega>$ is actually of the form

$$\begin{aligned}
\int d^4x e^{ip(x-y)} \;<\Omega|T(\hat{\psi}(x)\bar{\hat{\psi}}(y))|\Omega> &= \frac{i}{\gamma.p-m_e}+\frac{i}{\gamma.p-m_e}(-i\Sigma(p))\frac{i}{\gamma.p-m_e}\\
&+ \frac{i}{\gamma.p-m_e}(-i\Sigma(p))\frac{i}{\gamma.p-m_e}(-i\Sigma(p))\frac{i}{\gamma.p-m_e}\\
&+ ...\\
&= \frac{i}{\gamma.p-m_e-\Sigma(p)}.
\end{aligned} \tag{107}$$

The physical or renormalized mass $m_r$ is defined as the pole of the two-point function $\int d^4x e^{ip(x-y)} \;<\Omega|T(\hat{\psi}(x)\bar{\hat{\psi}}(y))|\Omega>$, viz

$$(\gamma.p-m_e-\Sigma(p))_{\gamma.p=m_r}=0. \tag{108}$$

Since $\Sigma(p)=\Sigma(\gamma.p)$ (see below) we have

$$m_r-m_e-\Sigma(m_r)=0. \tag{109}$$

We expand $\Sigma(p)=\Sigma(\gamma.p)$ as

$$\Sigma(p)=\Sigma(m_r)+(\gamma.p-m_r)\frac{d\Sigma}{d\gamma.p}|_{\gamma.p=m_r}+O((\gamma.p-m_r)^2). \tag{110}$$

Hence

$$\begin{aligned}
\gamma.p-m_e-\Sigma(p) &= (\gamma.p-m_r)\frac{1}{Z_2}-O((\gamma.p-m_r)^2)\\
&= (\gamma.p-m_r)\frac{1}{Z_2}(1+O^{'}((\gamma.p-m_r))).
\end{aligned} \tag{111}$$

$$Z_2^{-1} = 1 - \frac{d\Sigma}{d\gamma.p}|_{\gamma.p=m_r}. \tag{112}$$

Thus

$$\int d^4x e^{ip(x-y)} \ < \Omega|T(\hat{\psi}(x)\bar{\hat{\psi}}(y))|\Omega > \ = \ \frac{iZ_2}{\gamma.p - m_r}. \tag{113}$$

This is the desired form (101). The correction to the mass is given by (109) or equivalently

$$\delta m_r = m_r - m_e = \Sigma(m_r). \tag{114}$$

We are interested in just the one-loop correction. Thus

$$\delta m_r = m_r - m_e = \Sigma_2(m_r). \tag{115}$$

We evaluate the loop integral $\Sigma_2(p)$ by the same method used for the vertex correction, i.e. we introduce Feynman parameters, we Wick rotate and then we regularize the ultraviolet divergence using the Pauli-Villars method. Clearly the integral is infrared divergent so we will also add a small photon mass. In summary we would like to compute

$$-i\Sigma_2(p) \ = \ (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu \frac{i(\gamma.k + m_e)}{k^2 - m_e^2 + i\epsilon} \gamma_\mu \Big[\frac{-i}{(p-k)^2 - \mu^2 + i\epsilon} - \frac{-i}{(p-k)^2 - \Lambda^2 + i\epsilon}\Big]. \tag{116}$$

We have (with $L = k - (1-x_1)p$, $\Delta_\mu = -x_1(1-x_1)p^2 + x_1 m_e^2 + (1-x_1)\mu^2$)

$$\begin{aligned}\frac{1}{k^2 - m_e^2 + i\epsilon} \frac{1}{(p-k)^2 - \mu^2 + i\epsilon} &= \int dx_1 \frac{1}{\Big[x_1(k^2 - m_e^2 + i\epsilon) + (1-x_1)((p-k)^2 - \mu^2 + i\epsilon)\Big]^2} \\ &= \int dx_1 \frac{1}{(L^2 - \Delta_\mu + i\epsilon)^2}. \end{aligned} \tag{117}$$

Thus

$$\begin{aligned}-i\Sigma_2(p) &= -e^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu (\gamma.k + m_e) \gamma_\mu \Big[\int dx_1 \frac{1}{(L^2 - \Delta_\mu + i\epsilon)^2} - \int dx_1 \frac{1}{(L^2 - \Delta_\Lambda + i\epsilon)^2}\Big] \\ &= -e^2 \int \frac{d^4k}{(2\pi)^4} (-2\gamma.k + 4m_e) \Big[\int dx_1 \frac{1}{(L^2 - \Delta_\mu + i\epsilon)^2} - \int dx_1 \frac{1}{(L^2 - \Delta_\Lambda + i\epsilon)^2}\Big] \\ &= -e^2 \int dx_1 (-2(1-x_1)\gamma.p + 4m_e) \int \frac{d^4L}{(2\pi)^4} \Big[\frac{1}{(L^2 - \Delta_\mu + i\epsilon)^2} - \frac{1}{(L^2 - \Delta_\Lambda + i\epsilon)^2}\Big] \\ &= -ie^2 \int dx_1 (-2(1-x_1)\gamma.p + 4m_e) \int \frac{d^4L_E}{(2\pi)^4} \Big[\frac{1}{(L_E^2 + \Delta_\mu)^2} - \frac{1}{(L_E^2 + \Delta_\Lambda)^2}\Big] \\ &= -\frac{ie^2}{8\pi^2} \int dx_1 (-2(1-x_1)\gamma.p + 4m_e) \int r^3 dr \Big[\frac{1}{(r^2 + \Delta_\mu)^2} - \frac{1}{(r^2 + \Delta_\Lambda)^2}\Big] \\ &= -\frac{ie^2}{16\pi^2} \int dx_1 (-2(1-x_1)\gamma.p + 4m_e) \ln \frac{\Delta_\Lambda}{\Delta_\mu}. \end{aligned} \tag{118}$$

The final result is

$$\Sigma_2(p) = \frac{\alpha}{2\pi}\int dx_1(-(1-x_1)\gamma.p+2m_e)\ln\frac{(1-x_1)\Lambda^2}{-x_1(1-x_1)p^2+x_1m_e^2+(1-x_1)\mu^2}. \quad (119)$$

This is logarithmically divergent. Thus the mass correction or shift at one-loop is logarithmically divergent given by

$$\delta m_r = \Sigma_2(\gamma.p=m_r) = \frac{\alpha m_e}{2\pi}\int dx_1(2-x_1)\ln\frac{x_1\Lambda^2}{(1-x_1)^2m_e^2+x_1\mu^2}. \quad (120)$$

The physical mass is therefore given by

$$m_r = m_e\Big[1+\frac{\alpha}{2\pi}\int dx_1(2-x_1)\ln\frac{x_1\Lambda^2}{(1-x_1)^2m_e^2+x_1\mu^2}\Big]. \quad (121)$$

Clearly the bare mass $m_e$ must depend on the cutoff $\Lambda$ in such a way that in the limit $\Lambda\longrightarrow\infty$ the physical mass $m_r$ remains finite.

### 1.5.2 The Wave-Function Renormalization $Z_2$

At one-loop order we also need to compute the wave function renormalization. We have

$$\begin{aligned}
Z_2^{-1} &= 1-\frac{d\Sigma_2}{d\gamma.p}|_{\gamma.p=m_r}\\
&= 1-\frac{\alpha}{2\pi}\int dx_1\Big[-(1-x_1)\ln\frac{(1-x_1)\Lambda^2}{-x_1(1-x_1)p^2+x_1m_e^2+(1-x_1)\mu^2}\\
&+ (-(1-x_1)\gamma.p+2m_e)(2\gamma.p)\frac{x_1(1-x_1)}{-x_1(1-x_1)p^2+x_1m_e^2+(1-x_1)\mu^2}\Big]_{\gamma.p=m_r}\\
&= 1-\frac{\alpha}{2\pi}\int dx_1\Big[-(1-x_1)\ln\frac{(1-x_1)\Lambda^2}{x_1^2m_e^2+(1-x_1)\mu^2}+\frac{2m_e^2x_1(1-x_1)(1+x_1)}{x_1^2m_e^2+(1-x_1)\mu^2}\Big]. \quad (122)
\end{aligned}$$

Thus

$$Z_2 = 1+\delta Z_2. \quad (123)$$

$$\delta Z_2 = \frac{\alpha}{2\pi}\int_0^1 dx_1\Big[-(1-x_1)\ln\frac{(1-x_1)\Lambda^2}{x_1^2m_e^2+(1-x_1)\mu^2}+\frac{2m_e^2x_1(1-x_1)(1+x_1)}{x_1^2m_e^2+(1-x_1)\mu^2}\Big]. \quad (124)$$

