# THE QFT NOTES 5 

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## 1 The Electromagnetic Field

### 1.1 Covariant Formulation of Classical Electrodynamics

The Field Tensor The electric and magnetic fields $\vec{E}$ and $\vec{B}$ generated by a charge density $\rho$ and a current density $\vec{J}$ are given by the Maxwell's equations written in the Heaviside-Lorentz system as

$$
\begin{gather*}
\vec{\nabla} \vec{E}=\rho, \text { Gauss' s Law }  \tag{1}\\
\vec{\nabla} \vec{B}=0, \text { No }- \text { Magnetic Monopole Law. }  \tag{2}\\
\vec{\nabla} \times \vec{E}=-\frac{1}{c} \frac{\partial \vec{B}}{\partial t}, \text { Faraday' s Law. }  \tag{3}\\
\vec{\nabla} \times \vec{B}=\frac{1}{c}\left(\vec{J}+\frac{\partial \vec{E}}{\partial t}\right), \text { Ampere }- \text { Maxwell' s Law } \tag{4}
\end{gather*}
$$

The Lorentz force law expresses the force exerted on a charge $q$ moving with a velocity $\vec{u}$ in the presence of an electric and magnetic fields $\vec{E}$ and $\vec{B}$. This is given by

$$
\begin{equation*}
\vec{F}=q\left(\vec{E}+\frac{1}{c} \vec{u} \times \vec{B}\right) . \tag{5}
\end{equation*}
$$

The continuity equation expresses local conservation of the electric charge. It reads

$$
\begin{equation*}
\frac{\partial \rho}{\partial t}+\vec{\nabla} \vec{J}=0 \tag{6}
\end{equation*}
$$

We consider now the following Lorentz transformation

$$
\begin{align*}
x^{\prime} & =\gamma(x-v t) \\
y^{\prime} & =y \\
z^{\prime} & =z \\
t^{\prime} & =\gamma\left(t-\frac{v}{c^{2}} x\right) . \tag{7}
\end{align*}
$$

In other words (with $x^{0}=c t, x^{1}=x, x^{2}=y, x^{3}=z$ and signature $(+---)$ )

$$
x^{\mu^{\prime}}=\Lambda^{\mu}{ }_{\nu} x^{\nu}, \Lambda=\left(\begin{array}{cccc}
\gamma & -\gamma \beta & 0 & 0  \tag{8}\\
-\gamma \beta & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right) .
$$

The transformation laws of the electric and magnetic fields $\vec{E}$ and $\vec{B}$ under this Lorentz transformation are given by

$$
\begin{align*}
& E_{x}^{\prime}=E_{x}, E_{y}^{\prime}=\gamma\left(E_{y}-\frac{v}{c} B_{z}\right), E_{z}^{\prime}=\gamma\left(E_{z}+\frac{v}{c} B_{y}\right) \\
& B_{x}^{\prime}=B_{x}, B_{y}^{\prime}=\gamma\left(B_{y}+\frac{v}{c} E_{z}\right), B_{z}^{\prime}=\gamma\left(B_{z}-\frac{v}{c} E_{y}\right) . \tag{9}
\end{align*}
$$

Clearly $\vec{E}$ and $\vec{B}$ do not transform like the spatial part of a 4 -vector. In fact $\vec{E}$ and $\vec{B}$ are the components of a second-rank antisymmetric tensor. Let us recall that a second-rank tensor $F^{\mu \nu}$ is an abject carrying two indices which transforms under a Lorentz transformation $\Lambda$ as

$$
\begin{equation*}
F^{\mu \nu^{\prime}}=\Lambda^{\mu}{ }_{\lambda} \Lambda^{\nu}{ }_{\sigma} F^{\lambda \sigma} . \tag{10}
\end{equation*}
$$

This has 16 components. An antisymmetric tensor will satisfy the extra condition $F_{\mu \nu}=-F_{\mu \nu}$ so the number of independent components is reduced to 6 . Explicitly we write

$$
F^{\mu \nu}=\left(\begin{array}{cccc}
0 & F^{01} & F^{02} & F^{03}  \tag{11}\\
-F^{01} & 0 & F^{12} & F^{13} \\
-F^{02} & -F^{12} & 0 & F^{23} \\
-F^{03} & -F^{13} & -F^{23} & 0
\end{array}\right) .
$$

The transformation laws (10) can then be rewritten as

$$
\begin{align*}
& F^{01^{\prime}}=F^{01}, F^{02^{\prime}}=\gamma\left(F^{02}-\beta F^{12}\right), F^{03^{\prime}}=\gamma\left(F^{03}+\beta F^{31}\right) \\
& F^{23^{\prime}}=F^{23}, F^{31^{\prime}}=\gamma\left(F^{31}+\beta F^{03}\right), F^{12^{\prime}}=\gamma\left(F^{12}-\beta F^{02}\right) \tag{12}
\end{align*}
$$

By comparing (9) and (12) we obtain

$$
\begin{equation*}
F^{01}=-E_{x}, F^{02}=-E_{y}, F^{03}=-E_{z}, F^{12}=-B_{z}, F^{31}=-B_{y}, F^{23}=-B_{x} . \tag{13}
\end{equation*}
$$

Thus

$$
F^{\mu \nu}=\left(\begin{array}{cccc}
0 & -E_{x} & -E_{y} & -E_{z}  \tag{14}\\
E_{x} & 0 & -B_{z} & B_{y} \\
E_{y} & B_{z} & 0 & -B_{x} \\
E_{z} & -B_{y} & B_{x} & 0
\end{array}\right) .
$$

Let us remark that (9) remains unchanged under the duality transformation

$$
\begin{equation*}
\vec{E} \longrightarrow \vec{B}, \vec{B} \longrightarrow-\vec{E} \tag{15}
\end{equation*}
$$

The tensor (14) changes under the above duality transformation to the tensor

$$
\tilde{F}^{\mu \nu}=\left(\begin{array}{cccc}
0 & -B_{x} & -B_{y} & -B_{z}  \tag{16}\\
B_{x} & 0 & E_{z} & -E_{y} \\
B_{y} & -E_{z} & 0 & E_{x} \\
B_{z} & E_{y} & -E_{x} & 0
\end{array}\right) .
$$

It is not difficult to show that

$$
\begin{equation*}
\tilde{F}^{\mu \nu}=\frac{1}{2} \epsilon^{\mu \nu \alpha \beta} F^{\alpha \beta} . \tag{17}
\end{equation*}
$$

The 4-dimensional Levi-Civita antisymmetric tensor $\epsilon^{\mu \nu \alpha \beta}$ is defined in an obvious way.
The second-rank antisymmetric tensor $\tilde{F}$ is called the field tensor while the second-rank antisymmetric tensor $\tilde{F}$ is called the dual field tensor.

Covariant Formulation The proper charge density $\rho_{0}$ is the charge density measured in the inertial reference frame $O^{\prime}$ where the charge is at rest. This is given by $\rho_{0}=Q / V_{0}$ where $V_{0}$ is the proper volume. Because the dimension along the direction of the motion is Lorentz contracted the volume $V$ measured in the reference frame $O$ is given by $V=\sqrt{1-u^{2} / c^{2}} V_{0}$. Thus the charge density measured in $O$ is

$$
\begin{equation*}
\rho=\frac{Q}{V}=\frac{\rho_{0}}{\sqrt{1-\frac{u^{2}}{c^{2}}}} \tag{18}
\end{equation*}
$$

The current density $\vec{J}$ measured in $O$ is proportional to the velocity $\vec{u}$ and to the current density $\rho$, viz

$$
\begin{equation*}
\vec{J}=\rho \vec{u}=\frac{\rho_{0} \vec{u}}{\sqrt{1-\frac{u^{2}}{c^{2}}}} \tag{19}
\end{equation*}
$$

The 4 -vector velocity $\eta^{\mu}$ is defined by

$$
\begin{equation*}
\eta^{\mu}=\frac{1}{\sqrt{1-\frac{u^{2}}{c^{2}}}}(c, \vec{u}) . \tag{20}
\end{equation*}
$$

Hence we can define the current density 4 -vector $J^{\mu}$ by

$$
\begin{equation*}
J^{\mu}=\rho_{0} \eta^{\mu}=\left(c \rho, J_{x}, J_{y}, J_{z}\right) \tag{21}
\end{equation*}
$$

The continuity equation $\vec{\nabla} \vec{J}=-\partial \rho / \partial t$ which expresses charge conservation will take the form

$$
\begin{equation*}
\partial_{\mu} J^{\mu}=0 \tag{22}
\end{equation*}
$$

In terms of $F_{\mu \nu}$ and $\tilde{F}_{\mu \nu}$ Maxwell's equations will take the form

$$
\begin{equation*}
\partial_{\mu} F^{\mu \nu}=\frac{1}{c} J^{\nu}, \partial_{\mu} \tilde{F}^{\mu \nu}=0 . \tag{23}
\end{equation*}
$$

The first equation yields Gauss's and Ampere-Maxwell's laws whereas the second equation yields Maxwell's third equation $\vec{\nabla} \vec{B}=0$ and Faraday's law.

