Geophysical Journal (1988) **92,** 171–178

RESEARCH NOTE

Logarithmic Fourier transformation

G. V. Haines and Alan G. Jones
Geological Survey of Canada, 1 Observatory Crescent, Ottawa, Ontario, K1A OY3 *Canada*

Accepted 1987 July 13. Received 1987 July 3; in original form 1987 February 20

SUMMARY

We present an exact and analytical expression for the Fourier transform of a function that has been sampled logarithmically. The procedure is significantly more efficient computationally than the fast Fourier transformation (FFT) for transforming functions or measured responses which decay slowly with increasing abscissa value. We illustrate the proposed method with an example from electromagnetic geophysics, where the scaling is often such that our logarithmic Fourier transform (LFT) should be applied. For the example chosen, we are able to obtain results that agree with those from an FFT to within 0.5 per cent in a time that is a factor of 10^2 shorter. Potential applications of our LFT in geophysics include conversion of wide-band electromagnetic frequency responses to transient responses, glacial loading and unloading, aquifer recharge problems, normal mode and earth tide studies in seismology, and impulsive shock wave modelling.

Key words: Fourier transformation, line current, logarithmic sampling

INTRODUCTION

In many scientific problems it is necessary to compute the Fourier transform (FT) of a function or measured response that rises quickly then decays slowly with increasing abscissa (independent parameter) value (an 'inverse ramp'). In such cases, it is desirable to derive the function, or sample the response, with a small abscissa interval during the rise, but then increase the interval with increasing abscissa value. Various approximate techniques have previously been presented to accomplish effectively the required FT of a function or response that is sampled in such a manner. These include fitting the response with a set of functions for which the analytical FTs are known (Lee & Lewis 1974; Lamontagne 1975, see Holladay 1981), discrete Fourier transform (DFT) of a suitably interpolated response (Dey & Morrison 1973; Palacky & West 1973; Hostetter 1982), decomposition of a polynomial approximation of the response into partial fractions (Chen & Haas 1968), and FTs of segments of the response (Asten & Verma 1978).

If the function can be adequately described by uniform sampling in the logarithmic parameter domain, then the cosine and sine transforms can be written as Hankel transforms, and a fast Hankel transform (FHT) convolution filtering technique (Anderson 1979; Johansen & Sørensen 1979; Nissen & Enmark, 1986) can be employed. The FHT method has, in general, effectively superseded the previously mentioned schemes for geophysical problems since its advent as a special case in the excellent article on FHT by Johansen & Sørensen (1979).

The approach we present here also is applicable to functions and responses that are adequately represented by a logarithmic sampling scheme. The theoretical development is similar to that of Talman (1978) with the differences that we treat the Fourier integral directly, not sine and cosine integrals independently, and we introduce a trade-off parameter which must lie in the range max $(\alpha, 0)$ to min $(\beta, 1)$, where α and β are defined by the decay characteristics of the response that is to be transformed, whereas Talman (1978) considers the restrictive case of this parameter being set to $\frac{1}{2}$. Our approach leads to an exact, analytical expression for the logarithmic Fourier transform (LFT), or inverse logarithmic Fourier transform (ILFT) of that function. Because of the restriction in treating the logarithm of zero, the ILFT is only valid for functions that have zero amplitude at zero frequency (or wave number), and the LFT for functions that are zero at zero time (or space).

We will give the theoretical development, and then show an example of the application of the method for deriving the vertical magnetic field that would be observed over a line-current buried in a half-space of non-zero conductivity. Consideration will be given to the necessary sampling of the function, and to the effect of varying the trade-off parameter.

The main application that we are aware of for our LFT is in the field of electromagnetism, and is in particular the conversion of wide-band electromagnetic frequency responses to transient responses. Much effort has been expended in this area over the last 20 years, as is evidenced by the publications cited above. There are possible applications in other branches of geophysics, which might include glacial loading and unloading and the associated rebounding of the lithosphere, acquifer recharge problems, normal mode and earth tide studies in seismology, and inpulsive shock-wave modelling.

THEORY

We define the Fourier transform pair $[f(t), F(v)]$ as

$$F(v) = \mathrm{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t) \exp(-ivt)\, dt \tag{1a}$$

$$f(t) = \mathrm{F}^{-1}\{F(v)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(v) \exp(ivt)\, dv \tag{1b}$$

where F and F^{-1} represent forward and inverse Fourier transformations respectively.