A very deep observation is given by the identity $\delta Z_2=\delta F_1(0)=F_1(0)-1$ where $F_1(q^2)$ is given by (68). We have

$$\delta F_1(0) = \frac{\alpha}{2\pi}\int dxdydz\;\delta(x+y+z-1)\Big[\ln\frac{x\Lambda^2}{(1-x)^2m_e^2+x\mu^2}+\frac{m_e^2(1+x^2-4x)}{(1-x)^2m_e^2+x\mu^2}\Big]. \quad (125)$$

Clearly for $x=0$ we have $\int_0^1 dy\int_0^1 dz\ \delta(y+z-1)=1$ whereas for $x=1$ we have $\int_0^1 dy\int_0^1 dz\ \delta(y+z)=0$. In general

$$\int_0^1 dy\int_0^1 dz\ \delta(x+y+z-1)=1-x. \tag{126}$$

The proof is simple. Since $0\leq x\leq 1$ we have $0\leq 1-x\leq 1$ and $1/(1-x)\geq 1$. We shift the variables as $y=(1-x)y^{'}$ and $z=(1-x)z^{'}$. We have

$$\begin{aligned}\int_0^1 dy\int_0^1 dz\ \delta(x+y+z-1) &= (1-x)^2\int_0^{1/(1-x)}dy^{'}\int_0^{1/(1-x)}dz^{'}\ \frac{1}{1-x}\delta(y^{'}+z^{'}-1)\\ &= 1-x.\end{aligned} \tag{127}$$

By using this identity we get

$$\begin{aligned}\delta F_1(0) &= \frac{\alpha}{2\pi}\int dx(1-x)\Big[\ln\frac{x\Lambda^2}{(1-x)^2m_e^2+x\mu^2}+\frac{m_e^2(1+x^2-4x)}{(1-x)^2m_e^2+x\mu^2}\Big]\\ &= \frac{\alpha}{2\pi}\int dx\Big[x\ln\frac{x\Lambda^2}{(1-x)^2m_e^2+x\mu^2}+(1-2x)\ln\frac{x\Lambda^2}{(1-x)^2m_e^2+x\mu^2}+\frac{m_e^2(1-x)(1+x^2-4x)}{(1-x)^2m_e^2+x\mu^2}\Big]\\ &= \frac{\alpha}{2\pi}\int dx\Big[x\ln\frac{x\Lambda^2}{(1-x)^2m_e^2+x\mu^2}+\frac{d(x-x^2)}{dx}\ln\frac{x\Lambda^2}{(1-x)^2m_e^2+x\mu^2}+\frac{m_e^2(1-x)(1+x^2-4x)}{(1-x)^2m_e^2+x\mu^2}\Big]\\ &= \frac{\alpha}{2\pi}\int dx\Big[x\ln\frac{x\Lambda^2}{(1-x)^2m_e^2+x\mu^2}-(x-x^2)\frac{d}{dx}\ln\frac{x\Lambda^2}{(1-x)^2m_e^2+x\mu^2}+\frac{m_e^2(1-x)(1+x^2-4x)}{(1-x)^2m_e^2+x\mu^2}\Big]\\ &= \frac{\alpha}{2\pi}\int dx\Big[x\ln\frac{x\Lambda^2}{(1-x)^2m_e^2+x\mu^2}-\frac{m_e^2(1-x)(1-x^2)}{(1-x)^2m_e^2+x\mu^2}+\frac{m_e^2(1-x)(1+x^2-4x)}{(1-x)^2m_e^2+x\mu^2}\Big]\\ &= \frac{\alpha}{2\pi}\int dx\Big[x\ln\frac{x\Lambda^2}{(1-x)^2m_e^2+x\mu^2}-\frac{2m_e^2x(1-x)(2-x)}{(1-x)^2m_e^2+x\mu^2}\Big]\\ &= \frac{\alpha}{2\pi}\int dt\Big[(1-t)\ln\frac{(1-t)\Lambda^2}{t^2m_e^2+(1-t)\mu^2}-\frac{2m_e^2t(1-t)(1+t)}{t^2m_e^2+(1-t)\mu^2}\Big].\end{aligned} \tag{128}$$

We can immediately conclude that $\delta F_1(0)=-\delta Z_2$.

### 1.5.3 The Renormalization Constant $Z_1$

In our calculation of the vertex correction we have used the bare propagator $i/(\gamma.p-m_e)$ which has a pole at the bare mass $m=m_e$ which is as we have seen is actually a divergent quantity. This calculation should be repeated with the physical propagator $iZ_2/(\gamma.p-m_r)$. This propagator is obtained by taking the sum of the Feynman diagrams shown on SELF and ONEPARTICLE.

We reconsider the problem of scattering of an electron from an external electromagnetic field. The probability amplitude is given by the formula (13). We rewrite this formula as [1]

$$<\vec{p}^{'}s^{'}\ \text{out}|\vec{p}s\ \text{in}> \;=\; -\Big[\bar{u}^{s^{'}}(p^{'})(\gamma.p^{'}-m_e)\Big]_{\alpha^{'}}\int d^4x\int d^4x^{'}\ e^{-ipx+ip^{'}x^{'}}<\Omega|T(\hat{\psi}_{\alpha^{'}}(x^{'})\bar{\hat{\psi}}_{\alpha}(x))|\Omega>$$

[1] In writing this formula in this form we use the fact that $|0\ \text{out}>=|0\ \text{in}>=|0>$ and $|\Omega>=|0>/\sqrt{<0|0>}$. Recall that dividing by $<0|0>$ is equivalent to taking into account only connected Feynman graphs.

$$\times \quad \Big[(\gamma.p - m_e)u^s(p)\Big]_\alpha . \tag{129}$$

We sum up the quantum corrections to the two external legs by simply making the replacements

$$\gamma.p^{'} - m_e \longrightarrow (\gamma.p^{'} - m_r)/Z_2 \ , \ \gamma.p - m_e \longrightarrow (\gamma.p - m_r)/Z_2. \tag{130}$$

The probability for the spinor field to create or annihilate a particle is precisely $Z_2$ since $< \Omega|\hat{\psi}(0)|\vec{p}, s >= \sqrt{Z_2}u^s(p)$. Thus one must also replace $u^s(p)$ and $\bar{u}^{s^{'}}(p^{'})$ by $\sqrt{Z_2}u^s(p)$ and $\sqrt{Z_2}\bar{u}^{s^{'}}(p^{'})$.

Furthermore from our previous experience we know that the 2-point function $\int d^4x \int d^4x^{'} \ e^{-ipx+ip^{'}x^{'}} < \Omega|T(\hat{\psi}_{\alpha^{'}}(x^{'})\bar{\hat{\psi}}_\alpha(x))|\Omega >$ will be equal to the product of the two external propagators $iZ_2/(\gamma.p - m_r)$ and $iZ_2/(\gamma.p^{'} - m_r)$ times the amputated electron-photon vertex $\int d^4x \int d^4x^{'} \ e^{-ipx+ip^{'}x^{'}} < \Omega|T(\hat{\psi}_{\alpha^{'}}(x^{'})\bar{\hat{\psi}}_\alpha(x))|\Omega >_{\rm amp}$. Thus we make the replacement

$$< \Omega|T(\hat{\psi}_{\alpha^{'}}(x^{'})\bar{\hat{\psi}}_\alpha(x))|\Omega > \longrightarrow \frac{iZ_2}{\gamma.p^{'} - m_r} < \Omega|T(\hat{\psi}_{\alpha^{'}}(x^{'})\bar{\hat{\psi}}_\alpha(x))|\Omega > \frac{iZ_2}{\gamma.p - m_r}. \tag{131}$$

The formula of the probability amplitude $< \vec{p}^{'}s^{'} \text{ out}|\vec{p}s \text{ in} >$ becomes

$$< \vec{p}^{'}s^{'} \text{ out}|\vec{p}s \text{ in} > \ = \ Z_2\bar{u}^{s^{'}}(p^{'})_{\alpha^{'}} \int d^4x \int d^4x^{'} \ e^{-ipx+ip^{'}x^{'}} < \Omega|T(\hat{\psi}_{\alpha^{'}}(x^{'})\bar{\hat{\psi}}_\alpha(x))|\Omega >_{\rm amp} u^s(p)_\alpha. \tag{132}$$

The final result is that the amputated electron-photon vertex $\Gamma_\lambda(p^{'}, p)$ must be multiplied by $Z_2$, viz

$$< \vec{p}^{'}s^{'} \text{ out}|\vec{p}s \text{ in} > \ = \ -ie\Big(\bar{u}^{s^{'}}(p^{'})Z_2\Gamma_\lambda(p^{'}, p)u^s(p)\Big)A^{\lambda,{\rm backgr}}(q). \tag{133}$$

What we have done here is to add together the two Feynman diagrams VERTEXCOR. In the one-loop diagram the internal electron propagators are replaced by renormalized propagators.

In general an amputated Green's function with $n$ incoming lines and $m$ outgoing lines must be multiplied by a factor $(\sqrt{Z_2})^{n+m}$ in order to yield correctly the corresponding $S-$matrix element.