It remains to write down a covariant Lorentz force. We start with the 4 -vector proper force given by

$$
\begin{equation*}
K^{\mu}=\frac{q}{c} \eta_{\nu} F^{\mu \nu} \tag{24}
\end{equation*}
$$

This is called the Minkowski force. The spatial part of this force is

$$
\begin{equation*}
\vec{K}=\frac{q}{\sqrt{1-\frac{u^{2}}{c^{2}}}}\left(\vec{E}+\frac{1}{c} \vec{u} \times \vec{B}\right) . \tag{25}
\end{equation*}
$$

We have also

$$
\begin{equation*}
K^{\mu}=\frac{d p^{\mu}}{d \tau} \tag{26}
\end{equation*}
$$

In other words

$$
\begin{equation*}
\vec{K}=\frac{d \vec{p}}{d \tau}=\frac{d t}{d \tau} \vec{F}=\frac{1}{\sqrt{1-\frac{u^{2}}{c^{2}}}} \vec{F} \tag{27}
\end{equation*}
$$

This leads precisely to the Lorentz force law

$$
\begin{equation*}
\vec{F}=q\left(\vec{E}+\frac{1}{c} \vec{u} \times \vec{B}\right) . \tag{28}
\end{equation*}
$$

### 1.2 Gauge Potentials and Gauge Transformations

The electric and magnetic fields $\vec{E}$ and $\vec{B}$ can be expressed in terms of a scalar potential $V$ and a vector potential $\vec{A}$ as

$$
\begin{gather*}
\vec{B}=\vec{\nabla} \times \vec{A} .  \tag{29}\\
\vec{E}=-\frac{1}{c}\left(\vec{\nabla} V+\frac{\partial \vec{A}}{\partial t}\right) . \tag{30}
\end{gather*}
$$

We construct the 4 -vector potential $A^{\mu}$ as

$$
\begin{equation*}
A^{\mu}=(V / c, \vec{A}) \tag{31}
\end{equation*}
$$

The field tensor $F_{\mu \nu}$ can be rewritten in terms of $A_{\mu}$ as

$$
\begin{equation*}
F_{\mu \nu}=\partial_{\mu} A_{\nu}-\partial_{\nu} A_{\mu} . \tag{32}
\end{equation*}
$$

This equation is actually equivalent to the two equations (29) and (30). The homogeneous Maxwell's equation $\partial_{\mu} \tilde{F}^{\mu \nu}=0$ is automatically solved by this ansatz. The inhomogeneous Maxwell's equation $\partial_{\mu} F^{\mu \nu}=J^{\nu} / c$ becomes

$$
\begin{equation*}
\partial_{\mu} \partial^{\mu} A^{\nu}-\partial^{\nu} \partial_{\mu} A^{\mu}=\frac{1}{c} J^{\nu} \tag{33}
\end{equation*}
$$

We have a gauge freedom in choosing $A^{\mu}$ given by local gauge transformations of the form (with $\lambda$ any scalar function)

$$
\begin{equation*}
A^{\mu} \longrightarrow A^{\prime \mu}=A^{\mu}+\partial^{\mu} \lambda \tag{34}
\end{equation*}
$$

Indeed under this transformation we have

$$
\begin{equation*}
F^{\mu \nu} \longrightarrow F^{\prime \mu \nu}=F^{\mu \nu} \tag{35}
\end{equation*}
$$

These local gauge transformations form a (gauge) group. In this case the group is just the abelian $U(1)$ unitary group. The invariance of the theory under these transformations is termed a gauge invariance. The 4 -vector potential $A^{\mu}$ is called a gauge potential or a gauge field. We make use of the invariance under gauge transformations by working with a gauge potential $A^{\mu}$ which satisfies some extra conditions. This procedure is known as gauge fixing. Some of the gauge conditions so often used are

$$
\begin{gather*}
\partial_{\mu} A^{\mu}=0, \text { Lorentz Gauge. }  \tag{36}\\
\partial_{i} A^{i}=0, \text { Coulomb Gauge. }  \tag{37}\\
A^{0}=0, \text { Temporal Gauge. }  \tag{38}\\
A^{3}=0, \text { Axial Gauge. } \tag{39}
\end{gather*}
$$

In the Lorentz gauge the equations of motion (33) become

$$
\begin{equation*}
\partial_{\mu} \partial^{\mu} A^{\nu}=\frac{1}{c} J^{\nu} . \tag{40}
\end{equation*}
$$

Clearly we still have a gauge freedom $A^{\mu} \longrightarrow A^{\prime \mu}=A^{\mu}+\partial^{\mu} \phi$ where $\partial_{\mu} \partial^{\mu} \phi=0$. In other words if $A^{\mu}$ satisfies the Lorentz gauge $\partial_{\mu} A^{\mu}=0$ then $A^{\prime \mu}$ will also satisfy the Lorentz gauge, i.e. $\partial_{\mu} A^{\prime \mu}=$ 0 iff $\partial_{\mu} \partial^{\mu} \phi=0$. This residual gauge symmetry can be fixed by imposing another condition such as the temporal gauge $A^{0}=0$. We have therefore 2 constraints imposed on the components of the gauge potential $A^{\mu}$ which means that only two of them are really independent.

### 1.3 Maxwell's Lagrangian Density

The equations of motion of the gauge field $A^{\mu}$ is

$$
\begin{equation*}
\partial_{\mu} \partial^{\mu} A^{\nu}-\partial^{\nu} \partial_{\mu} A^{\mu}=\frac{1}{c} J^{\nu} \tag{41}
\end{equation*}
$$

These equations of motion should be derived from a local Lagrangian density $\mathcal{L}$, i.e. a Lagrangian which depends only on the fields and their first derivatives at the point $\vec{x}$. We have then

$$
\begin{equation*}
\mathcal{L}=\mathcal{L}\left(A_{\mu}, \partial_{\nu} A_{\mu}\right) \tag{42}
\end{equation*}
$$

The Lagrangian is the integral over $\vec{x}$ of the Lagrangian density, viz

$$
\begin{equation*}
L=\int d \vec{x} \mathcal{L} \tag{43}
\end{equation*}
$$

The action is the integral over time of $L$, namely

$$
\begin{equation*}
S=\int d t L=\int d^{4} x \mathcal{L} \tag{44}
\end{equation*}
$$

We compute

$$
\begin{align*}
\delta S & =\int d^{4} x \delta \mathcal{L} \\
& =\int d^{4} x\left[\delta A_{\nu} \frac{\delta \mathcal{L}}{\delta A_{\nu}}-\delta A_{\nu} \partial_{\mu} \frac{\delta \mathcal{L}}{\delta \partial_{\mu} A_{\nu}}+\partial_{\mu}\left(\delta A_{\nu} \frac{\delta \mathcal{L}}{\delta \partial_{\mu} A_{\nu}}\right)\right] . \tag{45}
\end{align*}
$$

The surface term is zero because the field $A_{\nu}$ at infinity is assumed to be zero and thus

$$
\begin{equation*}
\delta A_{\nu}=0, x^{\mu} \longrightarrow \pm \infty \tag{46}
\end{equation*}
$$

We get

$$
\begin{equation*}
\delta S=\int d^{4} x \delta A_{\nu}\left[\frac{\delta \mathcal{L}}{\delta A_{\nu}}-\partial_{\mu} \frac{\delta \mathcal{L}}{\delta \partial_{\mu} A_{\nu}}\right] \tag{47}
\end{equation*}
$$

The principle of least action $\delta S=0$ yields therefore the Euler-Lagrange equations

$$
\begin{equation*}
\frac{\delta \mathcal{L}}{\delta A_{\nu}}-\partial_{\mu} \frac{\delta \mathcal{L}}{\delta \partial_{\mu} A_{\nu}}=0 . \tag{48}
\end{equation*}
$$

Firstly the Lagrangian density $\mathcal{L}$ is a Lorentz scalar. Secondly the equations of motion (41) are linear in the field $A^{\mu}$ and hence the Lagrangian density $\mathcal{L}$ can at most be quadratic in $A^{\mu}$. The most general form of $\mathcal{L}$ which is quadratic in $A^{\mu}$ is

$$
\begin{equation*}
\mathcal{L}_{\text {Maxwell }}=\alpha\left(\partial_{\mu} A^{\mu}\right)^{2}+\beta\left(\partial_{\mu} A^{\nu}\right)\left(\partial^{\mu} A_{\nu}\right)+\gamma\left(\partial_{\mu} A^{\nu}\right)\left(\partial_{\nu} A^{\mu}\right)+\delta A_{\mu} A^{\mu}+\epsilon J_{\mu} A^{\mu} . \tag{49}
\end{equation*}
$$

We calculate

$$
\begin{gather*}
\frac{\delta \mathcal{L}_{\text {Maxwell }}}{\delta A_{\rho}}=2 \delta A^{\rho}+\epsilon J^{\rho} .  \tag{50}\\
\frac{\delta \mathcal{L}_{\text {Maxwell }}}{\delta \partial_{\sigma} A_{\rho}}=2 \alpha \eta^{\sigma \rho} \partial_{\mu} A^{\mu}+2 \beta \partial^{\sigma} A^{\rho}+2 \gamma \partial^{\rho} A^{\sigma} . \tag{51}
\end{gather*}
$$

Thus

$$
\begin{equation*}
\frac{\delta \mathcal{L}_{\text {Maxwell }}}{\delta A_{\rho}}-\partial_{\sigma} \frac{\delta \mathcal{L}_{\text {Maxwell }}}{\delta \partial_{\sigma} A_{\rho}}=0 \Leftrightarrow 2 \beta \partial_{\sigma} \partial^{\sigma} A^{\rho}+2(\alpha+\gamma) \partial^{\rho} \partial_{\sigma} A^{\sigma}-2 \delta A^{\rho}=\epsilon J^{\rho} . \tag{52}
\end{equation*}
$$

By comparing with the equations of motion (41) we obtain immediately (with $\zeta$ an arbitrary parameter)

$$
\begin{equation*}
2 \beta=-\zeta, 2(\alpha+\gamma)=\zeta, \delta=0, \epsilon=-\frac{1}{c} \zeta . \tag{53}
\end{equation*}
$$