The complex function $F(v)$, $-\infty < v < \infty$, is assumed to be defined for real frequency or wavenumber v, and we wish to transform from the v-domain to the logarithmic ω-domain using

$$v = \begin{cases} -v_0 \exp \omega, & \text{for } v < 0 \\ v_0 \exp \omega, & \text{for } v > 0. \end{cases}$$

The factor v_0 is introduced for dimensional consistency, and for simplicity is assumed to take the value of unity.

The function $F(v)$ thus transforms to two complex functions:

$$F(v) = \begin{cases} F(-v_0 \exp \omega) = G_1(\omega) & \text{for } v < 0 \\ F(v_0 \exp \omega) = G_2(\omega) & \text{for } v > 0. \end{cases} \tag{2}$$

The transformation $\mathrm{F} \rightarrow \mathrm{G}$ is obviously not possible for $v = 0$, and thus $F(v)$ must be zero at $v = 0$.

Similarly, we transform the real time variable t to τ by

$$t = \begin{cases} -t_0 \exp \tau, & \text{for } t < 0 \\ t_0 \exp \tau, & \text{for } t > 0 \end{cases}$$

where t_0 is also introduced for dimensional consistency, and the complex function $f(t)$ transforms to two complex functions:

$$f(t) = \begin{cases} f(-t_0 \exp \tau) = g_1(\tau) & \text{for } t < 0 \\ f(t_0 \exp \tau) = g_2(\tau) & \text{for } t > 0. \end{cases} \tag{3}$$

We wish to determine the transform in the ω-domain that is equivalent to the IFT of $F(v)$. Substituting transformations (2) into (1b) gives

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_1(\omega) \exp[\omega - (it \exp \omega)]\, d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty} G_2(\omega) \exp[\omega + (it \exp \omega)]\, d\omega. \tag{4}$$

Substituting transformations (3) into the above yields for $t < 0$

$$g_1(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_1(\omega) \exp[\omega + i \exp(\omega + \tau)]\, d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty} G_2(\omega) \exp[\omega - i \exp(\omega + \tau)]\, d\omega \tag{5a}$$

and for $t > 0$

$$g_2(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_1(\omega) \exp[\omega - i \exp(\omega + \tau)]\, d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty} G_2(\omega) \exp[\omega + i \exp(\omega + \tau)]\, d\omega. \tag{5b}$$

Accordingly, we need to solve integrals of the form

$$\int_{-\infty}^{\infty} G(\omega) \exp[\omega \pm i \exp(\omega + \tau)]\, d\omega.$$

For an arbitrary real parameter k (a 'trade-off' parameter), the integral may be rewritten as

$$\int_{-\infty}^{\infty} G(\omega) \exp[\omega \pm i \exp(\omega + \tau)]\, d\omega = \exp(-k\tau)\int_{-\infty}^{\infty} G(\omega) \exp[k(\omega + \tau) \pm i \exp(\omega + \tau)] \times \exp[(1 - k)\omega]\, d\omega. \tag{6}$$

Equation (6) now appears as a convolution type integral, which we solve formally by the usual method of FT, multiplication, then IFT, such that

$$\int_{-\infty}^{\infty} G(\omega)\exp[\omega \pm i\exp(\omega+\tau)]\,d\omega = \exp(-k\tau)\mathrm{F}^{-1}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\omega)\exp[k(\omega+\tau) \pm i\exp(\omega+\tau)] \times \exp[(1-k)\omega - is\tau]\,d\omega\,d\tau\right\}$$

$$= \exp(-k\tau)\mathrm{F}^{-1}\left\{\int_{-\infty}^{\infty} G(\omega)\exp[(1-k)\omega + is\omega]\,d\omega \int_{-\infty}^{\infty} \exp(kx \pm i\exp x)\times\exp(-isx)\,dx\right\}, \quad (7)$$

where $x = \omega + \tau$, and s is an intermediate variable of the inverse transform.

The first integral on the rhs is

$$\int_{-\infty}^{\infty} G(\omega)\exp[(1-k)\omega - is\omega]\,d\omega = 2\pi\mathrm{F}^{-1}\{G(\omega)\exp[(1-k)\omega]\}$$

which converges whenever $\alpha < 1-k < \beta$, where α and β are real and such that $|G(\omega)| \sim A\exp(-\alpha\omega)$ as $w \rightarrow -\infty$ and $|G(\omega)| \sim B\exp(-\beta\omega)$ as $\omega \rightarrow +\infty$. The second integral converges for $0 < 1-k < 1$:

$$\int_{-\infty}^{\infty} \exp(kx \pm i\exp x)\times\exp(-isx)\,dx = \int_{0}^{\infty} y^{k-1-is}\exp(\pm iy)\,dy = \exp\left[\pm\frac{\pi}{2}(s+ik)\right]\Gamma(k-is)$$