The calculation of the above probability amplitude will proceed exactly as before. The result by analogy with equation (71) must be of the form

$$< \vec{p}^{'}s^{'} \text{ out}|\vec{p}s \text{ in} > \ = \ -ie\bar{u}^{s^{'}}(p^{'})\Big[\gamma_\lambda F_1^{'}(q^2) + \frac{i\sigma_{\lambda\gamma}q^\gamma}{2m_r}F_2^{'}(q^2)\Big]u^s(p).A^{\lambda,{\rm backgr}}(q). \tag{134}$$

In other words

$$\begin{aligned} Z_2\Gamma_\lambda(p^{'}, p) &= \gamma_\lambda F_1^{'}(q^2) + \frac{i\sigma_{\lambda\gamma}q^\gamma}{2m_r}F_2^{'}(q^2) \\ &= \gamma_\lambda F_1(q^2) + \frac{i\sigma_{\lambda\gamma}q^\gamma}{2m_r}F_2(q^2) + \gamma_\lambda \Delta F_1(q^2) + \frac{i\sigma_{\lambda\gamma}q^\gamma}{2m_r}\Delta F_2(q^2). \end{aligned} \tag{135}$$

We are interested in order $\alpha$. Since $Z_2 = 1 + \delta Z_2$ where $\delta Z_2 = O(\alpha)$ we have $Z_2\Gamma_\lambda = \Gamma_\lambda + \delta Z_2\Gamma_\lambda = \Gamma_\lambda + \delta Z_2\gamma_\lambda$ to order $\alpha$. By using also the fact that $F_2^{'} = O(\alpha)$ we must have $\Delta F_2 = 0$. We conclude that we must have $\Delta F_1 = \delta Z_2$. Since $\delta Z_2 = -\delta F_1(0)$ we have the final result

$$\begin{aligned} F_1^{'}(q^2) &= F_1(q^2) + \Delta F_1(q^2) \\ &= F_1(q^2) + \delta Z_2 \\ &= F_1(q^2) - \delta F_1(0) \\ &= 1 + \delta F_1(q^2) - \delta F_1(0) \\ &= F_1^{\rm ren}(q^2). \end{aligned} \tag{136}$$

We introduce a new renormalization constant $Z_1$ by the relation

$$Z_1\Gamma_\lambda(q=0) = \gamma_\lambda. \tag{137}$$

The requirement that $F_1^{\rm ren}(0) = 1$ is equivalent to the statement that $Z_1 = Z_2$.

## 1.6 Ward-Takahashi Identities

**Ward-Takahashi Identities:** Let us start by considering the $3-$point function $\partial_\mu T(\hat{j}^\mu(x)\hat{\psi}(y)\bar{\hat{\psi}}(y^{'})$. For $y_0 > y_0^{'}$ we have explicitly

$$\begin{aligned} T(\hat{j}^\mu(x)\hat{\psi}(y)\bar{\hat{\psi}}(y^{'})) &= \theta(x_0 - y_0)\hat{j}^\mu(x)\hat{\psi}(y)\bar{\hat{\psi}}(y^{'}) + \theta(y_0^{'} - x_0)\hat{\psi}(y)\bar{\hat{\psi}}(y^{'})\hat{j}^\mu(x) \\ &+ \theta(y_0 - x_0)\theta(x_0 - y_0^{'})\hat{\psi}(y)\hat{j}^\mu(x)\bar{\hat{\psi}}(y^{'}). \end{aligned} \tag{138}$$

Recall that $\hat{j}^\mu = e\bar{\hat{\psi}}\gamma^\mu\hat{\psi}$. We compute immediately that (using current conservation $\partial_\mu \hat{j}^\mu = 0$)

$$\begin{aligned} \partial_\mu T(\hat{j}^\mu(x)\hat{\psi}(y)\bar{\hat{\psi}}(y^{'})) &= \delta(x_0 - y_0)\hat{j}^0(x)\hat{\psi}(y)\bar{\hat{\psi}}(y^{'}) - \delta(y_0^{'} - x_0)\hat{\psi}(y)\bar{\hat{\psi}}(y^{'})\hat{j}^0(x) \\ &- \delta(y_0 - x_0)\theta(x_0 - y_0^{'})\hat{\psi}(y)\hat{j}^0(x)\bar{\hat{\psi}}(y^{'}) + \theta(y_0 - x_0)\delta(x_0 - y_0^{'})\hat{\psi}(y)\hat{j}^0(x)\bar{\hat{\psi}}(y^{'}) \\ &= \delta(x_0 - y_0)[\hat{j}^0(x), \hat{\psi}(y)]\bar{\hat{\psi}}(y^{'}) - \delta(y_0^{'} - x_0)\hat{\psi}(y)[\bar{\hat{\psi}}(y^{'}), \hat{j}^0(x)]. \end{aligned} \tag{139}$$

We compute $[\hat{j}^0(x), \hat{\psi}(y)] = -e\delta^3(\vec{x} - \vec{y})\hat{\psi}(y)$ and $[\bar{\hat{\psi}}(y^{'}), \hat{j}^0(x)] = -e\delta^3(\vec{x} - \vec{y}^{'})\bar{\hat{\psi}}(y^{'})$. Hence we get

$$\partial_\mu T(\hat{j}^\mu(x)\hat{\psi}(y)\bar{\hat{\psi}}(y^{'})) = -e\delta^4(x - y)\hat{\psi}(y)\bar{\hat{\psi}}(y^{'}) + e\delta(y^{'} - x)\hat{\psi}(y)\bar{\hat{\psi}}(y^{'}). \tag{140}$$

The full result is clearly

$$\partial_\mu T(\hat{j}^\mu(x)\hat{\psi}(y)\bar{\hat{\psi}}(y^{'})) = \Big(-e\delta^4(x - y) + e\delta(y^{'} - x)\Big)T(\hat{\psi}(y)\bar{\hat{\psi}}(y^{'})). \tag{141}$$

In general we would have

$$\begin{aligned} \partial_\mu T(\hat{j}^\mu(x)\hat{\psi}(y_1)\bar{\hat{\psi}}(y_1^{'})...\hat{\psi}(y_n)\bar{\hat{\psi}}(y_n^{'})\hat{A}^{\alpha_1}(z_1)...) &= \sum_{i=1}^n \Big(-e\delta^4(x - y_i) + e\delta(y_i^{'} - x)\Big)T(\hat{\psi}(y_1)\bar{\hat{\psi}}(y_1^{'})... \\ &\times \hat{\psi}(y_n)\bar{\hat{\psi}}(y_n^{'})\hat{A}^{\alpha_1}(z_1)...). \end{aligned} \tag{142}$$

These are the Ward-Takahashi identities. Another important application of these identities is

$$\partial_\mu T(\hat{j}^\mu(x)\hat{A}^{\alpha_1}(z_1)...) = 0. \tag{143}$$

**Exact Photon Propagator:** The exact photon propagator is defined by

$$\begin{aligned} iD^{\mu\nu}(x-y) &= <0\ \text{out}|T(\hat{A}^{\mu}(x)\hat{A}^{\nu}(y))|0\ \text{in}> \\ &= <0\ \text{in}|T(\hat{A}^{\mu}_{\text{in}}(x)\hat{A}^{\nu}_{\text{in}}(y)S)|0\ \text{in}> \\ &= iD_F^{\mu\nu}(x-y)+\frac{(-i)^2}{2}\int d^4z_1\int d^4z_2 <0\ \text{in}|T(\hat{A}^{\mu}_{\text{in}}(x)\hat{A}^{\nu}_{\text{in}}(y)\hat{A}^{\rho_1}_{\text{in}}(z_1)\hat{A}^{\rho_2}_{\text{in}}(z_2))|0\ \text{in}> \\ &\times <0\ \text{in}|T(\hat{j}_{\text{in},\rho_1}(z_1)\hat{j}_{\text{in},\rho_2}(z_2))|0\ \text{in}>+... \\ &= iD_F^{\mu\nu}(x-y)+(-i)^2\int d^4z_1 iD_F^{\mu\rho_1}(x-z_1)\int d^4z_2 iD_F^{\nu\rho_2}(y-z_2) <0\ \text{in}|T(\hat{j}_{\text{in},\rho_1}(z_1) \\ &\times \hat{j}_{\text{in},\rho_2}(z_2))|0\ \text{in}>+... \end{aligned} \tag{144}$$

This can be rewritten as

$$\begin{aligned} iD^{\mu\nu}(x-y) &= iD_F^{\mu\nu}(x-y)-i\int d^4z_1 iD_F^{\mu\rho_1}(x-z_1) <0\ \text{in}|T(\hat{j}_{\text{in},\rho_1}(z_1)\hat{A}^{\nu}_{\text{in}}(y)\Big(-i\int d^4z_2\hat{A}^{\rho_2}_{\text{in}}(z_2) \\ &\times \hat{j}_{\text{in},\rho_2}(z_2)\Big))|0\ \text{in}>+... \end{aligned} \tag{145}$$