We get the Lagrangian density

$$
\begin{align*}
\mathcal{L}_{\text {Maxwell }} & =\alpha\left(\left(\partial_{\mu} A^{\mu}\right)^{2}-\partial_{\mu} A_{\nu} \partial^{\nu} A^{\mu}\right)-\frac{\zeta}{2}\left(\partial_{\mu} A_{\nu} \partial^{\mu} A^{\nu}-\partial_{\mu} A_{\nu} \partial^{\nu} A^{\mu}\right)-\frac{1}{c} \zeta J_{\mu} A^{\mu} \\
& =\alpha \partial_{\mu}\left(A^{\mu} \partial_{\nu} A^{\nu}-A^{\nu} \partial_{\nu} A^{\mu}\right)-\frac{\zeta}{4} F_{\mu \nu} F^{\mu \nu}-\frac{1}{c} \zeta J_{\mu} A^{\mu} \tag{54}
\end{align*}
$$

The first term is a total derivative which vanishes since the field $A_{\nu}$ vanishes at infinity. Thus we end up with the Lagrangian density

$$
\begin{equation*}
\mathcal{L}_{\text {Maxwell }}=-\frac{\zeta}{4} F_{\mu \nu} F^{\mu \nu}-\frac{1}{c} \zeta J_{\mu} A^{\mu} . \tag{55}
\end{equation*}
$$

In order to get a correctly normalized Hamiltonian density from this Lagrangian density we choose $\zeta=1$. We get finally the result

$$
\begin{equation*}
\mathcal{L}_{\text {Maxwell }}=-\frac{1}{4} F_{\mu \nu} F^{\mu \nu}-\frac{1}{c} J_{\mu} A^{\mu} \tag{56}
\end{equation*}
$$

### 1.4 Polarization Vectors

In this section we will consider a free electromagnetic gauge field $A^{\mu}$, i.e. we take $J^{\mu}=0$. In the Feynman gauge (see next section for detail) the equations of motion of the gauge field $A^{\mu}$ read

$$
\begin{equation*}
\partial_{\mu} \partial^{\mu} A^{\nu}=0 \tag{57}
\end{equation*}
$$

These are 4 massless Klein-Gordon equations. The solutions are plane-waves of the form

$$
\begin{equation*}
A^{\mu}=e^{ \pm \frac{i}{\hbar} p x} \epsilon_{\lambda}^{\mu}(\vec{p}) \tag{58}
\end{equation*}
$$

The $4-$ momentum $p^{\mu}$ is such that

$$
\begin{equation*}
p_{\mu} p^{\mu}=0 \tag{59}
\end{equation*}
$$

There are 4 independent polarization vectors $\epsilon_{\lambda}^{\mu}(\vec{p})$. The polarization vectors for $\lambda=1,2$ are termed transverse , the polarization vector for $\lambda=3$ is termed longitudinal and the polarization vector for $\lambda=0$ is termed scalar.

In the case of the Lorentz condition $\partial_{\mu} A^{\mu}=0$ the polarization vectors $\epsilon_{\lambda}^{\mu}(\vec{p})$ are found to satisfy $p_{\mu} \epsilon_{\lambda}^{\mu}(\vec{p})=0$. By imposing also the temporal gauge condition $A^{0}=0$ we get $\epsilon_{\lambda}^{0}(\vec{p})=0$ and the Lorentz condition becomes the Coulomb gauge $\vec{p} \cdot \vec{\epsilon}_{\lambda}(\vec{p})=0$.

Motivated by this we choose the polarization vectors $\epsilon_{\lambda}^{\mu}(\vec{p})$ as follows. We pick a fixed Lorentz frame in which the time axis is along some timelike unit 4 -vector $n^{\mu}$, viz

$$
\begin{equation*}
n_{\mu} n^{\mu}=1, n^{0}>0 \tag{60}
\end{equation*}
$$

The transverse polarization vectors will be chosen in the plane orthogonal to $n^{\mu}$ and to the $4-$ momentum $p^{\mu}$. The second requirement is equivalent to the Lorentz condition:

$$
\begin{equation*}
p_{\mu} \epsilon_{\lambda}^{\mu}(\vec{p})=0, \lambda=1,2 . \tag{61}
\end{equation*}
$$

The first requirement means that

$$
\begin{equation*}
n_{\mu} \epsilon_{\lambda}^{\mu}(\vec{p})=0, \lambda=1,2 . \tag{62}
\end{equation*}
$$

The transverse polarization vectors will furthermore be chosen to be spacelike (which is equivalent to the temporal gauge condition) and orthonormal, i.e.

$$
\begin{equation*}
\epsilon_{1}^{\mu}(\vec{p})=\left(0, \vec{\epsilon}_{1}(\vec{p})\right), \epsilon_{2}^{\mu}(\vec{p})=\left(0, \vec{\epsilon}_{2}(\vec{p})\right), \tag{63}
\end{equation*}
$$

and

$$
\begin{equation*}
\vec{\epsilon}_{i}(\vec{p}) \cdot \vec{\epsilon}_{j}(\vec{p})=\delta_{i j} . \tag{64}
\end{equation*}
$$

The longitudinal polarization vector is chosen in the plane $\left(n^{\mu}, p^{\mu}\right)$ orthogonal to $n^{\mu}$. More precisely we choose

$$
\begin{equation*}
\epsilon_{3}^{\mu}(\vec{p})=\frac{p^{\mu}-(n p) n^{\mu}}{n p} \tag{65}
\end{equation*}
$$

For $n^{\mu}=(1,0,0,0)$ we get $\epsilon_{3}^{\mu}(\vec{p})=(0, \vec{p} /|\vec{p}|)$. This longitudinal polarization vector satisfies

$$
\begin{equation*}
\epsilon_{3}^{\mu}(\vec{p}) \epsilon_{3 \mu}(\vec{p})=-1, \epsilon_{3}^{\mu}(\vec{p}) n_{\mu}=0, \epsilon_{3}^{\mu}(\vec{p}) \epsilon_{\lambda \mu}(\vec{p})=0, \lambda=1,2 . \tag{66}
\end{equation*}
$$

Let us also remark

$$
\begin{equation*}
p_{\mu} \epsilon_{3}^{\mu}(\vec{p})=-n^{\mu} p_{\mu} . \tag{67}
\end{equation*}
$$

Indeed for a massless vector field it is impossible to choose a third polarization vector which is transevrse. A massless particle can only have two polarization states regardless of its spin whereas a massive particle with spin $j$ can have $2 j+1$ polarization states.

The scalar polarization vector is chosen to be $n^{\mu}$ itself, namely

$$
\begin{equation*}
\epsilon_{0}^{\mu}(\vec{p})=n^{\mu} . \tag{68}
\end{equation*}
$$

In summary the polarization vectors $\epsilon_{\lambda}^{\mu}(\vec{p})$ are chosen such that they satisfy the orthonormalization condition

$$
\begin{equation*}
\epsilon_{\lambda}^{\mu}(\vec{p}) \epsilon_{\lambda^{\prime} \mu}(\vec{p})=\eta_{\lambda \lambda^{\prime}} . \tag{69}
\end{equation*}
$$

They also satisfy

$$
\begin{equation*}
p_{\mu} \epsilon_{1}^{\mu}(\vec{p})=p_{\mu} \epsilon_{2}^{\mu}(\vec{p})=0,-p_{\mu} \epsilon_{3}^{\mu}(\vec{p})=p_{\mu} \epsilon_{0}^{\mu}(\vec{p})=n^{\mu} p^{\mu} \tag{70}
\end{equation*}
$$

By choosing $n^{\mu}=(1,0,0,0)$ and $\vec{p}=(0,0, p)$ we obtain $\epsilon_{0}^{\mu}(\vec{p})=(1,0,0,0), \epsilon_{1}^{\mu}(\vec{p})=(0,1,0,0)$, $\epsilon_{2}^{\mu}(\vec{p})=(0,0,1,0)$ and $\epsilon_{3}^{\mu}(\vec{p})=(0,0,0,1)$.

We compute in the reference frame in which $n^{\mu}=(1,0,0,0)$ the completeness relations

$$
\begin{align*}
& \sum_{\lambda=0}^{3} \eta_{\lambda \lambda} \epsilon_{\lambda}^{0}(\vec{p}) \epsilon_{\lambda}^{0}(\vec{p})=\epsilon_{0}^{0}\left(\vec{p} \epsilon_{0}^{0}(\vec{p})=1\right.  \tag{71}\\
& \sum_{\lambda=0}^{3} \eta_{\lambda \lambda} \epsilon_{\lambda}^{0}(\vec{p}) \epsilon_{\lambda}^{i}(\vec{p})=\epsilon_{0}^{0}(\vec{p}) \epsilon_{0}^{i}(\vec{p})=0  \tag{72}\\
& \sum_{\lambda=0}^{3} \eta_{\lambda \lambda} \epsilon_{\lambda}^{i}(\vec{p}) \epsilon_{\lambda}^{j}(\vec{p})=-\sum_{\lambda=1}^{3} \epsilon_{\lambda}^{i}(\vec{p}) \epsilon_{\lambda}^{j}(\vec{p}) \tag{73}
\end{align*}
$$

The completeness relation for a 3 -dimensional orthogonal dreibein is

$$
\begin{equation*}
\sum_{\lambda=1}^{3} \epsilon_{\lambda}^{i}(\vec{p}) \epsilon_{\lambda}^{j}(\vec{p})=\delta^{i j} \tag{74}
\end{equation*}
$$

This can be checked for example by going to the reference frame in which $\vec{p}=(0,0, p)$. Hence we get

$$
\begin{equation*}
\sum_{\lambda=0}^{3} \eta_{\lambda \lambda} \epsilon_{\lambda}^{i}(\vec{p}) \epsilon_{\lambda}^{j}(\vec{p})=\eta^{i j} \tag{75}
\end{equation*}
$$