(Erdélyi 1953, §1.5, equations 32 (note typographical error), 33; Gradshteyn & Ryzhik 1980, 3.381.5). Thus, whenever $\max(\alpha, 0) < 1-k < \min(\beta, 1)$,

$$\int_{-\infty}^{\infty} G(\omega)\exp[\omega \pm i\exp(\omega+\tau)]\,d\omega = 2\pi\exp(-k\tau)\mathrm{F}^{-1}\left\{\exp\left[\pm\frac{\pi}{2}(s+ik)\right]\Gamma(k-is)\mathrm{F}^{-1}\{G(\omega)\exp[(1-k)\omega]\}\right\}. \quad (8)$$

Note that it is equally possible to derive the convolution by IFT, multiplication, then FT such that

$$\int_{-\infty}^{\infty} G(\omega)\exp[\omega \pm i\exp(\omega+\tau)]\,d\omega = \frac{1}{2\pi}\exp(-k\tau)\mathrm{F}\left\{\exp\left[\pm\frac{\pi}{2}(-s+ik)\right]\Gamma(k+is)\mathrm{F}\{G(\omega)\exp[(1-k)\omega]\}\right\}.$$

Substituting (8) into (5b) gives for $g_2(\tau)$

$$g_2(\tau) = \exp(-k\tau)\mathrm{F}^{-1}\left\{\Gamma(k-is)\left(\exp\left[-\frac{\pi}{2}(s+ik)\right]\right.\right.$$
$$\left.\left.\times\,\mathrm{F}^{-1}\{G_1(\omega)\exp[(1-k)\omega]\} + \exp\left[\frac{\pi}{2}(s+ik)\right]\mathrm{F}^{-1}\{G_2(\omega)\exp[(1-k)\omega]\}\right)\right\} \quad (9)$$

and the same for $g_1(\tau)$, but with a sign change in the exponential arguments $(\pi/2)(s+ik)$.

From comparison with our definitions of the FT and the IFT (equation 1), we call $f(t)$ derived by this procedure the inverse logarithmic Fourier transform (ILFT) of $F(v)$, and write it as $f(t) = \mathrm{L}^{-1}\{F(v)\}$. The logarithmic Fourier transform (LFT) can be constructed in exactly the same manner, where

$$G_2(\omega) = 2\pi\exp(-k\omega)\mathrm{F}^{-1}\left\{\Gamma(k-is)\left(\exp\left[\frac{\pi}{2}(s+ik)\right]\right.\right.$$
$$\left.\left.\times\,\mathrm{F}^{-1}\{g_1(\tau)\exp[(1-k)\tau]\} + \exp\left[-\frac{\pi}{2}(s+ik)\right]\mathrm{F}^{-1}\{g_2(\tau)\exp[(1-k)\tau]\}\right)\right\} \quad (10)$$

and the same for $G_1(\omega)$ except for the appropriate sign changes in the exponent term $(\pi/2)(s+ik)$. Note that for such a transformation to be realizable, $f(t)$ must be zero at $t=0$.

The choice of k is somewhat arbitrary, within the bounds set by α and β, and some numerical considerations are discussed below, but we note that for $|s|$ large then

$$|\Gamma(k-is)| \sim (2\pi)^{1/2}|s|^{k-1/2}\exp\left(-\frac{\pi|s|}{2}\right)$$

(Copson 1935, §9.55), and so

$$\left|\exp\left[\frac{\pi}{2}(s+ik)\right]\Gamma(k-is)\right| \sim \begin{cases}(2\pi)^{1/2}|s|^{k-1/2}\exp(-\pi|s|) & \text{as } s \rightarrow -\infty\\ (2\pi)^{1/2}|s|^{k-1/2} & \text{as } s \rightarrow +\infty\end{cases}$$

and

$$\left|\exp\left[-\frac{\pi}{2}(s+ik)\right]\Gamma(k-is)\right| \sim \begin{cases}(2\pi)^{1/2}|s|^{k-1/2} & \text{as } s \rightarrow -\infty\\ (2\pi)^{1/2}|s|^{k-1/2}\exp[-\pi|s|] & \text{as } s \rightarrow +\infty\end{cases}$$

and so the smaller the value of k, the more rapid the convergence of the above term with large s. However, at small values of k, the slower is its convergence at small s, the gamma function being singular at $k = s = 0$. Also, the smaller the value of k, the less rapid is the convergence of $F^{-1}\{G(\omega)\exp[(1-k)\omega]\}$.