This is indeed correct since we can write the exact photon propagator in the form

$$\begin{aligned} iD^{\mu\nu}(x-y) &= iD_F^{\mu\nu}(x-y)-i\int d^4z_1 iD_F^{\mu\rho_1}(x-z_1) <0\ \text{out}|T(\hat{j}_{\rho_1}(z_1)\hat{A}^{\nu}(y)|0\ \text{in}>. \\ &= iD_F^{\mu\nu}(x-y)-i\int d^4z_1 iD_F^{\mu\rho_1}(z_1) <0\ \text{out}|T(\hat{j}_{\rho_1}(z_1+x)\hat{A}^{\nu}(y)|0\ \text{in}>. \end{aligned} \tag{146}$$

See the Feynman diagram EXACTPHOTON. By using the identity (143) we see immediately that

$$i\partial_{\mu,x}D^{\mu\nu}(x-y) = i\partial_{\mu,x}D_F^{\mu\nu}(x-y). \tag{147}$$

In momentum space this reads

$$q_{\mu}D^{\mu\nu}(q) = q_{\mu}D_F^{\mu\nu}(q). \tag{148}$$

This expresses transversality of the vacuum polarization (more on this below).

**Exact Vertex Function:** Let us now discuss the exact vertex function $V^{\mu}(p^{'},p)$ defined by

$$-ie(2\pi)^4\delta^4(p^{'}-p-q)V^{\mu}(p^{'},p)=\int d^4x\int d^4x_1\int d^4y_1\ e^{i(p^{'}x_1-py_1-qx)}\ <\Omega|T(\hat{A}^{\mu}(x)\hat{\psi}(x_1)\bar{\hat{\psi}}(y_1))|\Omega>. \tag{149}$$

See the Feynman graph VERTEXEXACT1. We compute (with $D_F^{\mu\nu}(q)=-i\eta^{\mu\nu}/(q^2+i\epsilon)$)

$$\begin{aligned} \int d^4x e^{-iqx} <0\ \text{out}|T(\hat{A}^{\mu}(x)\hat{\psi}(x_1)\bar{\hat{\psi}}(y_1))|0\ \text{in}> &= \int d^4x e^{-iqx} <0\ \text{in}|T(\hat{A}^{\mu}_{\text{in}}(x)\hat{\psi}_{\text{in}}(x_1)\bar{\hat{\psi}}_{\text{in}}(y_1)S)|0\ \text{in}> \\ &= -i\int d^4x e^{-iqx}\int d^4z <0\ \text{in}|T(\hat{A}^{\mu}_{\text{in}}(x)\hat{A}^{\nu}_{\text{in}}(z)\hat{j}_{\text{in},\nu}(z) \end{aligned}$$

$$
\begin{aligned}
&\times \quad \hat{\psi}_{\rm in}(x_1)\bar{\hat{\psi}}_{\rm in}(y_1))|0 \text{ in} > +...\\
&= \quad -i\int d^4x e^{-iqx}\int d^4z i D_F^{\mu\nu}(x-z) < 0 \text{ in}|T(\hat{j}_{{\rm in},\nu}(z)\\
&\times \quad \hat{\psi}_{\rm in}(x_1)\bar{\hat{\psi}}_{\rm in}(y_1))|0 \text{ in} > +...\\
&= \quad -iD_F^{\mu\nu}(q)\int d^4x e^{-iqx} < 0 \text{ in}|T(\hat{j}_{{\rm in},\nu}(x)\hat{\psi}_{\rm in}(x_1)\\
&\times \quad \bar{\hat{\psi}}_{\rm in}(y_1))|0 \text{ in} > +... \qquad (150)
\end{aligned}
$$

This result holds to all orders of perturbation. In other words we must have

$$
\int d^4x e^{-iqx} < \Omega|T(\hat{A}^\mu(x)\hat{\psi}(x_1)\bar{\hat{\psi}}(y_1))|\Omega > \;\; = \;\; -iD^{\mu\nu}(q)\int d^4x e^{-iqx} < \Omega|T(\hat{j}_\nu(x)\hat{\psi}(x_1)\bar{\hat{\psi}}(y_1))|\Omega > . \qquad (151)
$$

It is understood that $D^{\mu\nu}(q)$ is the full photon propagator. We must then have

$$
\begin{aligned}
-ie(2\pi)^4\delta^4(p^{'}-p-q)V^\mu(p^{'},p) \;\; &= \;\; -iD^{\mu\nu}(q)\int d^4x\int d^4x_1\int d^4y_1\; e^{i(p^{'}x_1-py_1-qx)} \;\; < \Omega|T(\hat{j}_\nu(x)\hat{\psi}(x_1)\\
&\times \;\; \bar{\hat{\psi}}(y_1))|\Omega > . \qquad (152)
\end{aligned}
$$

In terms of the vertex function $\Gamma^\mu(p^{'},p)$ defined previously and the exact fermion propagators $S(p)$, $S(p^{'})$ and the exact photon propagator $D^{\mu\nu}(q)$ we have

$$
V^\mu(p^{'},p) \;\; = \;\; D^{\mu\nu}(q)S(p^{'})\Gamma_\nu(p^{'},p)S(p). \qquad (153)
$$

This expression means that the vertex function can be decomposed into the QED proper vertex dressed with the full electron and photon propagators. See the Feynman graph VERTEXEXACT.

We have then

$$
\begin{aligned}
-ie(2\pi)^4\delta^4(p^{'}-p-q)D^{\mu\nu}(q)S(p^{'})\Gamma_\nu(p^{'},p)S(p) \;\; &= \;\; -iD^{\mu\nu}(q)\int d^4x\int d^4x_1\int d^4y_1\; e^{i(p^{'}x_1-py_1-qx)}\\
&\times \;\; < \Omega|T(\hat{j}_\nu(x)\hat{\psi}(x_1)\bar{\hat{\psi}}(y_1))|\Omega > . \qquad (154)
\end{aligned}
$$

We contract this equation with $q_\mu$ we obtain

$$
\begin{aligned}
-ie(2\pi)^4\delta^4(p^{'}-p-q)q_\mu D^{\mu\nu}(q)S(p^{'})\Gamma_\nu(p^{'},p)S(p) \;\; &= \;\; -iq_\mu D^{\mu\nu}(q)\int d^4x\int d^4x_1\int d^4y_1\; e^{i(p^{'}x_1-py_1-qx)}\\
&\times \;\; < \Omega|T(\hat{j}_\nu(x)\hat{\psi}(x_1)\bar{\hat{\psi}}(y_1))|\Omega > . \qquad (155)
\end{aligned}
$$

By using the identity $q_\mu D^{\mu\nu}(q) = q_\mu D_F^{\mu\nu}(q) = -iq^\nu/(q^2+i\epsilon)$ we obtain

$$
\begin{aligned}
-ie(2\pi)^4\delta^4(p^{'}-p-q)S(p^{'})q^\nu\Gamma_\nu(p^{'},p)S(p) \;\; &= \;\; -iq^\nu\int d^4x\int d^4x_1\int d^4y_1\; e^{i(p^{'}x_1-py_1-qx)}\\
&\times \;\; < \Omega|T(\hat{j}_\nu(x)\hat{\psi}(x_1)\bar{\hat{\psi}}(y_1))|\Omega >\\
&= \;\; -\int d^4x\int d^4x_1\int d^4y_1\; e^{i(p^{'}x_1-py_1-qx)}\\
&\times \;\; \partial^{\nu,x} < \Omega|T(\hat{j}_\nu(x)\hat{\psi}(x_1)\bar{\hat{\psi}}(y_1))|\Omega > . \qquad (156)
\end{aligned}
$$

By using the identity (141) we get

$$\begin{aligned}
-ie(2\pi)^4\delta^4(p^{'}-p-q)S(p^{'})q^{\nu}\Gamma_{\nu}(p^{'},p)S(p) &= -\int d^4x\int d^4x_1\int d^4y_1\; e^{i(p^{'}x_1-py_1-qx)}\\
&\times\; (-e\delta^4(x-x_1)+e\delta^4(x-y_1)) < \Omega|T(\hat{\psi}(x_1)\bar{\hat{\psi}}(y_1))|\Omega >\\
&= e\int d^4x_1\int d^4y_1\; e^{i(p^{'}-q)x_1}\; e^{-ipy_1} < \Omega|T(\hat{\psi}(x_1)\bar{\hat{\psi}}(y_1))|\Omega >\\
&-\; e\int d^4x_1\int d^4y_1\; e^{ip^{'}x_1}\; e^{-i(p+q)y_1} < \Omega|T(\hat{\psi}(x_1)\bar{\hat{\psi}}(y_1))|\Omega >\\
&= e(2\pi)^4\delta^4(p^{'}-p-q)(S(p)-S(p^{'})). \qquad (157)
\end{aligned}$$

In the above equation we have made use of the Fourier transform

$$< \Omega|T(\hat{\psi}(x_1)\bar{\hat{\psi}}(y_1))|\Omega >=\int\frac{d^4k}{(2\pi)^4}S(k)\; e^{-ik(x_1-y_1)}. \qquad (158)$$

We derive then the fundamental result

$$-iS(p^{'})q^{\nu}\Gamma_{\nu}(p^{'},p)S(p) = S(p)-S(p^{'}). \qquad (159)$$

Equivalently we have

$$-iq^{\nu}\Gamma_{\nu}(p^{'},p) = S^{-1}(p^{'})-S^{-1}(p). \qquad (160)$$

For our purposes this is the most important of all Ward-Takahashi identities.