In summary we get the completeness relations

$$
\begin{equation*}
\sum_{\lambda=0}^{3} \eta_{\lambda \lambda} \epsilon_{\lambda}^{\mu}(\vec{p}) \epsilon_{\lambda}^{\nu}(\vec{p})=\eta^{\mu \nu} \tag{76}
\end{equation*}
$$

From this equation we derive that the sum over the transverse polarization states is given by

$$
\begin{equation*}
\sum_{\lambda=1}^{2} \epsilon_{\lambda}^{\mu}(\vec{p}) \epsilon_{\lambda}^{\nu}(\vec{p})=-\eta^{\mu \nu}-\frac{p^{\mu} p^{\nu}}{(n p)^{2}}+\frac{p^{\mu} n^{\nu}+p^{\nu} n^{\mu}}{n p} \tag{77}
\end{equation*}
$$

### 1.5 Quantization of The Electromagnetic Gauge Field

We start with the Lagrangian density

$$
\begin{equation*}
\mathcal{L}_{\text {Maxwell }}=-\frac{1}{4} F_{\mu \nu} F^{\mu \nu}-\frac{1}{c} J_{\mu} A^{\mu} . \tag{78}
\end{equation*}
$$

The field tensor is defined by $F_{\mu \nu}=\partial_{\mu} A_{\nu}-\partial_{\nu} A_{\mu}$. The equations of motion of the gauge field $A^{\mu}$ derived from the Lagrangian density $\mathcal{L}_{\text {Maxwell }}$ are given by

$$
\begin{equation*}
\partial_{\mu} \partial^{\mu} A^{\nu}-\partial^{\nu} \partial_{\mu} A^{\mu}=\frac{1}{c} J^{\nu} \tag{79}
\end{equation*}
$$

There is a freedom in the definition of the gauge field $A^{\mu}$ given by the gauge transformations

$$
\begin{equation*}
A^{\mu} \longrightarrow A^{\prime \mu}=A^{\mu}+\partial^{\mu} \lambda \tag{80}
\end{equation*}
$$

The form of the equations of motion (79) strongly suggest the Lorentz condition

$$
\begin{equation*}
\partial^{\mu} A_{\mu}=0 . \tag{81}
\end{equation*}
$$

We incorporate this constraint via a Lagrange multiplier $\zeta$ in order to obtain a gauge-fixed Lagrangian density, viz

$$
\begin{equation*}
\mathcal{L}_{\text {gauge-fixed }}=-\frac{1}{4} F_{\mu \nu} F^{\mu \nu}-\frac{1}{2} \zeta\left(\partial^{\mu} A_{\mu}\right)^{2}-\frac{1}{c} J_{\mu} A^{\mu} \tag{82}
\end{equation*}
$$

The added extra term is known as a gauge-fixing term. This modification was proposed first by Fermi. The equations of motion derived from this Lagrangian density are

$$
\begin{equation*}
\partial_{\mu} \partial^{\mu} A^{\nu}-(1-\zeta) \partial^{\nu} \partial_{\mu} A^{\mu}=\frac{1}{c} J^{\nu} \tag{83}
\end{equation*}
$$

These are equivalent to Maxwell's equations in the Lorentz gauge. To see this we remark first that

$$
\begin{equation*}
\partial_{\nu}\left(\partial_{\mu} \partial^{\mu} A^{\nu}-(1-\zeta) \partial^{\nu} \partial_{\mu} A^{\mu}\right)=\frac{1}{c} \partial_{\nu} J^{\nu} \tag{84}
\end{equation*}
$$

Gauge invariance requires current conservation, i.e. we must have $\partial_{\nu} J^{\nu}=0$. Thus we obtain

$$
\begin{equation*}
\partial_{\mu} \partial^{\mu} \phi=0, \phi=\partial_{\mu} A^{\mu} . \tag{85}
\end{equation*}
$$

This is a Cauchy initial-value problem for $\partial_{\mu} A^{\mu}$. In other words if $\partial_{\mu} A^{\mu}=0$ and $\partial_{0}\left(\partial_{\mu} A^{\mu}\right)=0$ at an initial time $t=t_{0}$ then $\partial_{\mu} A^{\mu}=0$ at all times. Hence (83) are equivalent to Maxwell's equations in the Lorentz gauge.

We will work in the so-called Feynman gauge which corresponds to $\zeta=1$ and for simplicity we will set $J^{\mu}=0$. The equations of motion become the massless Klein-Gordon equations

$$
\begin{equation*}
\partial_{\mu} \partial^{\mu} A^{\nu}=0 . \tag{86}
\end{equation*}
$$

These can be derived from the Lagrangian density

$$
\begin{equation*}
\mathcal{L}=-\frac{1}{2} \partial_{\mu} A_{\nu} \partial^{\mu} A^{\nu} . \tag{87}
\end{equation*}
$$

This Lagrangian density is equal to the gauge-fixed Lagrangian density $\mathcal{L}_{\text {gauge-fixed }}$ modulo a total derivative term, viz

$$
\begin{equation*}
\mathcal{L}_{\text {gauge-fixed }}=\mathcal{L}+\text { total derivative term. } \tag{88}
\end{equation*}
$$

The conjugate momentum field is defined by

$$
\begin{align*}
\pi_{\mu} & =\frac{\delta \mathcal{L}}{\delta \partial_{t} A^{\mu}} \\
& =-\frac{1}{c^{2}} \partial_{t} A_{\mu} \tag{89}
\end{align*}
$$

The Hamiltonian density is then given by

$$
\begin{align*}
\mathcal{H} & =\pi_{\mu} \partial_{t} A^{\mu}-\mathcal{L} \\
& =\frac{1}{2} \partial_{i} A_{\mu} \partial^{i} A^{\mu}-\frac{1}{2} \partial_{0} A_{\mu} \partial^{0} A^{\mu} \\
& =\frac{1}{2}\left(\partial_{0} \vec{A}\right)^{2}+\frac{1}{2}(\vec{\nabla} \vec{A})^{2}-\frac{1}{2}\left(\partial_{0} A^{0}\right)^{2}-\frac{1}{2}\left(\vec{\nabla} A^{0}\right)^{2} . \tag{90}
\end{align*}
$$

The contribution of the zero-component $A^{0}$ of the gauge field is negative. Thus the Hamiltonian density is not positive definite as it should be. This is potentially a severe problem which will be solved by means of the gauge condition.

We have already found that there are 4 independent polarization vectors $\epsilon_{\lambda}^{\mu}(\vec{p})$ for each momentm $\vec{p}$. The $4-$ momentum $p^{\mu}$ satisfies $p^{\mu} p_{\mu}=0$, i.e. $\left(p^{0}\right)^{2}=\vec{p}^{2}$. We define $\omega(\vec{p})=\frac{c}{\hbar} p^{0}=$ $\frac{c}{\hbar}|\vec{p}|$. The most general solution of the classical equations of motion in the Lorentz gauge can be put in the form

$$
\begin{equation*}
A^{\mu}=c \int \frac{d^{3} \vec{p}}{(2 \pi \hbar)^{3}} \frac{1}{\sqrt{2 \omega(\vec{p})}} \sum_{\lambda=0}^{3}\left(e^{-\frac{i}{\hbar} p x} \epsilon_{\lambda}^{\mu}(\vec{p}) a(\vec{p}, \lambda)+e^{\frac{i}{\hbar} p x} \epsilon_{\lambda}^{\mu}(\vec{p}) a(\vec{p}, \lambda)^{*}\right)_{p^{0}=|\vec{p}|} \tag{91}
\end{equation*}
$$

We compute

$$
\begin{align*}
\frac{1}{2} \int \partial_{i} A^{\mu} \partial^{i} A^{\mu} & =-c^{2} \int \frac{d^{3} \vec{p}}{(2 \pi \hbar)^{3}} \frac{1}{4 \omega(\vec{p}} \frac{p^{i} p^{i}}{\hbar^{2}} \sum_{\lambda, \lambda^{\prime}=0}^{3} \epsilon_{\lambda}^{\mu}(\vec{p}) \epsilon_{\lambda^{\prime} \mu}(\vec{p})\left(a(\vec{p}, \lambda) a\left(\vec{p}, \lambda^{\prime}\right)^{*}+a(\vec{p}, \lambda)^{*} a\left(\vec{p}, \lambda^{\prime}\right)\right) \\
& -c^{2} \int \frac{d^{3} \vec{p}}{(2 \pi \hbar)^{3}} \frac{1}{4 \omega(\vec{p})} \frac{p^{i} p^{i}}{\hbar^{2}} \sum_{\lambda, \lambda^{\prime}=0}^{3} \epsilon_{\lambda}^{\mu}(\vec{p}) \epsilon_{\lambda^{\prime} \mu}(-\vec{p})\left(e^{-\frac{2 i}{\hbar} p^{0} x^{0}} a(\vec{p}, \lambda) a\left(-\vec{p}, \lambda^{\prime}\right)\right. \\
& \left.+e^{+\frac{2 i}{\hbar} p^{0} x^{0}} a(\vec{p}, \lambda)^{*} a\left(-\vec{p}, \lambda^{\prime}\right)^{*}\right)  \tag{92}\\
\frac{1}{2} \int \partial_{0} A^{\mu} \partial^{0} A^{\mu} & =c^{2} \int \frac{d^{3} \vec{p}}{(2 \pi \hbar)^{3}} \frac{1}{4 \omega(\vec{p})} \frac{p^{0} p^{0}}{\hbar^{2}} \sum_{\lambda, \lambda^{\prime}=0}^{3} \epsilon_{\lambda}^{\mu}(\vec{p}) \epsilon_{\lambda^{\prime} \mu}(\vec{p})\left(a(\vec{p}, \lambda) a\left(\vec{p}, \lambda^{\prime}\right)^{*}+a(\vec{p}, \lambda)^{*} a\left(\vec{p}, \lambda^{\prime}\right)\right) \\
& -c^{2} \int \frac{d^{3} \vec{p}}{(2 \pi \hbar)^{3}} \frac{1}{4 \omega(\vec{p})} \frac{p^{0} p^{0}}{\hbar^{2}} \sum_{\lambda, \lambda^{\prime}=0}^{3} \epsilon_{\lambda}^{\mu}(\vec{p}) \epsilon_{\lambda^{\prime} \mu}(-\vec{p})\left(e^{-\frac{2 i}{\hbar} p^{0} x^{0}} a(\vec{p}, \lambda) a\left(-\vec{p}, \lambda^{\prime}\right)\right. \\
& \left.+e^{+\frac{2 i}{\hbar} p^{0} x^{0}} a(\vec{p}, \lambda)^{*} a\left(-\vec{p}, \lambda^{\prime}\right)^{*}\right) . \tag{93}
\end{align*}
$$