NUMERICAL CONSIDERATIONS

In a numerical problem, the inverse Fourier transforms are approximated by a discrete transform over a finite interval. If we sample $G(\omega)$ at N equidistant points within the bandwidth $\omega_{\min} < \omega < \omega_{\max}$, then the inverse Fourier transform of $G(\omega)$ is

$$f(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\exp(is\omega)\,d\omega$$
$$\approx \frac{1}{2\pi}\sum_{n=-N/2}^{N/2-1} G(\omega_n)\exp(is\omega_n)\Delta\omega$$
$$= \frac{\Delta\omega}{2\pi}\exp(is\bar{\omega})\sum_{n=-N/2}^{N/2-1} G(\omega_n)\exp(isn\Delta\omega)$$

where $\Delta\omega = (\omega_{\max} - \omega_{\min})/N$, $\bar{\omega} = (\omega_{\max} + \omega_{\min})/2$, and $\omega_n = \bar{\omega} + n\Delta\omega$, the number of sampling points N assumed to be even. The first sample is at $\omega_{\min}$, and the last at $\omega_{\max} - \Delta\omega$.

As a result of the sampling, the range of s is now restricted to $-\pi/\Delta\omega \le s < \pi/\Delta\omega$. In discrete Fourier analysis, the transform is computed at the same number of sampling points N, equi-spaced over the bandwidth in s, so that

$$f(s_j) = \frac{\omega_{\max} - \omega_{\min}}{2\pi}\exp(is_j\bar{\omega})\left\{\frac{1}{N}\sum_{n=-N/2}^{N/2-1} G(\omega_n)\exp(i2\pi jn/N)\right\},$$

where $s_j = 2\pi j/(\omega_{\max} - \omega_{\min})$, $j = -N/2, -N/2+1, -N/2+2, \ldots, N/2-1$. The term in braces is the discrete $1/N$-scaled inverse Fourier transform computed by conventional FFT routines (for example FFT by Singleton, 1979; SSP routine HARM). The $(\omega_{\max} - \omega_{\min})/2\pi$ is a factor that this particular discrete transform must be multiplied by in order to approximate the inverse transform defined by equation (1b). The $\exp(is_j\bar{\omega})$ results from the $\bar{\omega}$ shift in the ω-domain.

After multiplying the above function by $\exp[\pm(\pi/2)(s_j + ik)]\Gamma(k - is_j)$, the inverse transform is again approximated by a discrete inverse, the multiplying factor this time being $(s_{\max} - s_{\min})/2\pi = N/(\omega_{\max} - \omega_{\min})$. Thus, the two multiplying factors together give $N/2\pi$, and the frequency shift of $\bar{\omega}$ in the ω-domain results in a phase shift of the same amount, but of opposite sign, in the τ-domain. Table 1 gives the sampling in the various domains; in order to describe the procedure we introduce two more domains, ω' and τ', which are related to their respective domains by a shift along their abscissae, which introduces a phase factor of $\exp(-is\bar{\omega})$ in the s-domain.

The gamma function $\Gamma(k - is)$ is efficiently derived by calculating $\Gamma(l + k - is)$, l being a positive integer, using the asymptotic expansion formula which contains a summation involving Bernoulli's numbers (Abramov 1960, p. 3; Abramowitz & Stegun 1970, 6.1.40), then invoking the recurrence relation for gamma functions $\Gamma(l + k - 1 - is) = \Gamma(l + k - is)/(l + k - 1 - is)$ (Abramowitz & Stegun 1970, 6.1.15). (A value for l greater than four ensures that the gamma function is derived to seven significant figures when only the first three Bernoulli numbers are used. See also Abramov 1960, p. 5).

Table 1. Sampling and end points in the various domains

Domain	Relation	Minimum	Maximum	Mid-point	Sample interval
ν		$\nu_{\min}$	$\nu_{\max}$		logarithmic
ω	$\log(\nu)$	$\omega_{\min}$	$\omega_{\max}$	$\bar{\omega} = \frac{\omega_{\min} + \omega_{\max}}{2}$	$\frac{\omega_{\max} - \omega_{\min}}{N}$
ω'	$\omega - \bar{\omega}$	$-\frac{\omega_{\max} - \omega_{\min}}{2}$	$\frac{\omega_{\max} - \omega_{\min}}{2}$	0	$\Delta\omega$
s	$F^{-1}\{[\omega']\}$	$\frac{-\pi N}{\omega_{\max} - \omega_{\min}}$	$\frac{\pi N}{\omega_{\max} - \omega_{\min}}$	0	$\frac{2\pi}{\omega_{\max} - \omega_{\min}}$
τ'	$F^{-1}\{[s]\}$	$\frac{-\pi}{\Delta s}$	$\frac{\pi}{\Delta s}$	0	$\frac{2\pi}{2s_{\max}} (= \Delta\omega)$
τ	$\tau' + \bar{\tau}$	$-\omega_{\max}$	$-\omega_{\min}$	$-\bar{\omega}$	$\Delta\omega$
t	e^{τ}	$\frac{1}{\nu_{\max}}$	$\frac{1}{\nu_{\min}}$		logarithmic