We know that for $p$ near mass shell, i.e. $p^2=m_r^2$, the propagator $S(p)$ behaves as $S(p)=iZ_2/(\gamma.p-m_r)$. Since $p^{'}=p+q$ the momentum $p^{'}$ is near mass shell only if $p$ is near mass shell and $q$ goes to 0. Thus near mass shell we have

$$-iq^{\nu}\Gamma_{\nu}(p,p) = -iZ_2^{-1}q^{\nu}\gamma_{\nu}. \qquad (161)$$

In other words

$$\Gamma_{\nu}(p,p) = Z_2^{-1}\gamma^{\nu}. \qquad (162)$$

The renormalization constant $Z_1$ is defined precisely by

$$\Gamma_{\nu}(p,p) = Z_2^{-1}\gamma^{\nu}. \qquad (163)$$

In other words we have

$$Z_1=Z_2. \qquad (164)$$

The above Ward-Takahashi identity guarantees $F_1^{\rm ren}(0)=1$ to all orders in perturbation theory.

## 1.7 One-Loop Calculation III: Vacuum Polarization

### 1.7.1 The Renormalization Constant $Z_3$ and Renormalization of the Electric Charge

The next natural question we can ask is what is the structure of the exact 2−point photon function. At tree level we know that the answer is given by the bare photon propagator, viz

$$\int d^4x e^{iq(x-y)} \ < \Omega|T(\hat{A}^\mu(x)\hat{A}^\nu(y))|\Omega> \ = \ \frac{-i\eta^{\mu\nu}}{q^2+i\epsilon}+.... \tag{165}$$

Recall the case of the electron bare propagator which was corrected at one-loop by the electron self-energy $-i\Sigma_2(p)$. By analogy the above bare photon propagator will be corrected at one-loop by the photon self-energy $i\Pi_2^{\mu\nu}(q)$ shown on figure 2POINTPH. By using Feynman rules we have

$$i\Pi_2^{\mu\nu}(q)=(-1)\int\frac{d^4k}{(2\pi)^4}\mathrm{tr}(-ie\gamma^\mu)\frac{i(\gamma.k+m_e)}{k^2-m_e^2+i\epsilon}(-ie\gamma^\nu)\frac{i(\gamma.(k+q)+m_e)}{(k+q)^2-m_e^2+i\epsilon}. \tag{166}$$

This self-energy is the essential ingredient in vacuum polarization diagrams. See for example (7).

Similarly to the electron case, the photon self-energy $i\Pi_2^{\mu\nu}(q)$ is only the first diagram (which is of order $e^2$) among the one-particle irreducible (1PI) diagrams with 2 photon lines which we will denote by $i\Pi^{\mu\nu}(q)$. See figure 2POINTPH1. By Lorentz invariance $i\Pi^{\mu\nu}(q)$ must be a linear combination of $\eta^{\mu\nu}$ and $q^{\mu\nu}$. Now the full 2−point photon function will be obtained by the sum of all diagrams with an increasing number of insertions of the 1PI diagram $i\Pi^{\mu\nu}(q)$. This is shown on figure 2POINTPHE. The corresponding expression is

$$\begin{aligned}\int d^4x e^{iq(x-y)} \ < \Omega|T(\hat{A}^\mu(x)\hat{A}^\nu(y))|\Omega> \ &= \ \frac{-i\eta^{\mu\nu}}{q^2+i\epsilon}+\frac{-i\eta^\mu_\rho}{q^2+i\epsilon}i\Pi^{\rho\sigma}(q)\frac{-i\eta^\nu_\sigma}{q^2+i\epsilon}\\ &+ \ \frac{-i\eta^\mu_\rho}{q^2+i\epsilon}i\Pi^{\rho\sigma}(q)\frac{-i\eta_{\sigma\lambda}}{q^2+i\epsilon}i\Pi^{\lambda\eta}(q)\frac{-i\eta^\nu_\eta}{q^2+i\epsilon}+... \end{aligned}\tag{167}$$

By comparing with (146) we get

$$\begin{aligned}-i\int d^4x e^{iq(x-y)}\int d^4z_1 iD_F^{\mu\rho_1}(z_1)<0\text{ out}|T(\hat{j}_{\rho_1}(z_1+x)\hat{A}^\nu(y)|0\text{ in}> \ &= \ \frac{-i\eta^\mu_\rho}{q^2+i\epsilon}i\Pi^{\rho\sigma}(q)\frac{-i\eta^\nu_\sigma}{q^2+i\epsilon}\\ &+ \ ...\end{aligned}\tag{168}$$

By contracting both sides with $q_\mu$ and using current conservation $\partial_\mu\hat{j}^\mu=0$ we obtain the Ward identity

$$q^\mu\Pi_{\mu\nu}(q)=0. \tag{169}$$

Hence we must have

$$\Pi^{\mu\nu}(q)=(q^2\eta^{\mu\nu}-q^\mu q^\nu)\Pi(q^2). \tag{170}$$

It is straightforward to show that the exact 2−point photon function becomes

$$\int d^4x e^{iq(x-y)} \quad < \Omega|T(\hat{A}^\mu(x)\hat{A}^\nu(y))|\Omega > \quad = \quad \frac{-i\eta^{\mu\nu}}{q^2+i\epsilon} + \frac{-i\eta^\mu_\rho}{q^2+i\epsilon}(\eta^{\rho\nu} - \frac{q^\rho q^\nu}{q^2})(\Pi + \Pi^2 + ...)$$
$$= \quad \frac{-iq^\mu q^\nu}{(q^2)^2} + \frac{-i}{q^2+i\epsilon}\frac{1}{1-\Pi(q^2)}(\eta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}). \quad (171)$$

This propagator has a single pole at $q^2 = 0$ if the function $\Pi(q^2)$ is regular at $q^2 = 0$. This is indeed true to all orders in perturbation theory. Physically this means that the photon remains massless. We define the renormalization constant $Z_3$ as the residue at the $q^2 = 0$ pole, viz

$$Z_3 = \frac{1}{1-\Pi(0)}. \quad (172)$$

The terms proportional to $q^\mu q^\nu$ in the above exact propagator will lead to vanishing contributions inside a probability amplitude, i.e. when we connect the exact 2−point photon function to at least one electron line. This is another manifestation of the Ward-Takahashi identities. We give an example of this cancellation next.

The contribution of the tree level plus vacuum polarization diagrams to the probability amplitude of the process $e^- + e^+ \longrightarrow \mu^- + \mu^+$ was given by

$$-e^2(2\pi)^4\delta^4(k+p-k^{'}-p^{'})(\bar{u}^{s^{'}}(p^{'})\gamma_\mu u^s(p))\Big(\frac{-i\eta^{\mu\nu}}{q^2} + \frac{-i\eta^\mu_\rho}{q^2}i\Pi_2^{\rho\sigma}(q)\frac{-i\eta^\nu_\sigma}{q^2}\Big)(\bar{u}^{r^{'}}(k^{'})\gamma_\nu u^r(k)). \quad (173)$$

By using the exact 2−point photon function this becomes

$$-e^2(2\pi)^4\delta^4(k+p-k^{'}-p^{'})(\bar{u}^{s^{'}}(p^{'})\gamma_\mu u^s(p))\Big(\frac{-iq^\mu q^\nu}{(q^2)^2} + \frac{-i}{q^2+i\epsilon}\frac{1}{1-\Pi(q^2)}(\eta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2})\Big)(\bar{u}^{r^{'}}(k^{'})\gamma_\nu u^r(k)). \quad (174$$

We can check that $\bar{u}^{s^{'}}(p^{'})\gamma_\mu q^\mu u^s(p) = \bar{u}^{s^{'}}(p^{'})(\gamma_\mu p^\mu - \gamma_\mu p^{'\mu})u^s(p) = 0$. We get then the probability amplitude

$$-e^2(2\pi)^4\delta^4(k+p-k^{'}-p^{'})(\bar{u}^{s^{'}}(p^{'})\gamma_\mu u^s(p))\Big(\frac{-i}{q^2+i\epsilon}\frac{1}{1-\Pi(q^2)}\eta^{\mu\nu}\Big)(\bar{u}^{r^{'}}(k^{'})\gamma_\nu u^r(k)). \quad (175)$$