The Hamiltonian becomes (since $p^{0} p^{0}=p^{i} p^{i}$ )

$$
\begin{align*}
H & =\int d^{3} x\left(\frac{1}{2} \partial_{i} A_{\mu} \partial^{i} A^{\mu}-\frac{1}{2} \partial_{0} A_{\mu} \partial^{0} A^{\mu}\right) \\
& =-c^{2} \int \frac{d^{3} \vec{p}}{(2 \pi \hbar)^{3}} \frac{1}{2 \omega(\vec{p})} \frac{p^{0} p^{0}}{\hbar^{2}} \sum_{\lambda, \lambda^{\prime}=0}^{3} \epsilon_{\lambda}^{\mu}(\vec{p}) \epsilon_{\lambda^{\prime} \mu}(\vec{p})\left(a(\vec{p}, \lambda) a\left(\vec{p}, \lambda^{\prime}\right)^{*}+a(\vec{p}, \lambda)^{*} a\left(\vec{p}, \lambda^{\prime}\right)\right) \\
& =-\int \frac{d^{3} \vec{p}}{(2 \pi \hbar)^{3}} \frac{\omega(\vec{p})}{2} \sum_{\lambda, \lambda^{\prime}=0}^{3} \epsilon_{\lambda}^{\mu}(\vec{p}) \epsilon_{\lambda^{\prime} \mu}(\vec{p})\left(a(\vec{p}, \lambda) a\left(\vec{p}, \lambda^{\prime}\right)^{*}+a(\vec{p}, \lambda)^{*} a\left(\vec{p}, \lambda^{\prime}\right)\right) \\
& =-\int \frac{d^{3} \vec{p}}{(2 \pi \hbar)^{3}} \frac{\omega(\vec{p})}{2} \sum_{\lambda=0}^{3} \eta_{\lambda \lambda}\left(a(\vec{p}, \lambda) a(\vec{p}, \lambda)^{*}+a(\vec{p}, \lambda)^{*} a(\vec{p}, \lambda)\right) . \tag{94}
\end{align*}
$$

In the quantum theory $A^{\mu}$ becomes the operator

$$
\begin{equation*}
\hat{A}^{\mu}=c \int \frac{d^{3} \vec{p}}{(2 \pi \hbar)^{3}} \frac{1}{\sqrt{2 \omega(\vec{p})}} \sum_{\lambda=0}^{3}\left(e^{-\frac{i}{\hbar} p x} \epsilon_{\lambda}^{\mu}(\vec{p}) \hat{a}(\vec{p}, \lambda)+e^{\frac{i}{\hbar} p x} \epsilon_{\lambda}^{\mu}(\vec{p}) \hat{a}(\vec{p}, \lambda)^{+}\right)_{p^{0}=|\vec{p}|} \tag{95}
\end{equation*}
$$

The conjugate momentum $\pi^{\mu}$ becomes the operator

$$
\begin{align*}
\hat{\pi}^{\mu} & =-\frac{1}{c^{2}} \partial_{t} \hat{A}^{\mu} \\
& =\int \frac{d^{3} \vec{p}}{(2 \pi \hbar)^{3}} \frac{i}{c} \sqrt{\frac{\omega(\vec{p})}{2}} \sum_{\lambda=0}^{3}\left(e^{-\frac{i}{\hbar} p x} \epsilon_{\lambda}^{\mu}(\vec{p}) \hat{a}(\vec{p}, \lambda)-e^{\frac{i}{\hbar} p x} \epsilon_{\lambda}^{\mu}(\vec{p}) \hat{a}(\vec{p}, \lambda)^{+}\right)_{p^{0}=|\vec{p}|} \tag{96}
\end{align*}
$$

We impose the equal-time canonical commutation relations

$$
\begin{gather*}
{\left[\hat{A}^{\mu}\left(x^{0}, \vec{x}\right), \hat{\pi}^{\nu}\left(x^{0}, \vec{y}\right)\right]=i \hbar \eta^{\mu \nu} \delta^{3}(\vec{x}-\vec{y}) .}  \tag{97}\\
{\left[\hat{A}^{\mu}\left(x^{0}, \vec{x}\right), \hat{A}^{\nu}\left(x^{0}, \vec{y}\right)\right]=\left[\hat{\pi}^{\mu}\left(x^{0}, \vec{x}\right), \hat{\pi}^{\nu}\left(x^{0}, \vec{y}\right)\right]=0 .} \tag{98}
\end{gather*}
$$

The operators $\hat{a}^{+}$and $\hat{a}$ are expected to be precisely the creation and annihilation operators. In other words we expect that

$$
\begin{equation*}
\left[\hat{a}(\vec{p}, \lambda), \hat{a}\left(\vec{q}, \lambda^{\prime}\right)\right]=\left[\hat{a}(\vec{p}, \lambda)^{+}, \hat{a}\left(\vec{q}, \lambda^{\prime}\right)^{+}\right]=0 \tag{99}
\end{equation*}
$$

We compute then

$$
\begin{align*}
{\left[\hat{A}^{\mu}\left(x^{0}, \vec{x}\right), \hat{\pi}^{\nu}\left(x^{0}, \vec{y}\right)\right]=} & -i \int \frac{d^{3} \vec{p}}{(2 \pi \hbar)^{3}} \int \frac{d^{3} \vec{q}}{(2 \pi \hbar)^{3}} \frac{1}{\sqrt{2 \omega(\vec{p})}} \sqrt{\frac{\omega(\vec{q})}{2}} \sum_{\lambda, \lambda^{\prime}=0}^{3} \epsilon_{\lambda}^{\mu}(\vec{p}) \epsilon_{\lambda^{\prime}}^{\nu}(\vec{q})( \\
& \left.e^{-\frac{i}{\hbar} p x} e^{+\frac{i}{\hbar} q y}\left[\hat{a}(\vec{p}, \lambda), \hat{a}\left(\vec{q}, \lambda^{\prime}\right)^{+}\right]+e^{+\frac{i}{\hbar} p x} e^{-\frac{i}{\hbar} q y}\left[\hat{a}\left(\vec{q}, \lambda^{\prime}\right), \hat{a}(\vec{p}, \lambda)^{+}\right]\right) \tag{100}
\end{align*}
$$

We can immediately conclude that we must have

$$
\begin{equation*}
\left[\hat{a}(\vec{p}, \lambda), \hat{a}\left(\vec{q}, \lambda^{\prime}\right)^{+}\right]=-\eta_{\lambda \lambda^{\prime}} \hbar(2 \pi \hbar)^{3} \delta^{3}(\vec{p}-\vec{q}) . \tag{101}
\end{equation*}
$$

By using (99) and (101) we can also verify the equal-time canonical commutation relations (98). The minus sign in (101) causes serious problems. For transverse $(i=1,2)$ and longitudinal $(i=3)$ polarizations the number operator is given as usual by $\hat{a}(\vec{p}, i)^{+} \hat{a}(\vec{p}, i)$. Indeed we compute

$$
\begin{align*}
& {\left[\hat{a}(\vec{p}, i)^{+} \hat{a}(\vec{p}, i), \hat{a}(\vec{q}, i)\right]=-\hbar(2 \pi \hbar)^{3} \delta^{3}(\vec{p}-\vec{q}) \hat{a}(\vec{q}, i)} \\
& {\left[\hat{a}(\vec{p}, i)^{+} \hat{a}(\vec{p}, i), \hat{a}(\vec{q}, i)^{+}\right]=\hbar(2 \pi \hbar)^{3} \delta^{3}(\vec{p}-\vec{q}) \hat{a}(\vec{q}, i)^{+} .} \tag{102}
\end{align*}
$$

In the case of the scalar polarization $(\lambda=0)$ the number operator is given by $-\hat{a}(\vec{p}, 0)^{+} \hat{a}(\vec{p}, 0)$ since

$$
\begin{align*}
& {\left[-\hat{a}(\vec{p}, 0)^{+} \hat{a}(\vec{p}, 0), \hat{a}(\vec{q}, 0)\right]=-\hbar(2 \pi \hbar)^{3} \delta^{3}(\vec{p}-\vec{q}) \hat{a}(\vec{q}, 0)} \\
& {\left[-\hat{a}(\vec{p}, 0)^{+} \hat{a}(\vec{p}, 0), \hat{a}(\vec{q}, 0)^{+}\right]=\hbar(2 \pi \hbar)^{3} \delta^{3}(\vec{p}-\vec{q}) \hat{a}(\vec{q}, 0)^{+} .} \tag{103}
\end{align*}
$$