AN EXAMPLE OF THE METHOD

The example chosen to illustrate the above described method for LFT is taken from Jones (1986). The vertical magnetic field $H_z(y, f)$ observed on the surface of the Earth, of conductivity σ, due to a line-current flowing in the x-direction at depth d within the Earth is given by

$$H_z(y, f) = \frac{I}{2} \mathrm{F}\left\{\frac{iv}{\pi(\eta + |v|)} \exp(-\eta d)\right\}, \tag{11}$$

where $\eta = (v^2 + i2\pi f \mu \sigma)^{1/2}$, v is the wavenumber, μ is the permeability of the Earth, and I is the current flowing in the line at frequency f. It can be shown that $\alpha = -1$, $\beta = \infty$, and thus k can be chosen in the range $0 < k < 1$. The kernel of the FT is

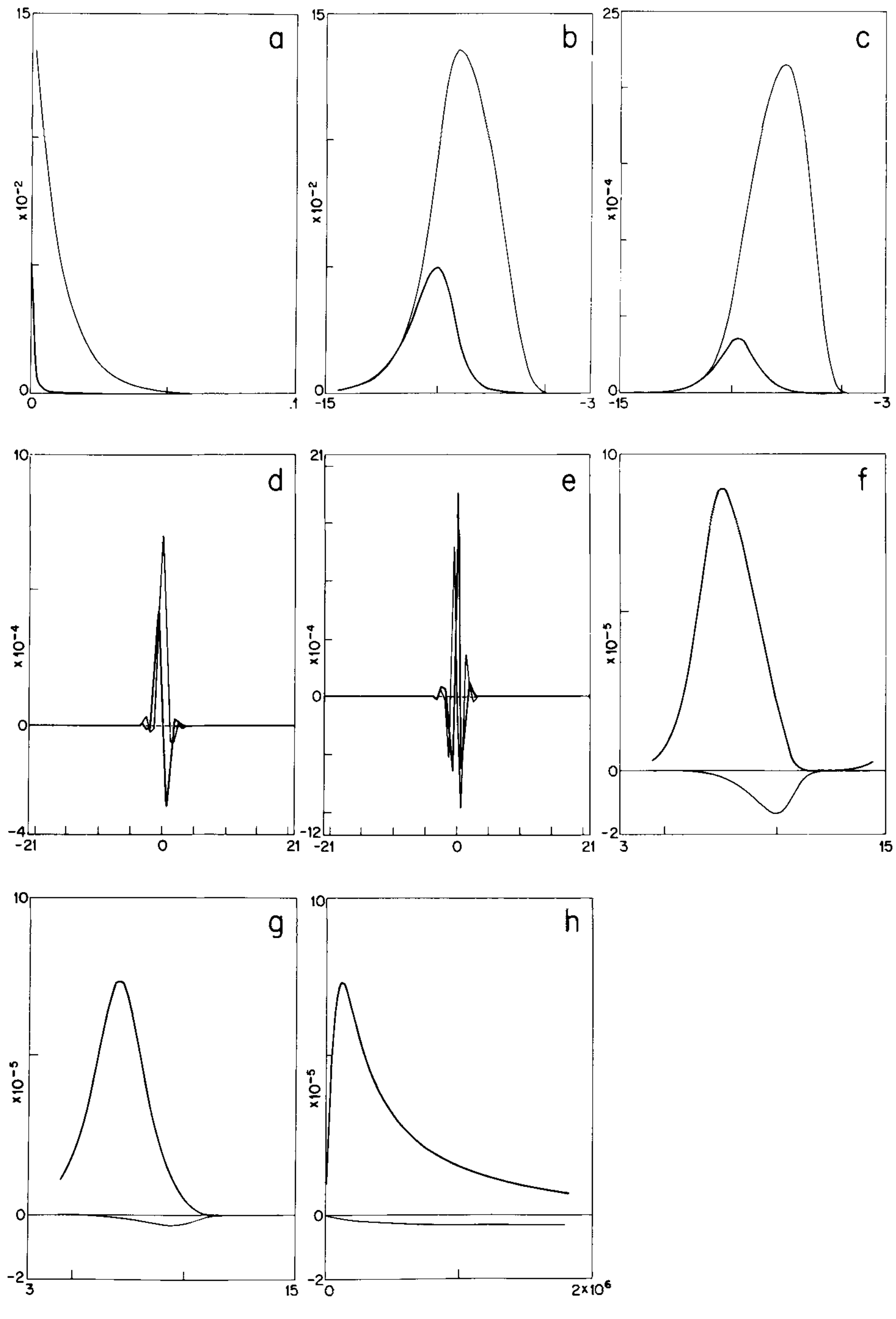

Figure 1. The procedure to compute the ILFT. (a) $F(v)$ for $v > 0$; (b) $G_2(\omega)(= -G_1(\omega)$ in this example); (c) $e^{\omega/2}[b]$; (d) $\mathrm{F}^{-1}\{[c]\}$; (e) $e^{\pi/2(s+1/2)}\Gamma(\frac{1}{2} - is)$ [d]; (f) $\mathrm{F}^{-1}\{[e]\}$; (g) $g_2(\tau) = e^{-\tau/2}[f]$; (h) $f(t)$, for $t > 0$. The heavy lines are the real parts, and the light lines are the imaginary parts.

antisymmetric about $v=0$, i.e. $G_1(\omega)=-G_2(\omega)$, and thus the LFT reduces to

$$g_2(\tau)=2\pi\exp(-k\tau)\mathrm{F}^{-1}\Big\{\Gamma(k-is)\Big(-\exp\Big[\frac{\pi}{2}(s+ik)\Big]+\exp\Big[-\frac{\pi}{2}(s+ik)\Big]\Big)\mathrm{F}^{-1}\{G_2(\omega)\exp[(1-k)\omega]\}\Big\}$$

$$=-4\pi\exp(-k\tau)\mathrm{F}^{-1}\Big\{\Gamma(k-is)\sinh\Big(\frac{\pi}{2}(s+ik)\Big)\mathrm{F}^{-1}\{G_2(\omega)\exp[(1-k)\omega]\}\Big\}.$$

The sequence of operations required to perform the ILFT is illustrated in Fig. 1(a)–(h).