For scattering with very low $q^2$ this becomes

$$-e^2(2\pi)^4\delta^4(k+p-k^{'}-p^{'})(\bar{u}^{s^{'}}(p^{'})\gamma_\mu u^s(p))\Big(\frac{-i}{q^2+i\epsilon}\frac{1}{1-\Pi(0)}\eta^{\mu\nu}\Big)(\bar{u}^{r^{'}}(k^{'})\gamma_\nu u^r(k)) \quad =$$
$$-e_R^2(2\pi)^4\delta^4(k+p-k^{'}-p^{'})(\bar{u}^{s^{'}}(p^{'})\gamma_\mu u^s(p))\Big(\frac{-i}{q^2+i\epsilon}\eta^{\mu\nu}\Big)(\bar{u}^{r^{'}}(k^{'})\gamma_\nu u^r(k)). \quad (176)$$

This looks exactly like the tree level contribution with an electric charge $e_R$ given by

$$e_R = e\sqrt{Z_3}. \quad (177)$$

The electric charge $e_R$ is called the renormalized electric charge. This shift of the electric charge relative to tree level is a general feautre since the amplitude for any process with very low momentum transfer $q^2$ when we replace the bare photon propagator with the exact photon propagator will appear as a tree level process with the renoramlized electric charge $e_R$.

Using the definition of the renormalized electric charge $e_R$ the above probability amplitude can now be put in the form

$$\begin{aligned}
-e^2(2\pi)^4\delta^4(k+p-k^{'}-p^{'})(\bar{u}^{s^{'}}(p^{'})\gamma_\mu u^s(p))\Big(\frac{-i}{q^2+i\epsilon}\frac{1}{1-\Pi(q^2)}\eta^{\mu\nu}\Big)(\bar{u}^{r^{'}}(k^{'})\gamma_\nu u^r(k)) &= \\
-e_R^2(2\pi)^4\delta^4(k+p-k^{'}-p^{'})(\bar{u}^{s^{'}}(p^{'})\gamma_\mu u^s(p))\Big(\frac{-i}{q^2+i\epsilon}\frac{1-\Pi(0)}{1-\Pi(q^2)}\eta^{\mu\nu}\Big)(\bar{u}^{r^{'}}(k^{'})\gamma_\nu u^r(k)) &= \\
-e_{\text{eff}}^2(2\pi)^4\delta^4(k+p-k^{'}-p^{'})(\bar{u}^{s^{'}}(p^{'})\gamma_\mu u^s(p))\Big(\frac{-i}{q^2+i\epsilon}\eta^{\mu\nu}\Big)(\bar{u}^{r^{'}}(k^{'})\gamma_\nu u^r(k)) & \quad (178)
\end{aligned}$$

The effective charge $e_{\text{eff}}$ is momentum dependent given by

$$e_{\text{eff}}^2 = e_R^2\frac{1-\Pi(0)}{1-\Pi(q^2)} = \frac{e^2}{1-\Pi(q^2)}. \quad (179)$$

At one-loop order we have $\Pi = \Pi_2$ and thus the effective charge becomes

$$e_{\text{eff}}^2 = \frac{e_R^2}{1-\Pi_2(q^2)+\Pi_2(0)}. \quad (180)$$

### 1.7.2 Dimensional Regularization

We now evaluate the loop integral $\Pi_2(q^2)$ given by

$$\Pi_2^{\mu\nu}(q) = ie^2\int\frac{d^4k}{(2\pi)^4}\text{tr}\gamma^\mu\frac{(\gamma.k+m_e)}{k^2-m_e^2+i\epsilon}\gamma^\nu\frac{(\gamma.(k+q)+m_e)}{(k+q)^2-m_e^2+i\epsilon}. \quad (181)$$

This integral is quadratically UV divergent as one can see from the rough estimate

$$\begin{aligned}
\Pi_2^{\mu\nu}(q) &\sim \int_0^\Lambda k^3dk\frac{1}{k}\frac{1}{k} \\
&\sim \frac{1}{2}\Lambda^2. \quad (182)
\end{aligned}$$

This can be made more precise using this naive cutoff procedure and we will indeed find that it is quadratically UV divergent. This is a severe divergence which is stronger than the logarithmic divergences we encountered in previous calculations. In any case a naive cutoff will break the Ward-Takahashi identity $Z_1 = Z_2$. As in previous cases the Pauli-Villars regularization can be used here and it will preserve the Ward-Takahashi identity $Z_1 = Z_2$. However this method is very complicated to implement in this case.

We will employ in this section a more powerful and more elegant regularization method known as dimensional regularization. The idea is simply to compute the loop integral $\Pi_2(q^2)$ not in 4 dimensions but in $d$ dimensions. The result will be an analytic function in $d$. We are clearly interested in the limit $d \longrightarrow 4$.

We start as before by introducing Feynman parameters, namely

$$\begin{aligned}
\frac{1}{k^2-m_e^2+i\epsilon}\frac{1}{(k+q)^2-m_e^2+i\epsilon} &= \int_0^1 dx\int_0^y \delta(x+y-1)\frac{1}{\Big[x(k^2-m_e^2+i\epsilon)+y((k+q)^2-m_e^2+i\epsilon)\Big]^2}\\
&= \int_0^1 dx\,\frac{1}{\Big[(k+(1-x)q)^2+x(1-x)q^2-m_e^2+i\epsilon\Big]^2}\\
&= \int_0^1 dx\,\frac{1}{\Big[l^2-\Delta+i\epsilon\Big]^2}. \qquad (183)
\end{aligned}$$

We have defined $l=k+(1-x)q$ and $\Delta=m_e^2-x(1-x)q^2$. Furthermore

$$\begin{aligned}
\mathrm{tr}\gamma^\mu(\gamma.k+m_e)\gamma^\nu(\gamma.(k+q)+m_e) &= 4k^\mu(k+q)^\nu+4k^\nu(k+q)^\mu-4\eta^{\mu\nu}(k.(k+q)-m_e^2)\\
&= 4(l^\mu-(1-x)q^\mu)(l^\nu+xq^\nu)+4(l^\nu-(1-x)q^\nu)(l^\mu+xq^\mu)\\
&- 4\eta^{\mu\nu}((l-(1-x)q).(l+xq)-m_e^2)\\
&= 4l^\mu l^\nu-4(1-x)xq^\mu q^\nu+4l^\nu l^\mu-4(1-x)xq^\nu q^\mu\\
&- 4\eta^{\mu\nu}(l^2-x(1-x)q^2-m_e^2)+... \qquad (184)
\end{aligned}$$

We have now the $d-$dimensional loop integral

$$\begin{aligned}
\Pi_2^{\mu\nu}(q) &= 4ie^2\int\frac{d^dl}{(2\pi)^d}\Big(l^\mu l^\nu+l^\nu l^\mu-2(1-x)xq^\nu q^\mu-\eta^{\mu\nu}(l^2-x(1-x)q^2-m_e^2)\Big)\\
&\times \int_0^1 dx\,\frac{1}{\Big[l^2-\Delta+i\epsilon\Big]^2}. \qquad (185)
\end{aligned}$$

By rotational invariance in $d$ dimensions we can replace $l^\mu l^\nu$ by $l^2\eta^{\mu\nu}/d$. Thus we get

$$\begin{aligned}
\Pi_2^{\mu\nu}(q) &= 4ie^2\int_0^1 dx\Big[(\frac{2}{d}-1)\eta^{\mu\nu}\int\frac{d^dl}{(2\pi)^d}\frac{l^2}{(l^2-\Delta+i\epsilon)^2}\\
&- (2(1-x)xq^\mu q^\nu-\eta^{\mu\nu}(x(1-x)q^2+m_e^2))\int\frac{d^dl}{(2\pi)^d}\frac{1}{(l^2-\Delta+i\epsilon)^2}\Big]. \qquad (186)
\end{aligned}$$

Next we Wick rotate ($d^dl=id^dl_E$ and $l^2=-l_E^2$) to obtain

$$\begin{aligned}
\Pi_2^{\mu\nu}(q) &= -4e^2\int_0^1 dx\Big[(-\frac{2}{d}+1)\eta^{\mu\nu}\int\frac{d^dl_E}{(2\pi)^d}\frac{l_E^2}{(l_E^2+\Delta)^2}\\
&- (2(1-x)xq^\mu q^\nu-\eta^{\mu\nu}(x(1-x)q^2+m_e^2))\int\frac{d^dl_E}{(2\pi)^d}\frac{1}{(l_E^2+\Delta)^2}\Big]. \qquad (187)
\end{aligned}$$