In the quantum theory the Hamiltonian becomes the operator

$$
\begin{equation*}
\hat{H}=-\int \frac{d^{3} \vec{p}}{(2 \pi \hbar)^{3}} \frac{\omega(\vec{p})}{2} \sum_{\lambda=0}^{3} \eta_{\lambda \lambda}\left(\hat{a}(\vec{p}, \lambda) \hat{a}(\vec{p}, \lambda)^{+}+\hat{a}(\vec{p}, \lambda)^{+} \hat{a}(\vec{p}, \lambda)\right) \tag{104}
\end{equation*}
$$

As before normal ordering yields the Hamiltonian operator

$$
\begin{align*}
\hat{H} & =-\int \frac{d^{3} \vec{p}}{(2 \pi \hbar)^{3}} \omega(\vec{p}) \sum_{\lambda=0}^{3} \eta_{\lambda \lambda} \hat{a}(\vec{p}, \lambda)^{+} \hat{a}(\vec{p}, \lambda) \\
& =\int \frac{d^{3} \vec{p}}{(2 \pi \hbar)^{3}} \omega(\vec{p})\left(\sum_{i=1}^{3} \hat{a}(\vec{p}, i)^{+} \hat{a}(\vec{p}, i)-\hat{a}(\vec{p}, 0)^{+} \hat{a}(\vec{p}, 0)\right) \tag{105}
\end{align*}
$$

Since $-\hat{a}(\vec{p}, 0)^{+} \hat{a}(\vec{p}, 0)$ is the number operator for scalar polarization the Hamiltonian $\hat{H}$ can only have positive eigenvalues. Let $\mid 0>$ be the vacuum state, viz

$$
\begin{equation*}
\hat{a}(\vec{p}, \lambda) \mid 0>=0, \forall \vec{p} \text { and } \forall \lambda . \tag{106}
\end{equation*}
$$

The one-particle states are defined by

$$
\begin{equation*}
\left|\vec{p}, \lambda>=\hat{a}(\vec{p}, \lambda)^{+}\right| 0>. \tag{107}
\end{equation*}
$$

Let us compute the expectation value

$$
\begin{equation*}
<\vec{p}, \lambda|\hat{H}| \vec{p}, \lambda> \tag{108}
\end{equation*}
$$

By using $\hat{H} \mid 0>=0$ and $\left[\hat{H}, \hat{a}(\vec{p}, \lambda)^{+}\right]=\hbar \omega(\vec{p}) \hat{a}(\vec{p}, \lambda)^{+}$we find

$$
\begin{align*}
<\vec{p}, \lambda|\hat{H}| \vec{p}, \lambda> & =<\vec{p}, \lambda\left|\left[\hat{H}, \hat{a}(\vec{p}, \lambda)^{+}\right]\right| 0> \\
& =\hbar \omega(\vec{p})<\vec{p}, \lambda \mid \vec{p}, \lambda> \tag{109}
\end{align*}
$$

However

$$
\begin{align*}
<\vec{p}, \lambda \mid \vec{p}, \lambda> & =<0\left|\left[\hat{a}(\vec{p}, \lambda), \hat{a}(\vec{p}, \lambda)^{+}\right]\right| 0> \\
& =-\eta_{\lambda \lambda} \hbar(2 \pi \hbar)^{3} \delta^{3}(\vec{p}-\vec{q})<0 \mid 0> \\
& =-\eta_{\lambda \lambda} \hbar(2 \pi \hbar)^{3} \delta^{3}(\vec{p}-\vec{q}) . \tag{110}
\end{align*}
$$

This is negative for the scalar polarization $\lambda=0$ which is potentially a severe problem. As a consequence the expectation value of the Hamiltonian operator in the one-particle state with scalar polarization is negative. The resolution of these problems lies in the Lorentz gauge fixing condition which needs to be taken into consideration.

### 1.6 Gupta-Bleuler Method

In the quantum theory the Lorentz gauge fixing condition $\partial_{\mu} A^{\mu}=0$ becomes the operator equation

$$
\begin{equation*}
\partial_{\mu} \hat{A}^{\mu}=0 \tag{111}
\end{equation*}
$$

Explicitly we have

$$
\begin{equation*}
\partial_{\mu} \hat{A}^{\mu}=-c \int \frac{d^{3} \vec{p}}{(2 \pi \hbar)^{3}} \frac{1}{\sqrt{2 \omega(\vec{p})}} \frac{i}{\hbar} p_{\mu} \sum_{\lambda=0}^{3}\left(e^{-\frac{i}{\hbar} p x} \epsilon_{\lambda}^{\mu}(\vec{p}) \hat{a}(\vec{p}, \lambda)-e^{\frac{i}{\hbar} p x} \epsilon_{\lambda}^{\mu}(\vec{p}) \hat{a}(\vec{p}, \lambda)^{+}\right)_{p^{0}=|\vec{p}|}=0 . \tag{112}
\end{equation*}
$$

However

$$
\begin{align*}
{\left[\partial_{\mu} \hat{A}^{\mu}\left(x^{0}, \vec{x}\right), \hat{A}^{\nu}\left(x^{0}, \vec{y}\right)\right] } & =\left[\partial_{0} \hat{A}^{0}\left(x^{0}, \vec{x}\right), \hat{A}^{\nu}\left(x^{0}, \vec{y}\right)\right]+\left[\partial_{i} \hat{A}^{i}\left(x^{0}, \vec{x}\right), \hat{A}^{\nu}\left(x^{0}, \vec{y}\right)\right] \\
& =-c\left[\hat{\pi}^{0}\left(x^{0}, \vec{x}\right), \hat{A}^{\nu}\left(x^{0}, \vec{y}\right)\right]+\partial_{i}^{x}\left[\hat{A}^{i}\left(x^{0}, \vec{x}\right), \hat{A}^{\nu}\left(x^{0}, \vec{y}\right)\right] \\
& =i \hbar c \eta^{0 \nu} \delta^{3}(\vec{x}-\vec{y}) . \tag{113}
\end{align*}
$$

In other words in the quantum theory we can not impose the Lorentz condition as the operator identity (111).

The problem we faced in the previous section was the fact that the Hilbert space of quantum states has an indefinite metric, i.e. the norm was not positive-definite. As we said the solution of this problem consists in imposing the Lorentz gauge condition but clearly this can not be done in the operator form (111). Obviously there are physical states in the Hilbert space associated with the photon transverse polarization states and unphysical states associated with the longitudinal and scalar polarization states. It is therefore natural to impose the Lorentz gauge condition only on the physical states $\mid \phi>$ associated with the transverse photons. We may require for example that the expectation value $\langle\phi| \partial_{\mu} \hat{A}^{\mu} \mid \phi>$ vanishes, viz

$$
\begin{equation*}
<\phi\left|\partial_{\mu} \hat{A}^{\mu}\right| \phi>=0 \tag{114}
\end{equation*}
$$

Let us recall that the gauge field operator is given by

$$
\begin{equation*}
\hat{A}^{\mu}=c \int \frac{d^{3} \vec{p}}{(2 \pi \hbar)^{3}} \frac{1}{\sqrt{2 \omega(\vec{p})}} \sum_{\lambda=0}^{3}\left(e^{-\frac{i}{\hbar} p x} \epsilon_{\lambda}^{\mu}(\vec{p}) \hat{a}(\vec{p}, \lambda)+e^{\frac{i}{\hbar} p x} \epsilon_{\lambda}^{\mu}(\vec{p}) \hat{a}(\vec{p}, \lambda)^{+}\right)_{p^{0}=|\vec{p}|} \tag{115}
\end{equation*}
$$

This is the sum of a positive-frequency part $\hat{A}_{+}^{\mu}$ and a negative-frequency part $\hat{A}_{-}^{\mu}$, viz

$$
\begin{equation*}
\hat{A}^{\mu}=\hat{A}_{+}^{\mu}+\hat{A}_{-}^{\mu} . \tag{116}
\end{equation*}
$$

These parts are given respectively by

$$
\begin{align*}
& \hat{A}_{+}^{\mu}=c \int \frac{d^{3} \vec{p}}{(2 \pi \hbar)^{3}} \frac{1}{\sqrt{2 \omega(\vec{p})}} \sum_{\lambda=0}^{3} e^{-\frac{i}{\hbar} p x} \epsilon_{\lambda}^{\mu}(\vec{p}) \hat{a}(\vec{p}, \lambda) .  \tag{117}\\
& \hat{A}_{-}^{\mu}=c \int \frac{d^{3} \vec{p}}{(2 \pi \hbar)^{3}} \frac{1}{\sqrt{2 \omega(\vec{p})}} \sum_{\lambda=0}^{3} e^{\frac{i}{\hbar} p x} \epsilon_{\lambda}^{\mu}(\vec{p}) \hat{a}(\vec{p}, \lambda)^{+} . \tag{118}
\end{align*}
$$

Instead of (114) we choose to impose the Lorentz gauge condition as the eigenvalue equation

$$
\begin{equation*}
\partial_{\mu} \hat{A}_{+}^{\mu} \mid \phi>=0 \tag{119}
\end{equation*}
$$

This is equivalent to

$$
\begin{equation*}
<\phi \mid \partial_{\mu} \hat{A}_{-}^{\mu}=0 \tag{120}
\end{equation*}
$$