For a line-current excited at 1 Hz at a depth of 1 km in an earth of conductivity 0.001 S m^{-1}, and free-space permeability, the kernel $F(v)$ of the Fourier transform in (11) is as shown in Fig. 1(a). Fig. 1(b) illustrates $G_2(\omega)(=K_{Hz}(\exp\omega))$, and the bandwidth, given by $\omega_B=\omega_{max}-\omega_{min}$, is chosen to be 10. Thus $\Delta s=2\pi/\omega_B=0.628318$, and for $N=64$ then $s_{max}=\pi N/\omega_B=20.106$ (see Table 1). For this example we choose $k=\frac{1}{2}$, and Fig. 1(c) illustrates $G_2(\omega)\exp(\omega/2)$. The IFT of this function is illustrated in Fig. 1(d), and for the function under consideration there is obviously little 'high-frequency' content in that the contributions for $|s|>\pi$ are smaller than 1 per cent of the maximum value. Thus, the gamma and exponential functions need only be computed out to this value if speed is of the essence. The multiplication of this function with the gamma and exponential functions for $k=\frac{1}{2}$ is illustrated in Fig. 1(e), and the IFT of this in Fig. 1(f). The function $g_2(\tau)$, given by $\exp(\tau/2)$ times the function shown in Fig. 1(f), is illustrated in Fig. 1(g), and finally the conversion to $f(t)$ is in Fig. 1(h).

The numerical result is dependent on the choice of k and N, with the dependence on k decreasing with increasing N. For $k=0.1$, 0.5, 0.9 and $N=64$, the ILFT gives the results for Re $\{g_2(\tau)\}$ as shown in Fig. 2(a). Keeping k constant at 0.5, and varying N from 64, 128, 256, 512 gives ILFTs as shown in Fig. 2(b). At this larger value of N, the variation with varying values of k is less than 1 per cent. It must be noted that for $k=0.9$ and $N=64$, the result at small distances was unphysical in that the derived imaginary part was positive.

In Fig. 3 is illustrated a comparison of the ILFT, for $N=512$, with an FFT result derived by Jones (1986) using a 16 384-point FFT. Also shown in the figure is the result of a 65 536-point FFT in order to obtain estimates at short distances. Note that the 16 384-point FFT 'clipped' the peak of the response, whereas the ILFT and the 65 536-point FFT values agree to better than 0.5 per cent.