We need to compute two $d-$dimensional integrals. These are

$$\begin{aligned}\int \frac{d^d l_E}{(2\pi)^d}\frac{l_E^2}{(l_E^2+\Delta)^2} &= \frac{1}{(2\pi)^d}\int d\Omega_d \int r^{d-1}dr\frac{r^2}{(r^2+\Delta)^2}\\ &= \frac{1}{(2\pi)^d}\frac{1}{2}\int d\Omega_d \int (r^2)^{\frac{d}{2}}dr^2\frac{1}{(r^2+\Delta)^2}\\ &= \frac{1}{(2\pi)^d}\frac{1}{2}\frac{1}{\Delta^{1-\frac{d}{2}}}\int d\Omega_d \int_0^1 dx\ x^{-\frac{d}{2}}(1-x)^{\frac{d}{2}}. \end{aligned} \quad (188)$$

$$\begin{aligned}\int \frac{d^d l_E}{(2\pi)^d}\frac{1}{(l_E^2+\Delta)^2} &= \frac{1}{(2\pi)^d}\int d\Omega_d \int r^{d-1}dr\frac{1}{(r^2+\Delta)^2}\\ &= \frac{1}{(2\pi)^d}\frac{1}{2}\int d\Omega_d \int (r^2)^{\frac{d-2}{2}}dr^2\frac{1}{(r^2+\Delta)^2}\\ &= \frac{1}{(2\pi)^d}\frac{1}{2}\frac{1}{\Delta^{2-\frac{d}{2}}}\int d\Omega_d \int_0^1 dx\ x^{1-\frac{d}{2}}(1-x)^{\frac{d}{2}-1}. \end{aligned} \quad (189)$$

In the above two equations we have used the change of variable $x = \Delta/(r^2+\Delta)$ and $dx/\Delta = -dr^2/(r^2+\Delta)^2$. We can also use the definition of the so-called beta function

$$B(\alpha,\beta) = \int_0^1 dx\ x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \quad (190)$$

Also we can use the area of a $d-$dimensional unit sphere given by

$$\int d\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}. \quad (191)$$

We get then

$$\int \frac{d^d l_E}{(2\pi)^d}\frac{l_E^2}{(l_E^2+\Delta)^2} = \frac{1}{(4\pi)^{\frac{d}{2}}}\frac{1}{\Delta^{1-\frac{d}{2}}}\frac{\Gamma(2-\frac{d}{2})}{\frac{2}{d}-1}. \quad (192)$$

$$\int \frac{d^d l_E}{(2\pi)^d}\frac{1}{(l_E^2+\Delta)^2} = \frac{1}{(4\pi)^{\frac{d}{2}}}\frac{1}{\Delta^{2-\frac{d}{2}}}\Gamma(2-\frac{d}{2}). \quad (193)$$

With these results the loop integral $\Pi_2^{\mu\nu}(q)$ becomes

$$\begin{aligned}\Pi_2^{\mu\nu}(q) &= -4e^2\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\int_0^1 dx\frac{1}{\Delta^{2-\frac{d}{2}}}\Big[-\Delta\eta^{\mu\nu} - (2(1-x)xq^\mu q^\nu - \eta^{\mu\nu}(x(1-x)q^2+m_e^2))\Big]\\ &= -4e^2\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\int_0^1 dx\frac{2x(1-x)}{\Delta^{2-\frac{d}{2}}}(q^2\eta^{\mu\nu}-q^\mu q^\nu). \end{aligned} \quad (194)$$

Therefore we conclude that the Ward-Takahashi identity is indeed maintained in dimensional regularization. The function $\Pi_2(q^2)$ is then given by

$$\Pi_2(q^2) = -4e^2\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\int_0^1 dx\frac{2x(1-x)}{\Delta^{2-\frac{d}{2}}}. \quad (195)$$

We want now to take the limit $d \longrightarrow 4$. We define the small parameter $\epsilon = 4 - d$. We use the expansion of the gamma function near its pole $z = 0$ given by

$$\Gamma(2 - \frac{d}{2}) = \Gamma(\frac{\epsilon}{2}) = \frac{2}{\epsilon} - \gamma + O(\epsilon). \tag{196}$$

The number $\gamma$ is given by $\gamma = 0.5772$ and is called the Euler-Mascheroni constant. It is not difficult to convince ourselves that the $1/\epsilon$ divergence in dimensional regularization corresponds to the logarithmic divergence $\ln \Lambda^2$ in Pauli-Villars regularization.

Thus near $d = 4$ (equivalently $\epsilon = 0$) we get

$$\begin{aligned}
\Pi_2(q^2) &= -\frac{4e^2}{(4\pi)^2}(\frac{2}{\epsilon} - \gamma + O(\epsilon)) \int_0^1 dx \; 2x(1-x)(1 - \frac{\epsilon}{2}\ln\Delta + O(\epsilon^2)) \\
&= -\frac{2\alpha}{\pi}\int_0^1 dx \; x(1-x)(\frac{2}{\epsilon} - \ln\Delta - \gamma + O(\epsilon)) \\
&= -\frac{2\alpha}{\pi}\int_0^1 dx \; x(1-x)(\frac{2}{\epsilon} - \ln(m_e^2 - x(1-x)q^2) - \gamma + O(\epsilon)).
\end{aligned} \tag{197}$$

We will also need

$$\Pi_2(0) = -\frac{2\alpha}{\pi}\int_0^1 dx \; x(1-x)(\frac{2}{\epsilon} - \ln(m_e^2) - \gamma + O(\epsilon)). \tag{198}$$

Thus

$$\Pi_2(q^2) - \Pi_2(0) = -\frac{2\alpha}{\pi}\int_0^1 dx \; x(1-x)(\ln\frac{m_e^2}{m_e^2 - x(1-x)q^2} + O(\epsilon)). \tag{199}$$

This is finite in the limit $\epsilon \longrightarrow 0$. At very high energies (small distances) corresponding to $-q^2 >> m_e^2$ we get

$$\begin{aligned}
\Pi_2(q^2) - \Pi_2(0) &= -\frac{2\alpha}{\pi}\int_0^1 dx \; x(1-x)(-\ln(1 + x(1-x)\frac{-q^2}{m_e^2}) + O(\epsilon)) \\
&= \frac{\alpha}{3\pi}\Big[\ln\frac{-q^2}{m_e^2} - \frac{5}{3} + O(\frac{m_e^2}{-q^2})\Big] \\
&= \frac{\alpha_R}{3\pi}\Big[\ln\frac{-q^2}{m_e^2} - \frac{5}{3} + O(\frac{m_e^2}{-q^2})\Big].
\end{aligned} \tag{200}$$

At one-loop order the effective electric charge is

$$e_{\rm eff}^2 = \frac{e_R^2}{1 - \frac{\alpha_R}{3\pi}[\ln\frac{-q^2}{m_e^2} - \frac{5}{3} + O(\frac{m_e^2}{-q^2})]}. \tag{201}$$

The electromagnetic coupling constant depends therefore on the energy as follows

$$\alpha_{\rm eff}(\frac{-q^2}{m_e^2}) = \frac{\alpha_R}{1 - \frac{\alpha_R}{3\pi}[\ln\frac{-q^2}{m_e^2} - \frac{5}{3} + O(\frac{m_e^2}{-q^2})]} \tag{202}$$

The effective electromagnetic coupling constant becomes large at high energies. We say that the electromagnetic coupling constant runs with energy or equivalently with distance.

## 1.8 Renormalization of QED

In this last section we will summarize all our results. The starting Lagrangian was

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - e\bar{\psi}\gamma_\mu\psi A^\mu. \tag{203}$$

We know that the electron and photon two-point functions behave as

$$\int d^4x e^{ip(x-y)} \ <\Omega|T(\hat{\psi}(x)\bar{\hat{\psi}}(y))|\Omega> = \frac{iZ_2}{\gamma.p - m_r + i\epsilon} + ... \tag{204}$$

$$\int d^4x e^{iq(x-y)} \ <\Omega|T(\hat{A}^\mu(x)\hat{A}^\nu(y))|\Omega> = \frac{-i\eta^{\mu\nu}Z_3}{q^2 + i\epsilon} + .... \tag{205}$$

Let us absorb the field strength renormalization constants $Z_2$ and $Z_3$ in the fields as follows

$$\hat{\psi}_r = \hat{\psi}/\sqrt{Z_2} \ , \ \hat{A}^\mu_r = \hat{A}^\mu/\sqrt{Z_3}. \tag{206}$$

The QED Lagrangian becomes

$$\mathcal{L} = -\frac{Z_3}{4}F_{r\mu\nu}F_r^{\mu\nu} + Z_2\bar{\psi}_r(i\gamma^\mu\partial_\mu - m)\psi_r - eZ_2\sqrt{Z_3}\bar{\psi}_r\gamma_\mu\psi_r A^\mu_r. \tag{207}$$