The condition (119) is stronger than (114). Indeed we can check that $<\phi\left|\partial_{\mu} \hat{A}^{\mu}\right| \phi>=<$ $\phi\left|\partial_{\mu} \hat{A}_{+}^{\mu}\right| \phi>+<\phi\left|\partial_{\mu} \hat{A}_{-}^{\mu}\right| \phi>=0$. In this way the physical states are defined precisely as the eigenvectors of the operator $\partial_{\mu} \hat{A}_{+}^{\mu}$ with eigenvalue 0 . In terms of the annihilation operators $\hat{a}(\vec{p}, \lambda)$ the condition (119) reads

$$
\begin{equation*}
\left.c \int \frac{d^{3} \vec{p}}{(2 \pi \hbar)^{3}} \frac{1}{\sqrt{2 \omega(\vec{p})}} \sum_{\lambda=0}^{3} e^{-\frac{i}{\hbar} p x}\left(-\frac{i}{\hbar} p_{\mu} \epsilon_{\lambda}^{\mu}(\vec{p})\right) \hat{a}(\vec{p}, \lambda) \right\rvert\, \phi>=0 . \tag{121}
\end{equation*}
$$

Since $p_{\mu} \epsilon_{i}^{\mu}(\vec{p})=0, i=1,2$ and $p_{\mu} \epsilon_{3}^{\mu}(\vec{p})=-p_{\mu} \epsilon_{0}^{\mu}(\vec{p})=-n^{\mu} p_{\mu}$ we get

$$
\begin{equation*}
\left.c \int \frac{d^{3} \vec{p}}{(2 \pi \hbar)^{3}} \frac{1}{\sqrt{2 \omega(\vec{p})}} e^{-\frac{i}{\hbar} p x} \frac{i}{\hbar} p_{\mu} n^{\mu}(\hat{a}(\vec{p}, 3)-\hat{a}(\vec{p}, 0)) \right\rvert\, \phi>=0 . \tag{122}
\end{equation*}
$$

We immediately conclude that

$$
\begin{equation*}
(\hat{a}(\vec{p}, 3)-\hat{a}(\vec{p}, 0)) \mid \phi>=0 . \tag{123}
\end{equation*}
$$

Hence we deduce the crucial identity

$$
\begin{align*}
& <\phi\left|\hat{a}(\vec{p}, 3)^{+} \hat{a}(\vec{p}, 3)\right| \phi>=<\phi\left|\hat{a}(\vec{p}, 0)^{+} \hat{a}(\vec{p}, 0)\right| \phi>.  \tag{124}\\
<\phi|\hat{H}| \phi>= & \int \frac{d^{3} \vec{p}}{(2 \pi \hbar)^{3}} \omega(\vec{p})\left(\sum_{i=1}^{2}<\phi\left|\hat{a}(\vec{p}, i)^{+} \hat{a}(\vec{p}, i)\right| \phi>+<\phi\left|\hat{a}(\vec{p}, 3)^{+} \hat{a}(\vec{p}, 3)\right| \phi>\right. \\
- & \left.<\phi\left|\hat{a}(\vec{p}, 0)^{+} \hat{a}(\vec{p}, 0)\right| \phi>\right) \\
= & \int \frac{d^{3} \vec{p}}{(2 \pi \hbar)^{3}} \omega(\vec{p}) \sum_{i=1}^{2}<\phi\left|\hat{a}(\vec{p}, i)^{+} \hat{a}(\vec{p}, i)\right| \phi>. \tag{125}
\end{align*}
$$

This is always positive definite and only transverse polarization states contribute to the expectation value of the Hamiltonian operator. This same thing will happen for all other physical observables such as the momentum operator and the angular momentum operator. Let us define

$$
\begin{equation*}
L(\vec{p})=\hat{a}(\vec{p}, 3)-\hat{a}(\vec{p}, 0) . \tag{126}
\end{equation*}
$$

We have

$$
\begin{equation*}
L(\vec{p}) \mid \phi>=0 . \tag{127}
\end{equation*}
$$

It is trivial to show that

$$
\begin{equation*}
\left[L(\vec{p}), L(\vec{p})^{+}\right]=0 . \tag{128}
\end{equation*}
$$

Thus

$$
\begin{equation*}
L(\vec{p}) \mid \phi_{c}>=0 \tag{129}
\end{equation*}
$$

where $\mid \phi_{c}>$ is also a physical state defined by

$$
\begin{equation*}
\left|\phi_{c}>=f_{c}\left(L^{+}\right)\right| \phi> \tag{130}
\end{equation*}
$$

The operator $f_{c}\left(L^{+}\right)$can be expanded as

$$
\begin{equation*}
f_{c}\left(L^{+}\right)=1+\int d^{3} \vec{p} c(\vec{p}) L(\vec{p})^{+}+\int d^{3} \vec{p} \int d^{3} \vec{p}^{\prime \prime} c\left(\vec{p}^{\prime}, \vec{p}^{\prime \prime}\right) L(\vec{p})^{+} L\left(\vec{p}^{\prime \prime}\right)^{+}+\ldots \tag{131}
\end{equation*}
$$

It is also trivial to show that

$$
\begin{equation*}
\left[f_{c}\left(L^{+}\right)^{+}, f_{c^{\prime}}\left(L^{+}\right)\right]=0 \tag{132}
\end{equation*}
$$

The physical state $\mid \phi_{c}>$ is completely equivalent to the state $\mid \phi>$ although $\mid \phi_{c}>$ contains longitudinal and scalar polarization states while $\mid \phi>$ contains only transverse polarization states. Indeed

$$
\begin{align*}
<\phi_{c} \mid \phi_{c^{\prime}}> & =<\phi\left|f_{c}\left(L^{+}\right)^{+} f_{c^{\prime}}\left(L^{+}\right)\right| \phi> \\
& =<\phi\left|f_{c^{\prime}}\left(L^{+}\right) f_{c}\left(L^{+}\right)^{+}\right| \phi> \\
& =<\phi \mid \phi> \tag{133}
\end{align*}
$$

Thus the scalar product between any two states $\mid \phi_{c}>$ and $\mid \phi_{c^{\prime}}>$ is fully determined by the norm of the state $\mid \phi>$. The state $\mid \phi_{c}>$ constructed from a given physical state $\mid \phi>$ defines an equivalence class. Clearly the state $\mid \phi>$ can be taken to be the representative of this equivalence class. The members of this equivalence class are related by gauge transformations. This can be checked explicitly as follows. We compute

$$
\begin{equation*}
<\phi_{c}\left|\hat{A}_{\mu}\right| \phi_{c}>=<\phi\left|f_{c}\left(L^{+}\right)^{+}\left[\hat{A}_{\mu}, f_{c}\left(L^{+}\right)\right]\right| \phi>+<\phi\left|\left[f_{c}\left(L^{+}\right)^{+}, \hat{A}_{\mu}\right]\right| \phi>+<\phi\left|\hat{A}_{\mu}\right| \phi>. \tag{134}
\end{equation*}
$$

By using the fact that the commutators of $\hat{A}^{\mu}$ with $L(\vec{p})$ and $L(\vec{p})^{+}$are $c$-numbers we obtain

$$
\begin{equation*}
<\phi_{c}\left|\hat{A}_{\mu}\right| \phi_{c}>=\int d^{3} \vec{p} c(\vec{p})\left[\hat{A}_{\mu}, L(\vec{p})^{+}\right]+\int d^{3} \vec{p} c(\vec{p})^{*}\left[L(\vec{p}), \hat{A}_{\mu}\right]+\langle\phi| \hat{A}_{\mu} \mid \phi> \tag{135}
\end{equation*}
$$

We compute

$$
\begin{equation*}
\left[\hat{A}^{\mu}, L(\vec{p})^{+}\right]=\frac{\hbar c}{\sqrt{2 \omega(\vec{p})}} e^{-\frac{i}{\hbar} p x}\left(\epsilon_{3}^{\mu}(\vec{p})+\epsilon_{0}^{\mu}(\vec{p})\right) \tag{136}
\end{equation*}
$$

Thus

$$
\begin{align*}
<\phi_{c}\left|\hat{A}^{\mu}\right| \phi_{c}> & \left.=\hbar c \int \frac{d^{3} \vec{p}}{\sqrt{2 \omega(\vec{p})}}\left(\epsilon_{3}^{\mu}(\vec{p})+\epsilon_{0}^{\mu}(\vec{p})\right)\left(c(\vec{p}) e^{-\frac{i}{\hbar} p x}+c(\vec{p})^{*} e^{\frac{i}{\hbar} p x}\right)+\langle\phi| \hat{A}^{\mu} \right\rvert\, \phi> \\
& =\hbar c \int \frac{d^{3} \vec{p}}{\sqrt{2 \omega(\vec{p})}}\left(\frac{p^{\mu}}{n \cdot p}\right)\left(c(\vec{p}) e^{-\frac{i}{\hbar} p x}+c(\vec{p})^{*} e^{\frac{i}{\hbar} p x}\right)+<\phi\left|\hat{A}^{\mu}\right| \phi> \\
& =\hbar c\left(-\frac{\hbar}{i} \partial^{\mu}\right) \int \frac{d^{3} \vec{p}}{\sqrt{2 \omega(\vec{p})}}\left(\frac{1}{n \cdot p}\right)\left(c(\vec{p}) e^{-\frac{i}{\hbar} p x}-c(\vec{p})^{*} e^{\frac{i}{\hbar} p x}\right)+<\phi\left|\hat{A}^{\mu}\right| \phi> \\
& =\partial^{\mu} \Lambda+<\phi\left|\hat{A}^{\mu}\right| \phi>  \tag{137}\\
& \Lambda=i \hbar^{2} c \int \frac{d^{3} \vec{p}}{\sqrt{2 \omega(\vec{p})}}\left(\frac{1}{n \cdot p}\right)\left(c(\vec{p}) e^{-\frac{i}{\hbar} p x}-c(\vec{p})^{*} e^{\frac{i}{\hbar} p x}\right) \tag{138}
\end{align*}
$$

Since $p^{0}=|\vec{p}|$ we have $\partial_{\mu} \partial^{\mu} \Lambda=0$, i.e. the gauge function $\Lambda$ is consistent with the Lorentz gauge condition.