CONCLUSIONS

We have described how the Fourier transform integral may be performed if the sampling of the parameter is logarithmic. The theory presented is exact and analytical, not approximate as were some previously suggested methods. It is important to realize that only certain functions or responses can be transformed both ways. These functions must have zero contribution at zero in

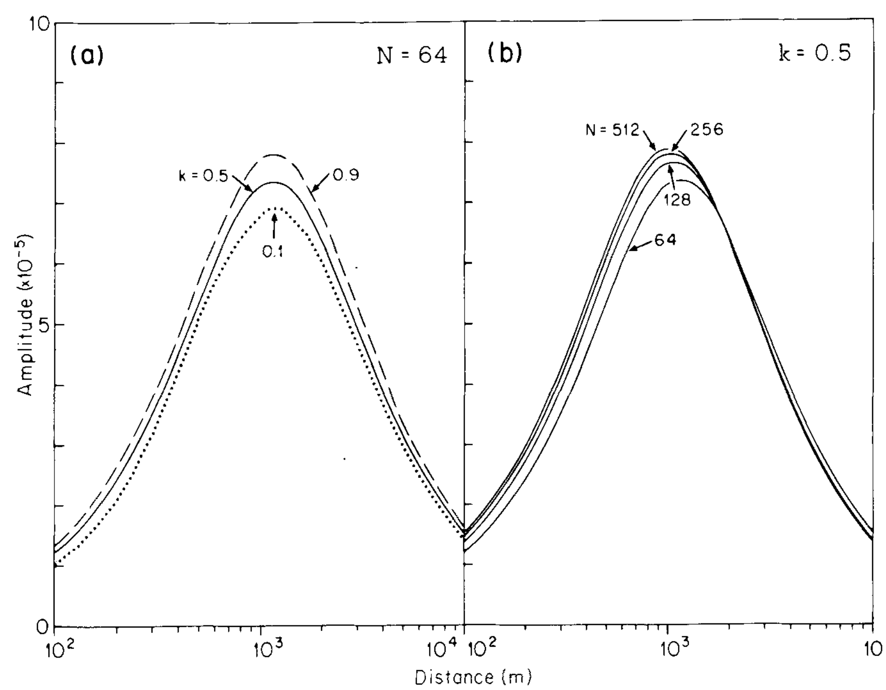

Figure 2. (a) Comparison of varying k for Re $\{g_2(\tau)\}$ with N constant at 64, $k=0.1$ in dotted line, $k=0.5$ in full line, $k=0.9$ in dashed line. (b) Comparison of varying N for k held constant at $\frac{1}{2}$.

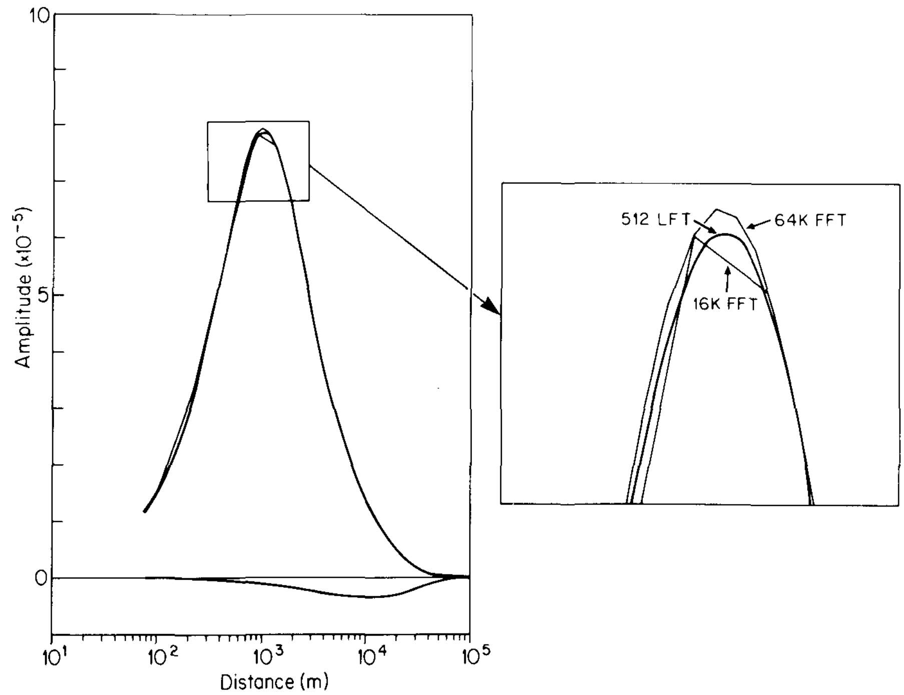

Figure 3. Comparison of 16 384 and 65 536 point conventional FFTs to a 512 point LFT. Note that the 16 384 point FFT has 'clipped' the top of the response (insert).

both domains, e.g., *odd* functions. The 'trade-off' parameter k must lie within bounds set by $\max(\alpha, 0) < 1 - k < \min(\beta, 1)$, and thus the appropriate α and β values must be determined. In Talman's (1978) approach, this factor was chosen to be $\frac{1}{2}$, however there can exist functions for which $\alpha > \frac{1}{2}$ or $\beta < \frac{1}{2}$.