The renormalized electric charge is defined by

$$eZ_2\sqrt{Z_3} = e_R Z_1. \tag{208}$$

This reduces to the previous definition $e_R = e\sqrt{Z_3}$ by using Ward identity in the form

$$Z_1 = Z_2. \tag{209}$$

We introduce the counter-terms

$$Z_1 = 1 + \delta_1 \ , \ Z_2 = 1 + \delta_2 \ , \ Z_3 = 1 + \delta_3. \tag{210}$$

We also introduce the renormalized mass $m_r$ and the counter-term $\delta_m$ by

$$Z_2 m = m_r + \delta_m. \tag{211}$$

We have

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{r\mu\nu}F_r^{\mu\nu} + \bar{\psi}_r(i\gamma^\mu\partial_\mu - m_r)\psi_r - e_R\bar{\psi}_r\gamma_\mu\psi_r A^\mu_r \\ &- \frac{\delta_3}{4}F_{r\mu\nu}F_r^{\mu\nu} + \bar{\psi}_r(i\delta_2\gamma^\mu\partial_\mu - \delta_m)\psi_r - e_R\delta_1\bar{\psi}_r\gamma_\mu\psi_r A^\mu_r.\end{aligned} \tag{212}$$

By dropping total derivative terms we find

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{r\mu\nu}F_r^{\mu\nu} + \bar{\psi}_r(i\gamma^\mu\partial_\mu - m_r)\psi_r - e_R\bar{\psi}_r\gamma_\mu\psi_r A^\mu_r \\ &- \frac{\delta_3}{2}A_{r\mu}(-\partial.\partial\ \eta^{\mu\nu} + \partial^\mu\partial^\nu)A_{r\nu} + \bar{\psi}_r(i\delta_2\gamma^\mu\partial_\mu - \delta_m)\psi_r - e_R\delta_1\bar{\psi}_r\gamma_\mu\psi_r A^\mu_r.\end{aligned} \tag{213}$$

There are three extra Feynman diagrams associated with the counter-terms $\delta_1$, $\delta_2$, $\delta_3$ and $\delta_m$ besides the usual three Feynman diagrams associated with the photon and electron propagators and the QED vertex. The Feynman diagrams of renormalized QED are shown on figure RENQED.

The counter-terms will be determined from renormalization conditions. There are four counter-terms and thus one must have 4 renormalization conditions. The first two renormalization conditions correspond to the fact that the electron and photon field-strength renormalization constants are equal 1. Indeed we have by construction

$$\int d^4x e^{ip(x-y)} \quad <\Omega|T(\hat{\psi}_r(x)\bar{\hat{\psi}}_r(y))|\Omega>= \frac{i}{\gamma.p-m_r+i\epsilon}+... \tag{214}$$

$$\int d^4x e^{iq(x-y)} \quad <\Omega|T(\hat{A}_r^{\mu}(x)\hat{A}_r^{\nu}(y))|\Omega> \quad = \quad \frac{-i\eta^{\mu\nu}}{q^2+i\epsilon}+.... \tag{215}$$

Let us recall that the one-particle irreducible (1PI) diagrams with 2 photon lines is $i\Pi^{\mu\nu}(q)=i(\eta^{\mu\nu}q^2-q^{\mu}q^{\nu})\Pi(q^2)$. We know that the residue of the photon propagator at $q^2=0$ is $1/(1-\Pi(0))$. Thus the first renormalization constant is

$$\Pi(q^2=0)=1. \tag{216}$$

The one-particle irreducible (1PI) diagrams with 2 electron lines is $-i\Sigma(\gamma.p)$. The residue of the electron propagator at $\gamma.p=m_r$ is $1/(1-(d\Sigma(\gamma.p)/d\gamma.p)|_{\gamma.p=m_r})$. Thus the second renormalization constant is

$$\frac{d\Sigma(\gamma.p)}{\gamma.p}|_{\gamma.p=m_r}=0. \tag{217}$$

Clearly the renormalized mass $m_r$ must be defined by setting the self-energy $-i\Sigma(\gamma.p)$ at $\gamma.p=m_r$ to zero so it is not shifted by quantum effects in renormalized QED. In other words we must have the renormalization constant

$$\Sigma(\gamma.p=m_r)=0. \tag{218}$$

Lastly the renormalized electric charge $e_R$ must also not be shifted by quantum effects in renormalized QED. The quantum correction to the electric charge is contained in the exact vertex function (the QED proper vertex) $-ie\Gamma^{\mu}(p^{'},p)$. Thus we must impose

$$\Gamma^{\mu}(p^{'}-p=0)=\gamma^{\mu}. \tag{219}$$

## 1.9 Problems and Exercises

**Mott Formula and Bhabha Scattering:**

- Use Feynman rules to write down the tree level probability amplitude for electron-muon scattering.
- Derive the unpolarized cross section of the electron-muon scattering at tree level in the limit $m_\mu \longrightarrow \infty$. The result is known as Mott formula.
- Repeat the above two questions for electron-electron scattering. This is known as Bhabha scattering.

**Scattering from an External Electromagnetic Field:** Compute the Feynman diagrams corresponding to the three first terms of equation (21).

**Spinor Technology:**

- Prove Gordon's identity (with $q = p - p^{'}$)

$$\bar{u}^{s^{'}}(p^{'})\gamma^\mu u^s(p) = \frac{1}{2m_e}\bar{u}^{s^{'}}(p^{'})\Big[(p+p^{'})^\mu - i\sigma^{\mu\nu}q_\nu\Big]u^s(p). \quad (220)$$

- Show that we can make the replacement

$$\begin{aligned}\bar{u}^{s^{'}}(p^{'})\Big[(x\gamma.p + y\gamma.q)\gamma^\mu(x\gamma.p+(y-1)\gamma.q)\Big]u^s(p) \quad &\longrightarrow \quad \bar{u}^{s^{'}}(p^{'})\Big[m_e(x+y)(x+y-1)(2p^\mu - m_e\gamma^\mu)\\ &- \quad (x+y)(y-1)\Big(2m_e(p+p^{'})^\mu + q^2\gamma^\mu - 3m_e^2\\ &\times \quad \gamma^\mu\Big) - m_e^2 y(x+y-1)\gamma^\mu + m_e y(y-1)\\ &\times \quad (2p^{'\mu} - m_e\gamma^\mu)\Big]u^s(p). \quad (221)\end{aligned}$$

**Spheres in $d$ Dimensions:** Show that the area of a $d-$dimensional unit sphere is given by

$$\int d\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}. \quad (222)$$

**Renormalization Constant $Z_2$:** Show that the probability for the spinor field to create or annihilate a particle is precisely $Z_2$.

**Ward Identity:** Consider a QED process which involves a single external photon with momentum $k$ and polarization $\epsilon_\mu$. The probability amplitude of this process is of the form $i\mathcal{M}^\mu(k)\epsilon_\mu(k)$. Show that current conservation leads to the Ward identity $k_\mu\mathcal{M}^\mu(k) = 0$.

Hint: See Peskin and Schroeder.

**Pauli-Villars Regulator Fields:** Show that Pauli-Villars regularization is equivalent to the introduction of regulator fields with large masses. The number of regulator fields can be anything.

Hint: See Zinn-Justin.

**Pauli-Villars Regularization:**

- Use Pauli-Villars Regularization to compute $\Pi_2^{\mu\nu}(q^2)$.
- Show that the $1/\epsilon$ divergence in dimensional regularization corresponds to the logarithmic divergence $\ln\Lambda^2$ in Pauli-Villars regularization. Compare for example the value of the integral (193) in both schemes.

**Uehling Potential and Lamb Shift:**

- Show that the electrostatic potential can be given by the integral

$$V(\vec{x}) = \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{-e^2 e^{i\vec{q}\vec{x}}}{\vec{q}^2}. \tag{223}$$

- Compute the one-loop correction to the above potential due to the vacuum polarization.
- By approximating the Uehling potential by a delta function determine the Lamb shift of the levels of the Hydrogen atom.

**Hard Cutoff Regulator:**

- Use a naive cutoff to evaluate $\Pi_2^{\mu\nu}(q^2)$. What do you conclude.
- Show that a naive cutoff will not preserve the Ward-Takahashi identity $Z_1 = Z_2$.

**Dimensional Regularization and QED Counter-terms:**

- Reevaluate the electron self-energy $-i\Sigma(\gamma.p)$ at one-loop in dimensional regularization.
- Compute the counter-terms $\delta_m$ and $\delta_2$ at one-loop.
- Use the expression of the photon self-energy $i\Pi^{\mu\nu}$ at one-loop computed in the lecture in dimensional regularization to evaluate the counter term $\delta_3$.
- Reevaluate the vertex function $-ie\Gamma^{\mu}(p^{'}, p)$ at one-loop in dimensional regularization.
- Compute the counter-term $\delta_1$ at one-loop.
- Show explicitly that dimensional regularization will preserve the Ward-Takahashi identity $Z_1 = Z_2$.