### 1.7 Propagator

The probability amplitudes for a gauge particle to propagate from the spacetime point $y$ to the spacetime $x$ is

$$
\begin{equation*}
i D^{\mu \nu}(x-y)=<0\left|\hat{A}^{\mu}(x) \hat{A}^{\nu}(y)\right| 0>. \tag{139}
\end{equation*}
$$

We compute

$$
\begin{align*}
i D^{\mu \nu}(x-y) & =c^{2} \int \frac{d^{3} \vec{q}}{(2 \pi \hbar)^{3}} \int \frac{d^{3} \vec{p}}{(2 \pi \hbar)^{3}} \frac{1}{\sqrt{2 \omega(\vec{q})}} \frac{1}{\sqrt{2 \omega(\vec{p})}} e^{-\frac{i}{\hbar} q x} e^{+\frac{i}{\hbar} p y} \sum_{\lambda^{\prime}, \lambda=0}^{3} \epsilon_{\lambda^{\prime}}^{\mu}(\vec{q}) \epsilon_{\lambda}^{\nu}(\vec{p}) \\
& \times<0\left|\left[\hat{a}\left(\vec{q}, \lambda^{\prime}\right), \hat{a}(\vec{p}, \lambda)^{+}\right]\right| 0> \\
& =c^{2} \hbar^{2} \int \frac{d^{3} \vec{p}}{(2 \pi \hbar)^{3}} \frac{1}{2 E(\vec{p})} e^{-\frac{i}{\hbar} p(x-y)} \sum_{\lambda=0}^{3}\left(-\eta_{\lambda \lambda} \epsilon_{\lambda}^{\mu}(\vec{q}) \epsilon_{\lambda}^{\nu}(\vec{p})\right) \\
& =c^{2} \hbar^{2} \int \frac{d^{3} \vec{p}}{(2 \pi \hbar)^{3}} \frac{1}{2 E(\vec{p})} e^{-\frac{i}{\hbar} p(x-y)}\left(-\eta^{\mu \nu}\right) \\
& =\hbar^{2} D(x-y)\left(-\eta^{\mu \nu}\right) . \tag{140}
\end{align*}
$$

The function $D(x-y)$ is the probability amplitude for a massless real scalar particle to propagate from $y$ to $x$. The retarded Green's function of the gauge field can be defined by

$$
\begin{align*}
i D_{R}^{\mu \nu}(x-y) & =\hbar^{2} D_{R}(x-y)\left(-\eta^{\mu \nu}\right) \\
& =\theta\left(x^{0}-y^{0}\right)<0\left|\left[\hat{A}^{\mu}(x), \hat{A}^{\nu}(y)\right]\right| 0> \tag{141}
\end{align*}
$$

The second line follows from the fact that $D_{R}(x-y)=\theta\left(x^{0}-y^{0}\right)<0|[\hat{\phi}(x), \hat{\phi}(y)]| 0>$. In momentum space this retarded Green's function reads

$$
\begin{equation*}
i D_{R}^{\mu \nu}(x-y)=\hbar^{2}\left(c \hbar \int \frac{d^{4} p}{(2 \pi \hbar)^{4}} \frac{i}{p^{2}} e^{-\frac{i}{\hbar} p(x-y)}\right)\left(-\eta^{\mu \nu}\right) \tag{142}
\end{equation*}
$$

Since $\partial_{\alpha} \partial^{\alpha} D_{R}(x-y)=(-i c / \hbar) \delta^{4}(x-y)$ we must have

$$
\begin{equation*}
\left(\partial_{\alpha} \partial^{\alpha} \eta_{\mu \nu}\right) D_{R}^{\nu \lambda}(x-y)=\hbar c \delta^{4}(x-y) \eta_{\mu}^{\lambda} . \tag{143}
\end{equation*}
$$

Another solution of this equation is the so-called Feynman propagator for a gauge field given by

$$
\begin{align*}
i D_{F}^{\mu \nu}(x-y) & =\hbar^{2} D_{F}(x-y)\left(-\eta^{\mu \nu}\right) \\
& =<0\left|T \hat{A}^{\mu}(x) \hat{A}^{\nu}(y)\right| 0>. \tag{144}
\end{align*}
$$

In momentum space this reads

$$
\begin{equation*}
i D_{F}^{\mu \nu}(x-y)=\hbar^{2}\left(c \hbar \int \frac{d^{4} p}{(2 \pi \hbar)^{4}} \frac{i}{p^{2}+i \epsilon} e^{-\frac{i}{\hbar} p(x-y)}\right)\left(-\eta^{\mu \nu}\right) . \tag{145}
\end{equation*}
$$

### 1.8 Problems and Exercises

## Maxwell's Equations

1) Derive Maxwell's equations from

$$
\begin{equation*}
\partial_{\mu} F^{\mu \nu}=\frac{1}{c} J^{\nu}, \partial_{\mu} \tilde{F}^{\mu \nu}=0 \tag{146}
\end{equation*}
$$

2) Derive from the expression of the field tensor $F_{\mu \nu}$ in terms of $A^{\mu}$ the electric and magnetic fields in terms of the scalar and vector potentials.

## Noether's Theorem

1) Prove Noether's theorem for an infinitesimal transformation of the form

$$
\begin{equation*}
\phi(x) \longrightarrow \phi^{\prime}(x)=\phi(x)+\delta \phi(x) . \tag{147}
\end{equation*}
$$

2) Determine the conserved current of the Dirac Lagrangian density under the local gauge transformation

$$
\begin{equation*}
\psi \longrightarrow \psi^{\prime}=e^{i \alpha} \psi \tag{148}
\end{equation*}
$$

3) What is the significance of the corresponding conserved charge.

## Polarization Vectors

1) Write down the polarization vectors in the reference frame where $n^{\mu}=(1,0,0,0)$.
2) Verify that

$$
\begin{equation*}
\sum_{\lambda=1}^{2} \epsilon_{\lambda}^{\mu}(\vec{p}) \epsilon_{\lambda}^{\nu}(\vec{p})=-\eta^{\mu \nu}-\frac{p^{\mu} p^{\nu}}{(n p)^{2}}+\frac{p^{\mu} n^{\nu}+p^{\nu} n^{\mu}}{n p} . \tag{149}
\end{equation*}
$$

## Gauge Invariance and Current Conservation

1) Show that current conservation $\partial^{\mu} J_{\mu}=0$ is a necessary and sufficient condition for gauge invariance. Consider the Lagrangian density

$$
\begin{equation*}
\mathcal{L}=-\frac{1}{4} F_{\mu \nu} F^{\mu \nu}+J_{\mu} A^{\mu} . \tag{150}
\end{equation*}
$$

2) The gauge-fixed equations of motion are given by

$$
\begin{equation*}
\partial_{\mu} \partial^{\mu} A^{\nu}-(1-\zeta) \partial^{\nu} \partial_{\mu} A^{\mu}=\frac{1}{c} J^{\nu} . \tag{151}
\end{equation*}
$$

Show that for $\zeta \neq 0$ these equations of motion are equivalent to Maxwell's equations in the Lorentz gauge.

Commutation Relations Verify

$$
\begin{equation*}
\left[\hat{a}(\vec{p}, \lambda), \hat{a}\left(\vec{q}, \lambda^{\prime}\right)^{+}\right]=-\eta_{\lambda \lambda^{\prime}} \hbar(2 \pi \hbar)^{3} \delta^{3}(\vec{p}-\vec{q}) . \tag{152}
\end{equation*}
$$

## Hamiltonian Operator

1) Show that the classical Hamiltonian of the electromagnetic field is given by

$$
\begin{equation*}
H=\int d^{3} x\left(\frac{1}{2} \partial_{i} A_{\mu} \partial^{i} A^{\mu}-\frac{1}{2} \partial_{0} A_{\mu} \partial^{0} A^{\mu}\right) . \tag{153}
\end{equation*}
$$

2) Show that in the quantum theory the Hamiltonian operator is of the form

$$
\begin{equation*}
\hat{H}=\int \frac{d^{3} \vec{p}}{(2 \pi \hbar)^{3}} \omega(\vec{p})\left(\sum_{i=1}^{3} \hat{a}(\vec{p}, i)^{+} \hat{a}(\vec{p}, i)-\hat{a}(\vec{p}, 0)^{+} \hat{a}(\vec{p}, 0)\right) . \tag{154}
\end{equation*}
$$

3) Impose the Lorentz gauge condition using the Gupta-Bleuler method. What are the physical states. What happens to the expectation values of $\hat{H}$.

Physical States Let us define

$$
\begin{equation*}
L(\vec{p})=\hat{a}(\vec{p}, 3)-\hat{a}(\vec{p}, 0) . \tag{155}
\end{equation*}
$$

Physical states are defined by

$$
\begin{equation*}
L(\vec{p}) \mid \phi>=0 . \tag{156}
\end{equation*}
$$

Define

$$
\begin{equation*}
\left|\phi_{c}>=f_{c}\left(L^{+}\right)\right| \phi>. \tag{157}
\end{equation*}
$$

1) Show that the physical state $\left|\phi_{c}\right\rangle$ is completely equivalent to the physical state $|\phi\rangle$.
2) Show that the two states $\mid \phi>$ and $\mid \phi_{c}>$ are related by a gauge transformation. Determine the gauge parameter.

## Photon Propagator

1) Compute the photon amplitude $i D^{\mu \nu}(x-y)=<0\left|\hat{A}^{\mu}(x) \hat{A}^{\nu}(y)\right| 0>$ in terms of the scalar amplitude $D(x-y)$.
2) Derive the photon propagator in a general gauge $\xi$.