If the LFT can be applied, then obviously the saving in computational cost is enormous. By the 'traditional' FFT method, assuming there are m operations required to derive the kernel function at one frequency, for N^* points there are mN^* operations to derive the kernel function, and $N^* \log_2 (N^*)$ to perform the FFT. By the LFT method, for N points, the kernel function need only be derived mN times, effecting an immediate saving of a factor of N^*/N. Also, the two IFFTs only require $N \log_2 (N)$ operations. The other operations are a function of N, and these can be reduced if the gamma and exponential functions are only derived for necessary s values or if they are kept in a look-up table. For the example given here, we can satisfactorily replace a 65 536-point FFT, requiring over 1 million operations for the FFT alone, by an ILFT requiring of the order of 5000 in total, effecting a speed increase of over two orders of magnitude.

Numerical implementation of the LFT requires due care in the choices of the k and N values for a given problem. In our example, for $k = 0.9$ and $N = 64$ we derived an unphysical imaginary part of the response at short distances, although the real part was closer to that given by the conventional FFT. As N increased, the results for varying k converged.

ACKNOWLEDGMENT

Geological Survey of Canada contribution 45786.

REFERENCES

Abramov, A. A., 1960. *Tables of ln Γ[z] for Complex Argument*, Pergamon Press, New York.

Abramowitz, M. and Stegun, I. A., 1970. *Handbook of Mathematical Functions*, Dover, New York.

Anderson, W. L., 1979. Numerical integration of related Hankel transforms of orders 0 and 1 by adaptive digital filtering, *Geophysics*, **44**, 1287–1305.

Asten, M. W. & Verma, S. K., 1978. Conversion of wideband EM frequency response to transient response using segmented transformation, *Geophysics*, **43**, 189–193.

Chen, C. & Haas, I. J., 1968. *Elements of Control Systems Analysis*, Prentice Hall, New York.

Copson, E. T., 1935. *An Introduction to the Theory of Functions of a Complex Variable*, Oxford University Press, London.

Dey, A. & Morrison, H. F., 1973. Electromagnetic coupling in frequency and time-domain induced polarization surveys over a multilayered earth, *Geophysics*, **38**, 380–405.

Erdélyi, A., 1953. *Higher Transcendental Functions, Vol.* 1, McGraw-Hill, New York.

Gradshteyn, I. S. & Ryzhik, I. M., 1980. *Tables of Integrals, Series and Products*, Academic Press, New York.

Holladay, J. S., 1981. YVESFT and CHANNEL: a subroutine package for stable transformation of sparse frequency domain electromagnetic data to the time domain, *Research in Applied Geophysics Rep. No.* 17, University of Toronto.

Hostetter, G. H., 1982. Recursive discrete Fourier transformation with unevenly spaced data, *IEEE Trans. Acoust., Speech, Signal Processing,* **ASSP 31,** 206–209.
Johansen, H. K. & Sørensen, K., 1979. Fast Hankel transforms, *Geophys. Prospect.,* **27,** 876–991.
Jones, A. G., 1986. On the use of line current analogues in geomagnetic depth sounding, *J. Geophys.,* **60,** 56–62.
Lamontagne, Y., 1975. Applications of wideband, time-domain electromagnetic measurements in mineral exploration, *PhD thesis,* University of Toronto.
Lee, T. & Lewis, R., 1974. Transient EM response of a large loop on a layered ground, *Geophys. Prospect.,* **22,** 430–444.
Nissen, J. & Enmark, T., 1986. An optimized digital filter for the Fourier transform, *Geophys. Prospect.,* **34,** 897–903.
Palacky, G. J. & West, G. F., 1973. Quantitative interpretation of input AEM measurements, *Geophysics,* **38,** 1145–1158.
Singleton, R. C., 1979. Mixed radix fast Fourier transforms, in *Programs for Digital Signal Processing,* pp. 1.4-1–1.4-18, ed. Digital Signal Processing Committee, IEEE-ASSP, IEEE Press, New York.
Talman, J. D., 1978. Numerical Fourier and Bessel transforms in logarithmic variables, *J. Comput. Phys.,* **29,** 35–48.